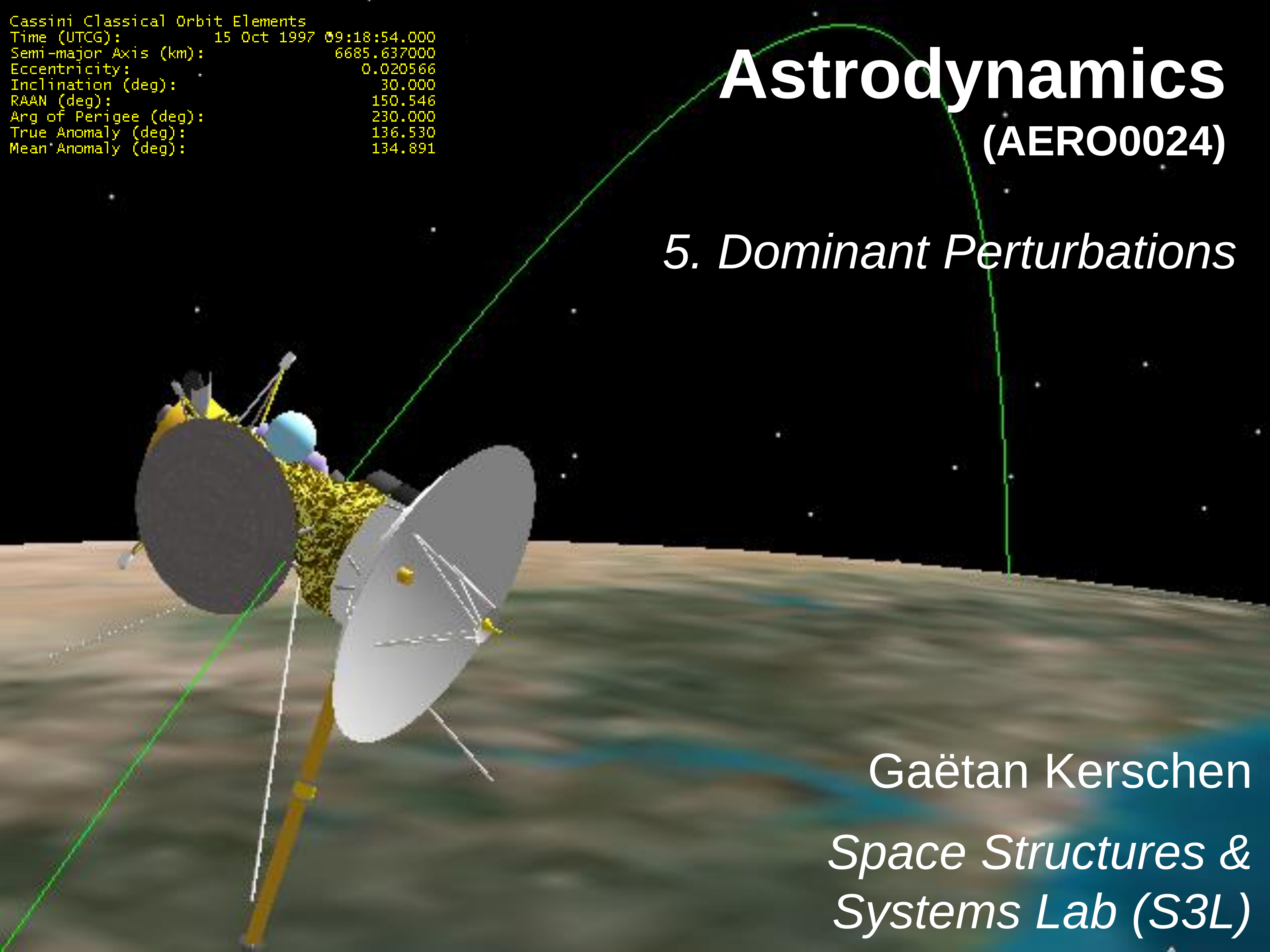
Astrodynamics
(AERO0024)

5. Dominant Perturbations



Gaëtan Kerschen

Space Structures & Systems Lab (S3L)

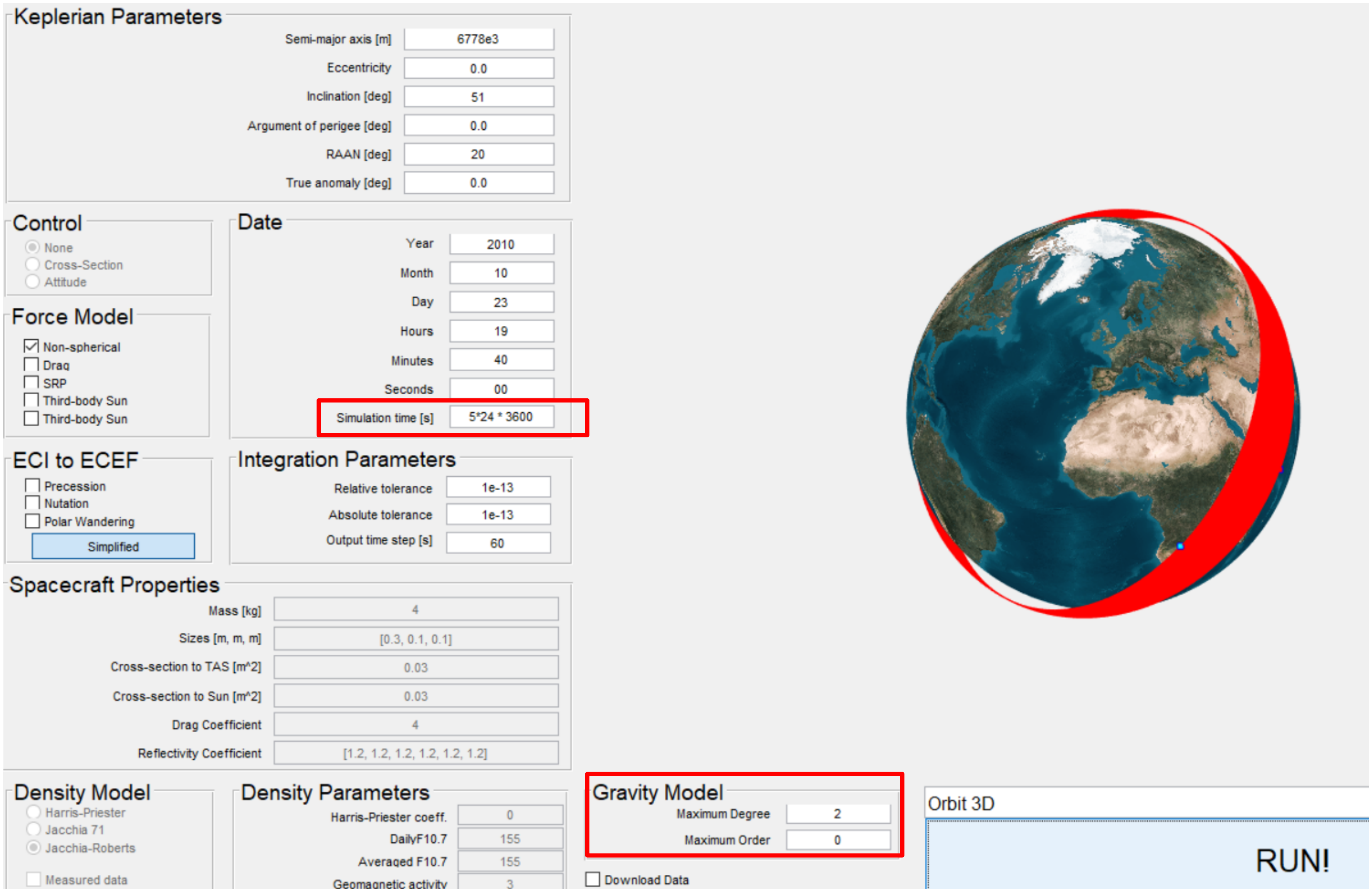
Motivation

Assumption of a two-body system in which the central body acts gravitationally as a point mass.

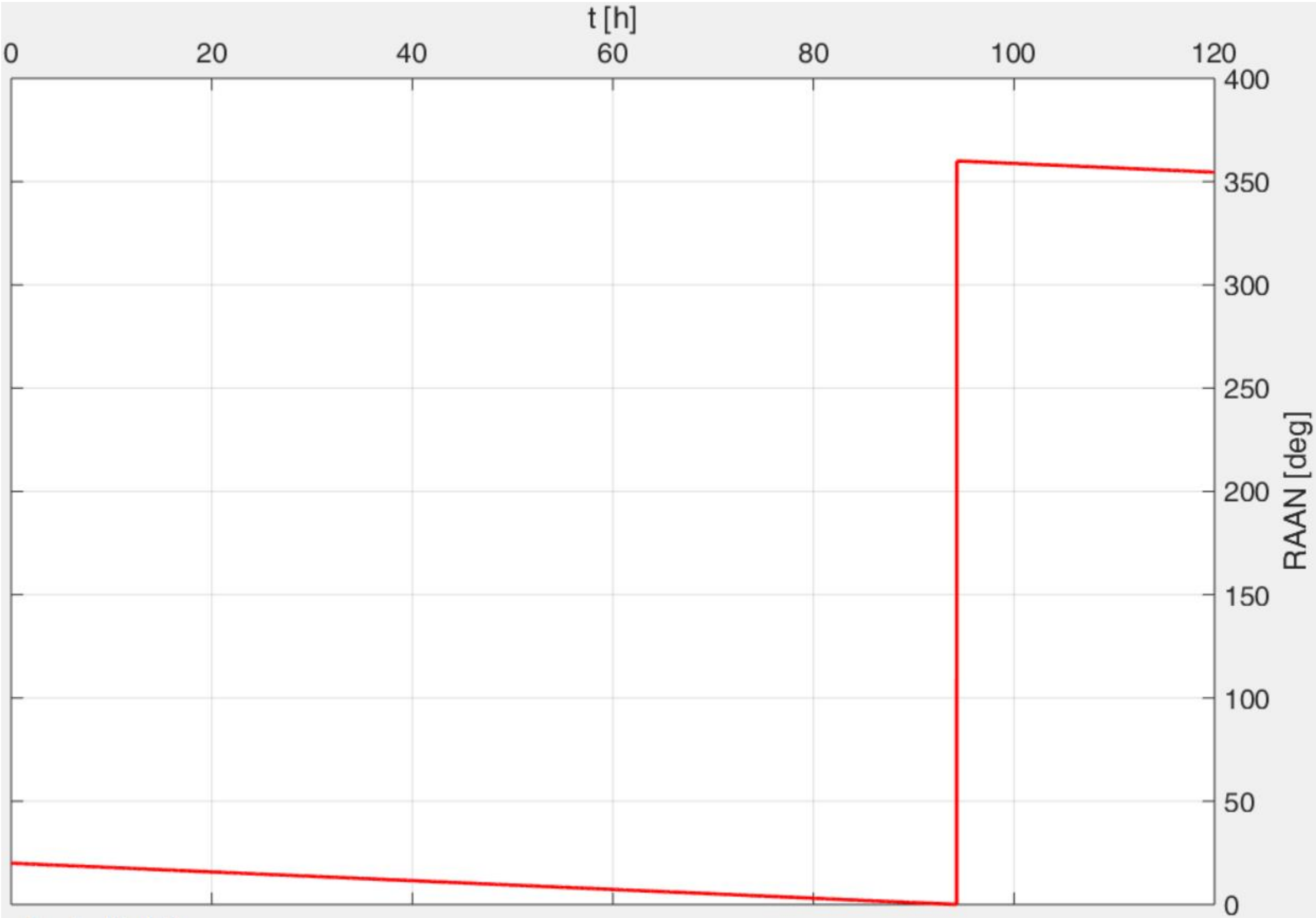
In many practical situations, a satellite experiences significant perturbations (accelerations).

These perturbations are sufficient to cause predictions of the position of the satellite based on a Keplerian approach to be in significant error in a brief time.

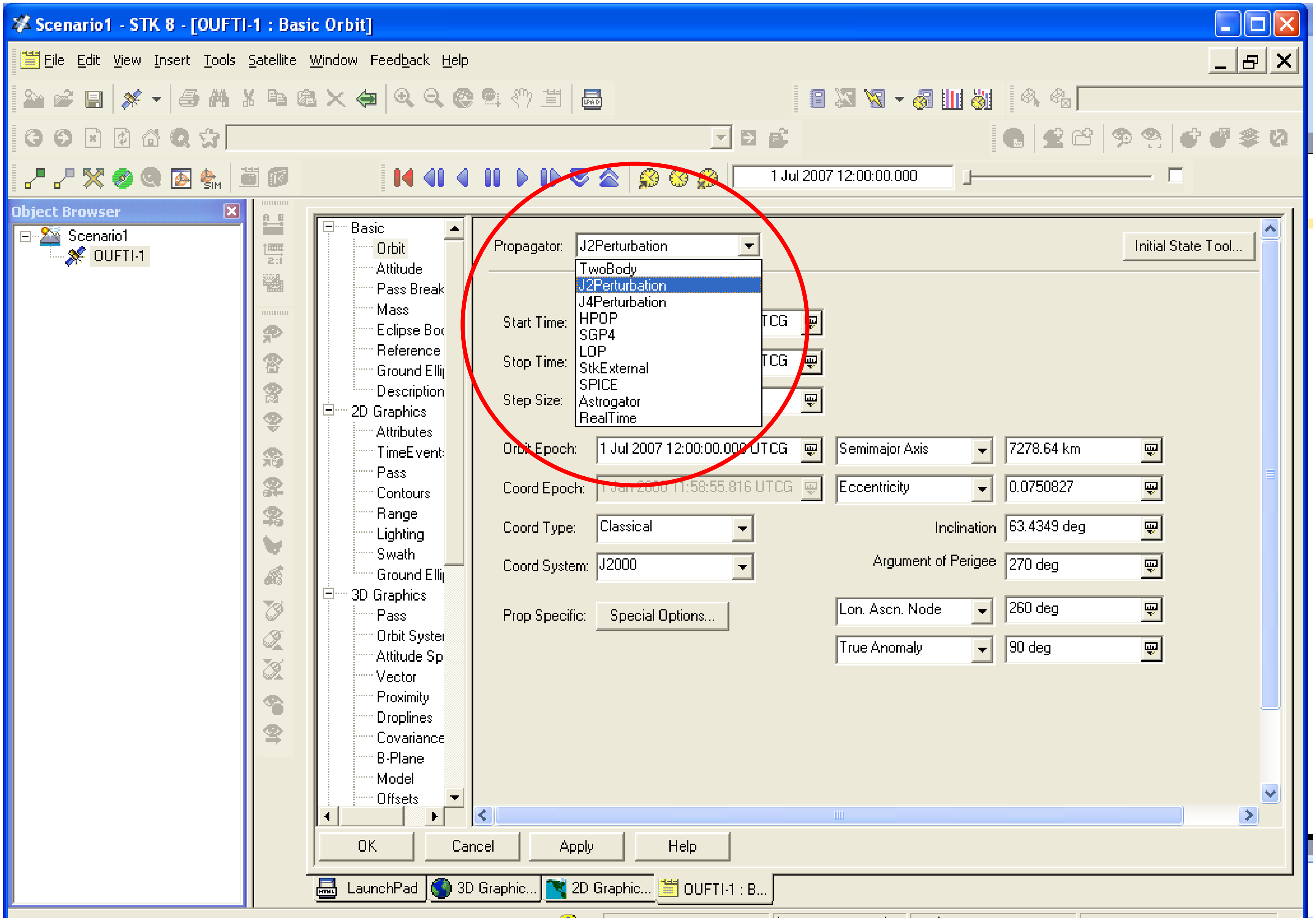
The Effect of Earth Oblateness



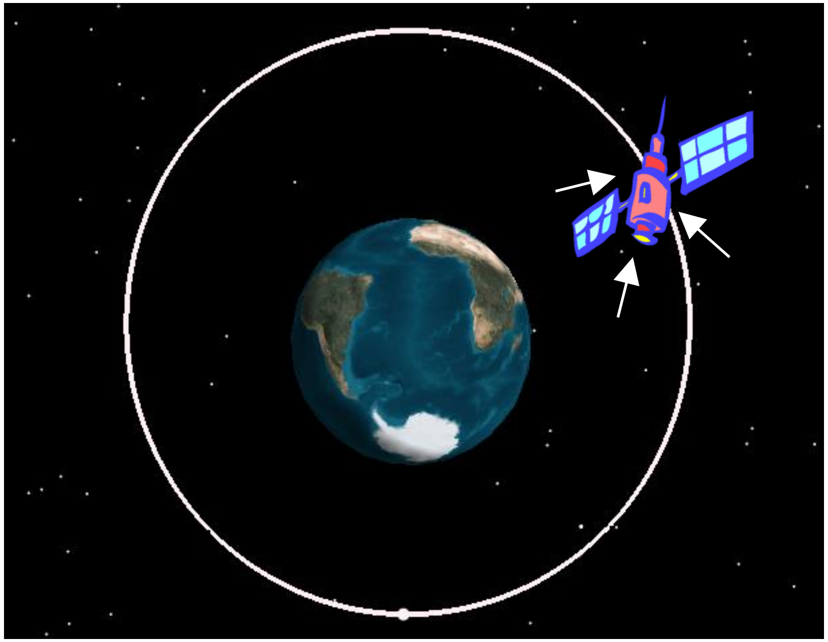
The Effect of Earth Oblateness



STK: Different Propagators



Non-Keplerian Motion



Dominant perturbations

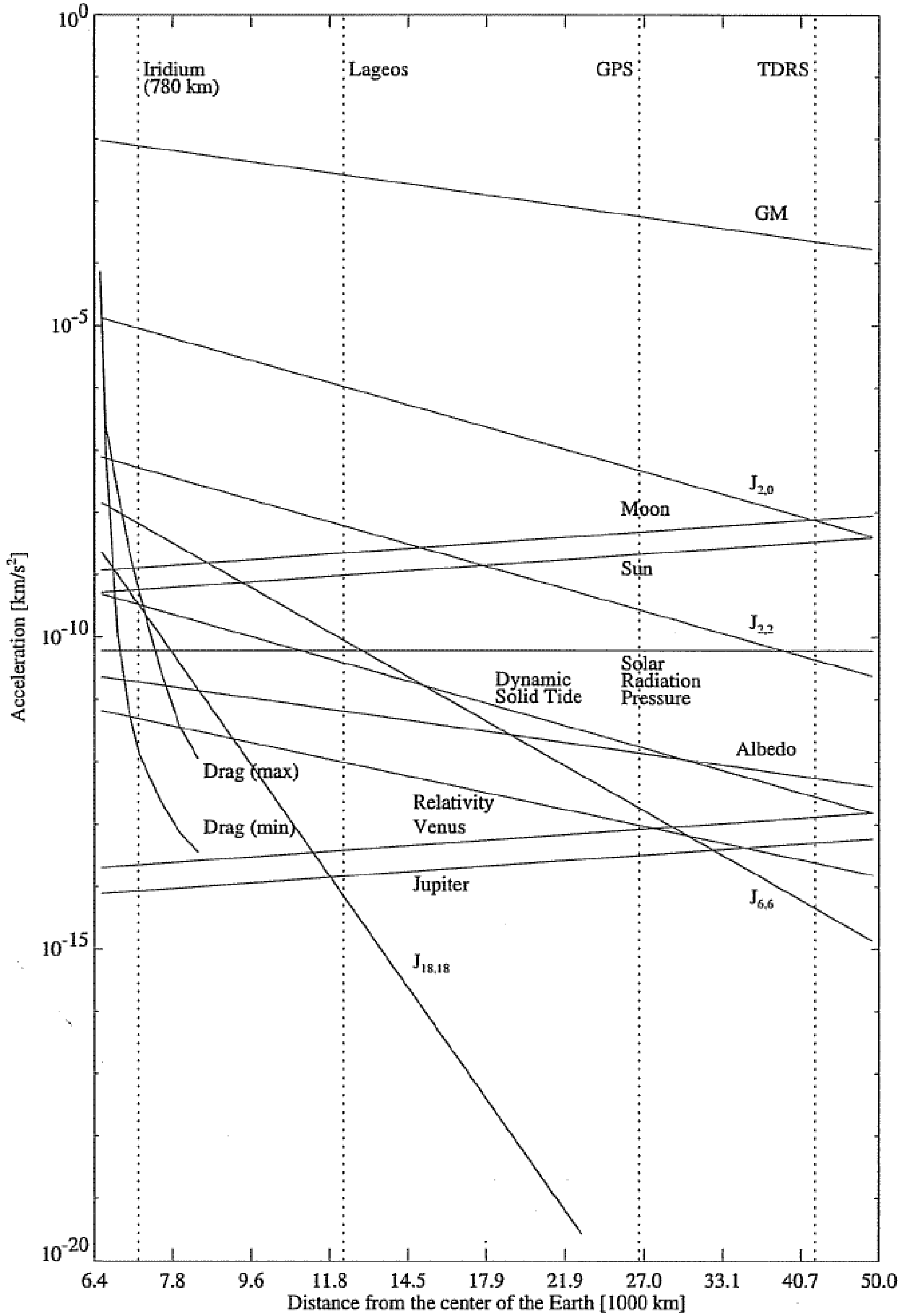
- Earth's gravity field
- Atmospheric drag
- Third-body perturbations
- Solar radiation pressure

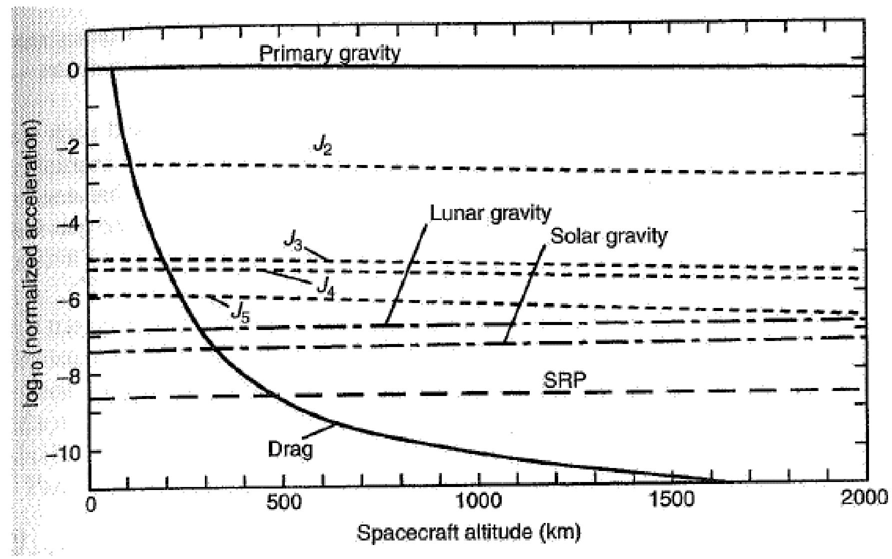
Different Perturbations and Importance ?

In low-earth orbit (LEO) ?

In geostationary orbit (GEO) ?

Satellite dependent !





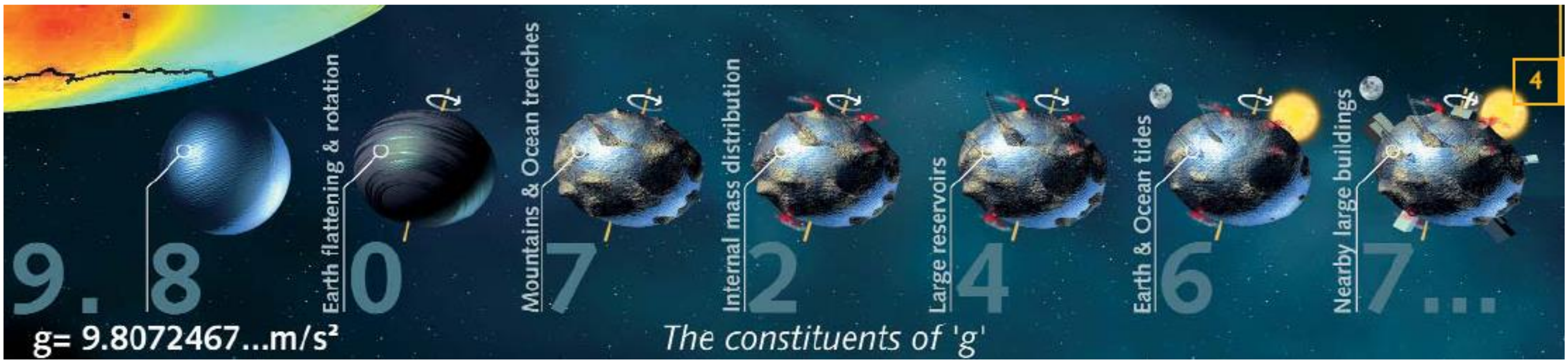
Montenbruck and Gill, *Satellite orbits,* Springer, 2000

Fortescue et al., *Spacecraft systems engineering*, 2003

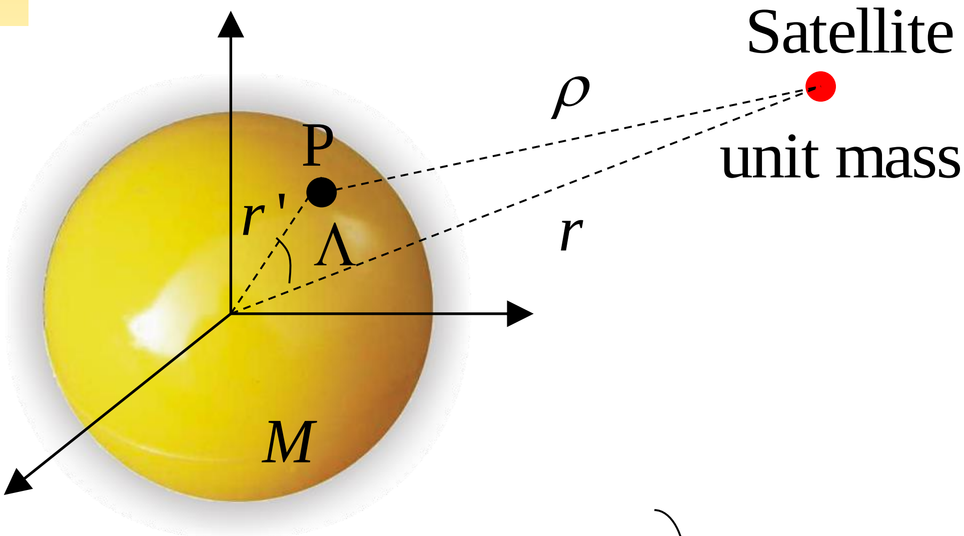
Orders of Magnitude

400 kms	1000 kms	36000 kms
Oblateness	**Oblateness**	Oblateness
Drag	Sun and moon	Sun and moon
		SRP

The Earth is not a Sphere…



Mathematical Modeling



$$r = \sqrt{x^2 + y^2 + z^2}$$

$$r' = \sqrt{\xi^2 + \eta^2 + \zeta^2}$$

U=-V, potential, $\ddot{\mathbf{r}} = \nabla U$

V, potential energy

$$U = G\int_{body} \frac{dm}{\rho}$$

$$\rho = \sqrt{r^2 + r'^2 - 2r'r\cos\Lambda}$$

$$\cos\Lambda = \frac{\mathbf{r}.\mathbf{r}'}{r.r'} \qquad \alpha = \frac{r'}{r} < 1$$

$$U = G\int_{body} \frac{dm}{r\sqrt{1 - 2\alpha\cos\Lambda + \alpha^2}}$$

Legendre Polynomials

First introduced in 1782 by Legendre

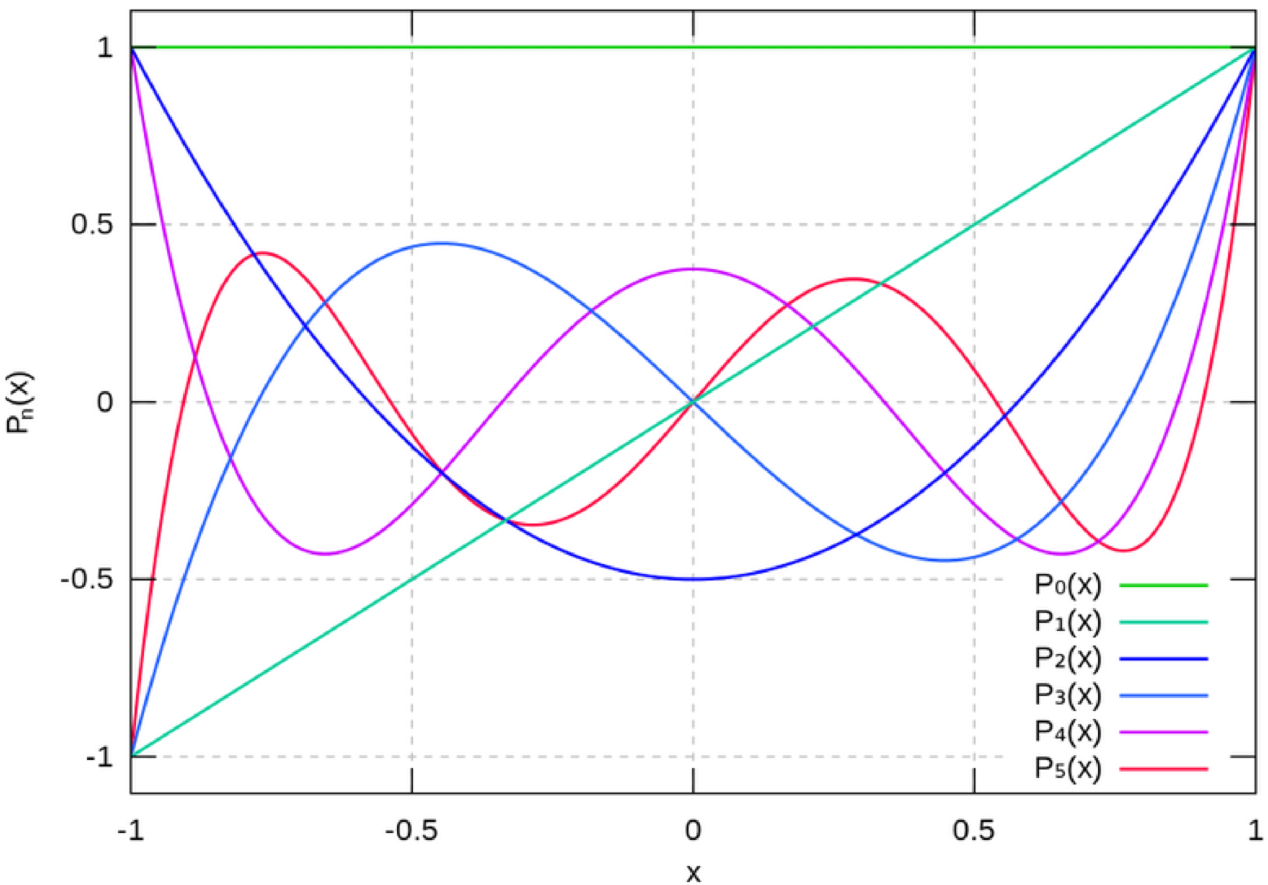
$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}\left(3x^2 - 1\right)$
3	$\frac{1}{2}\left(5x^3 - 3x\right)$
4	$\frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n$$

Legendre Polynomials Are Orthogonal

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0 \quad \text{if } n \neq m.$$



Let's Use Them

$$U = G\int_{body} \frac{dm}{r\sqrt{1-2\alpha\cos\Lambda+\alpha^2}}$$

$$\Downarrow \qquad \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

$$U = \frac{G}{r}\int_{body} \sum_{l=0}^{\infty} \alpha^l P_l[\cos(\Lambda)]\, dm$$

Summing Up…

$$U = \frac{G}{r}\int_{body}\sum_{l=0}^{\infty}\alpha^{l}P_{l}[\cos(\Lambda)]\,dm$$

Geometric method (intuitive feel for gravity and inertia)

$$U = U_0 + U_1 + U_2 + \ldots$$

Theory

Spherical-harmonic expansion

Experiments

Geometric Method: First Term

$$U_0 = \frac{G}{r}\int dm = \frac{\mu}{r}$$

Two-body potential

Geometric Method: Second Term

$$U_1 = \frac{G}{r}\int \cos(\Lambda)\alpha\, dm = \frac{G}{r}\int \frac{x\xi + y\eta + z\zeta}{r^2}\, dm$$

$$= \frac{G}{r^3}\left(x\int \xi\, dm + y\int \eta\, dm + z\int \zeta\, dm\right) = 0$$

Center of mass at the origin of
the coordinate frame

Geometric Method: Third Term

$$U_2 = \frac{G}{r}\int \frac{\alpha^2}{2}\left(3\cos^2\Lambda - 1\right) dm$$

$$= \frac{G}{2r^3}\int 2r'^2\, dm - \frac{G}{2r^3}\int 3r'^2 \sin^2\Lambda\, dm$$

$$= \frac{G}{2r^3}\left(A + B + C - 3I\right)$$

$$\int 2r'^2\, dm = \int\left(\eta^2 + \zeta^2\right) dm + \int\left(\xi^2 + \zeta^2\right) dm + \int\left(\eta^2 + \xi^2\right) dm$$

$= A + B + C$ Moments of inertia

$\int r'^2 \sin^2\Lambda\, dm = I$ Polar moment of inertia

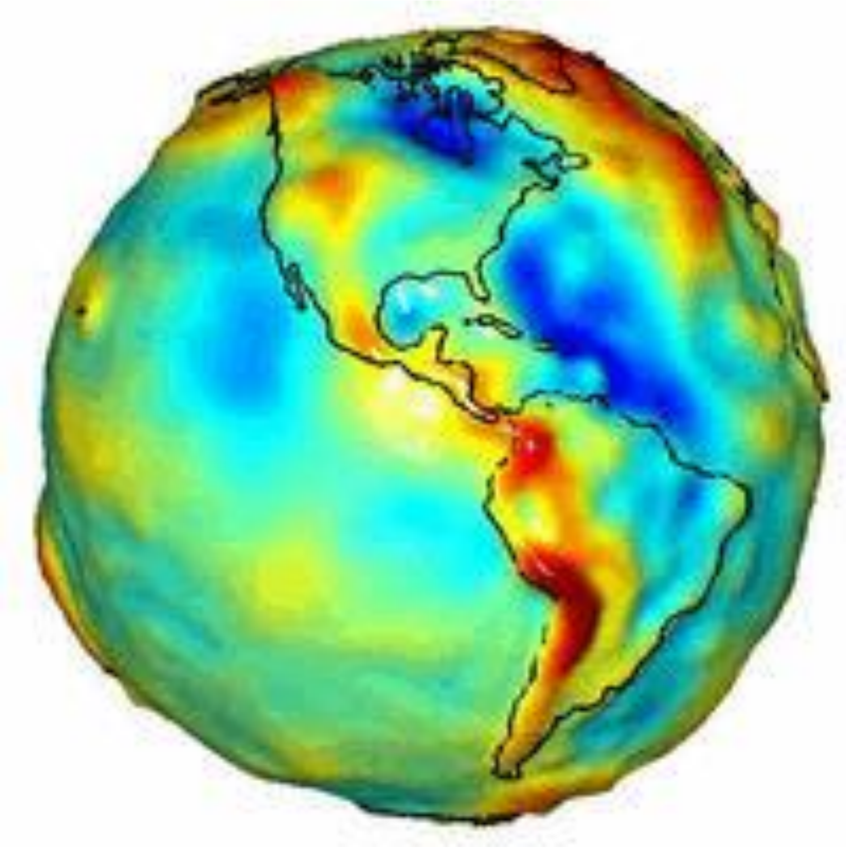
Geometric Method: MacCullagh's Formula

$$U = \frac{Gm_{\oplus}}{r} + \frac{G}{2r^3}\left(A + B + C - 3I\right) + \ldots$$

Some of the simplest assumptions are

- the ellipsoidal Earth (oblate spheroid) with uniform density (a=b>c).
- triaxial ellipsoid (a>b>c).

Geometric Method: Difficult to Go Further…



Summing up…

$$U = \frac{G}{r}\int_{body}\sum_{l=0}^{\infty}\alpha^l P_l[\cos(\Lambda)]\,dm$$

Geometric method (intuitive feel for gravity and inertia)

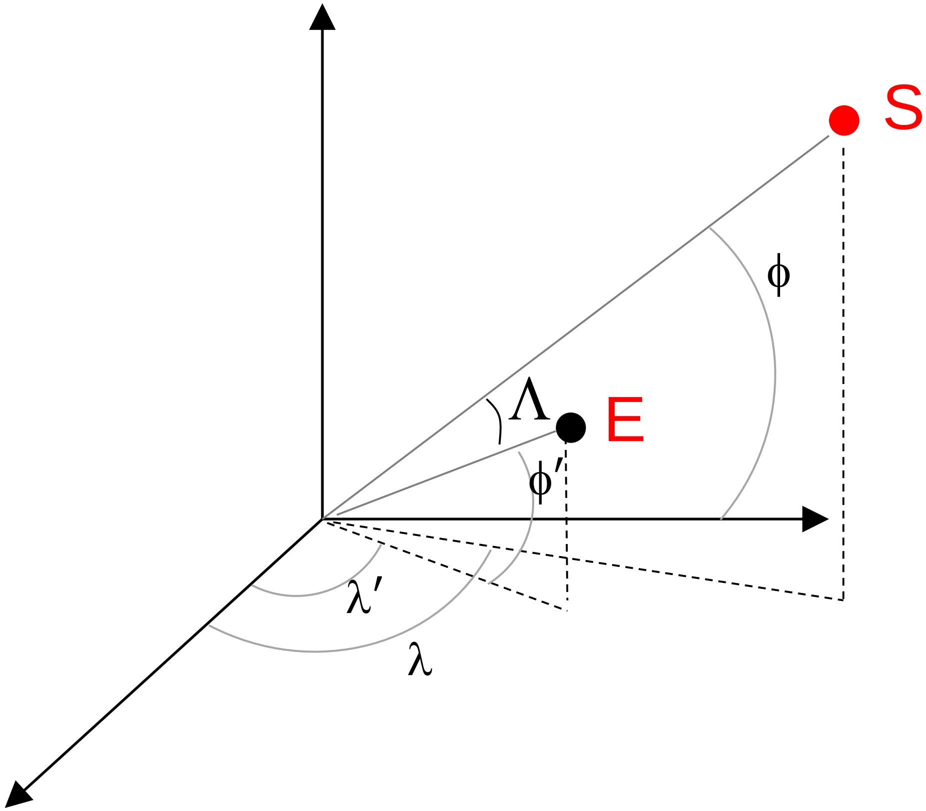
$$U = U_0 + U_1 + U_2 + \ldots$$

Theory

Spherical-harmonic expansion

Experiments

Spherical Trigonometry



$\phi \rightarrow$ latitude sat
$\lambda \rightarrow$ longitude sat
$\phi' \rightarrow$ latitude Earth
$\lambda' \rightarrow$ longitude Earth

$$\cos \Lambda = \cos(90 - \phi') \cos(90 - \phi) + \sin(90 - \phi') \sin(90 - \phi) \cos(\lambda - \lambda')$$

Addition Theorem for Spherical Harmonics

If $\cos\Lambda = \cos(90-\phi')\cos(90-\phi)+\sin(90-\phi')\sin(90-\phi)\cos(\lambda-\lambda')$

Then,

$$P_l(\cos\Lambda) = \sum_{m=0}^{l}(2-\delta_{0m})\frac{(l-m)!}{(l+m)!}P_{lm}(\sin\phi)P_{lm}(\sin\phi')\cos\big(m(\lambda-\lambda')\big)$$

where $P_{lm}(u) = (1-u^2)^{m/2}\dfrac{d}{du^m}P_n(u)$

Associated Legendre polynomial of degree *l* and order *m*

$$U = \frac{G}{r}\int_{body}\sum_{l=0}^{\infty}\alpha^l P_l(\cos\Lambda)dm = \frac{G}{r}\int_{body}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\Lambda)dm$$

$$+$$

$$P_l(\cos\Lambda) = \sum_{m=0}^{l}(2-\delta_{0m})\frac{(l-m)!}{(l+m)!}P_{lm}(\sin\phi)P_{lm}(\sin\phi')\cos\big(m(\lambda-\lambda')\big)$$

$$\swarrow$$

$$(\cos m\lambda \cos m\lambda' + \sin m\lambda \sin m\lambda')$$

$$\Downarrow$$

Depends only on the satellite (r,ϕ,λ)

$$U = \frac{GM_\oplus}{r}\left\{\sum_{l=0}^{\infty}\sum_{m=0}^{l}\left(\frac{R_\oplus}{r}\right)^l P_{lm}(\sin\phi)\left(C_{lm}\cos m\lambda + S_{lm}\sin m\lambda\right)\right\}$$

Depends only on the Earth (ϕ',λ'): spherical harmonics

$$C_{lm} = \frac{(2-\delta_{0m})}{M_\oplus}\frac{(l-m)!}{(l+m)!}\int\left(\frac{r'}{R_\oplus}\right)^l P_{lm}(\sin\phi')\cos m\lambda' dm$$

$$S_{lm} = \frac{(2-\delta_{0m})}{M_\oplus}\frac{(l-m)!}{(l+m)!}\int\left(\frac{r'}{R_\oplus}\right)^l P_{lm}(\sin\phi')\sin m\lambda' dm$$

Normalization: End Result

$$U = \frac{GM_{\oplus}}{r}\left\{\sum_{l=0}^{\infty}\sum_{m=0}^{l}\left(\frac{R_{\oplus}}{r}\right)^{l}\bar{P}_{lm}(\sin\phi)\left(\bar{C}_{lm}\cos m\lambda + \bar{S}_{lm}\sin m\lambda\right)\right\}$$

$$\begin{Bmatrix}\bar{C}_{lm}\\ \bar{S}_{lm}\end{Bmatrix} = \sqrt{\frac{(l+m)!}{(2-\delta_{0m})(2n+1)(l-m)!}}\begin{Bmatrix}C_{lm}\\ S_{lm}\end{Bmatrix}$$

$$\bar{P}_{lm} = \sqrt{\frac{(2-\delta_{0m})(2n+1)(l-m)!}{(l+m)!}}P_{lm}$$

Very Important Remark

Many different expressions exist in the literature:

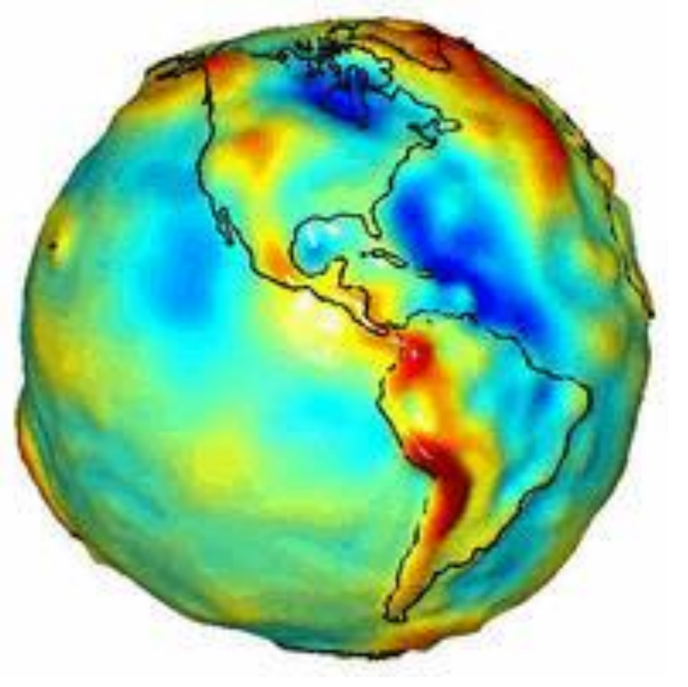
- ⇒ $V = \pm V$
- ⇒ $P_l^m = (-1)^m P_{lm}$
- ⇒ Normalized or non-normalized coefficients
- ⇒ Latitude or colatitude (sin ϕ or cosϕ)
- ⇒ …

Be always aware of the conventions/definitions used !

Spherical Harmonics

A set of functions used to represent functions on the surface of the sphere. They are a higher-dimensional analogy of Fourier series.

So any object that looks « kind-of-spherical » can be decomposed into an infinite sum of basic functions, as long as you multiply each basic function by the right coefficient



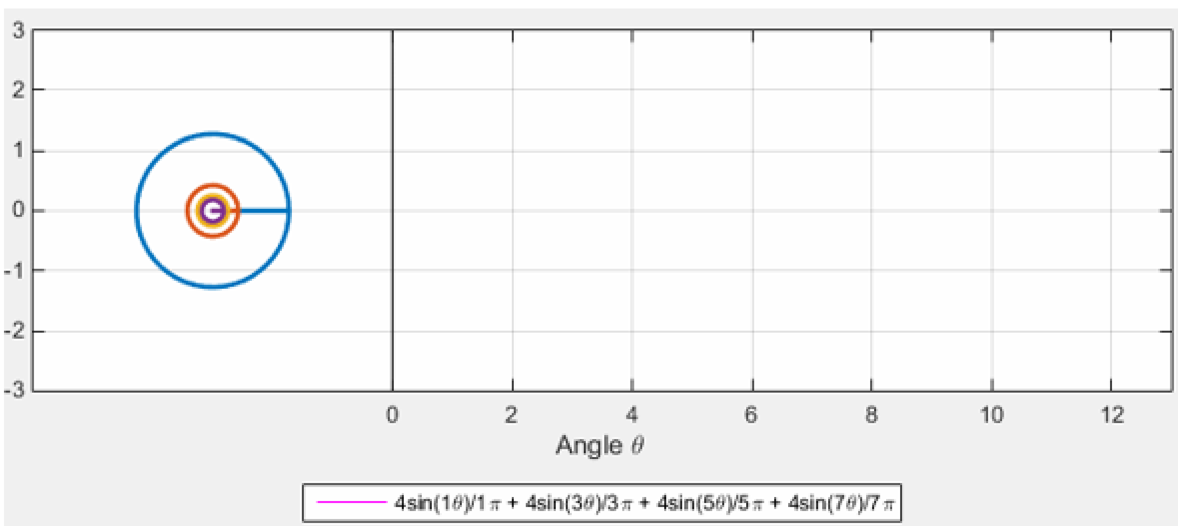
Our objective !

Fourier Series

A Fourier series is an expansion of a periodic function in terms of an infinite sum of sines and cosines.

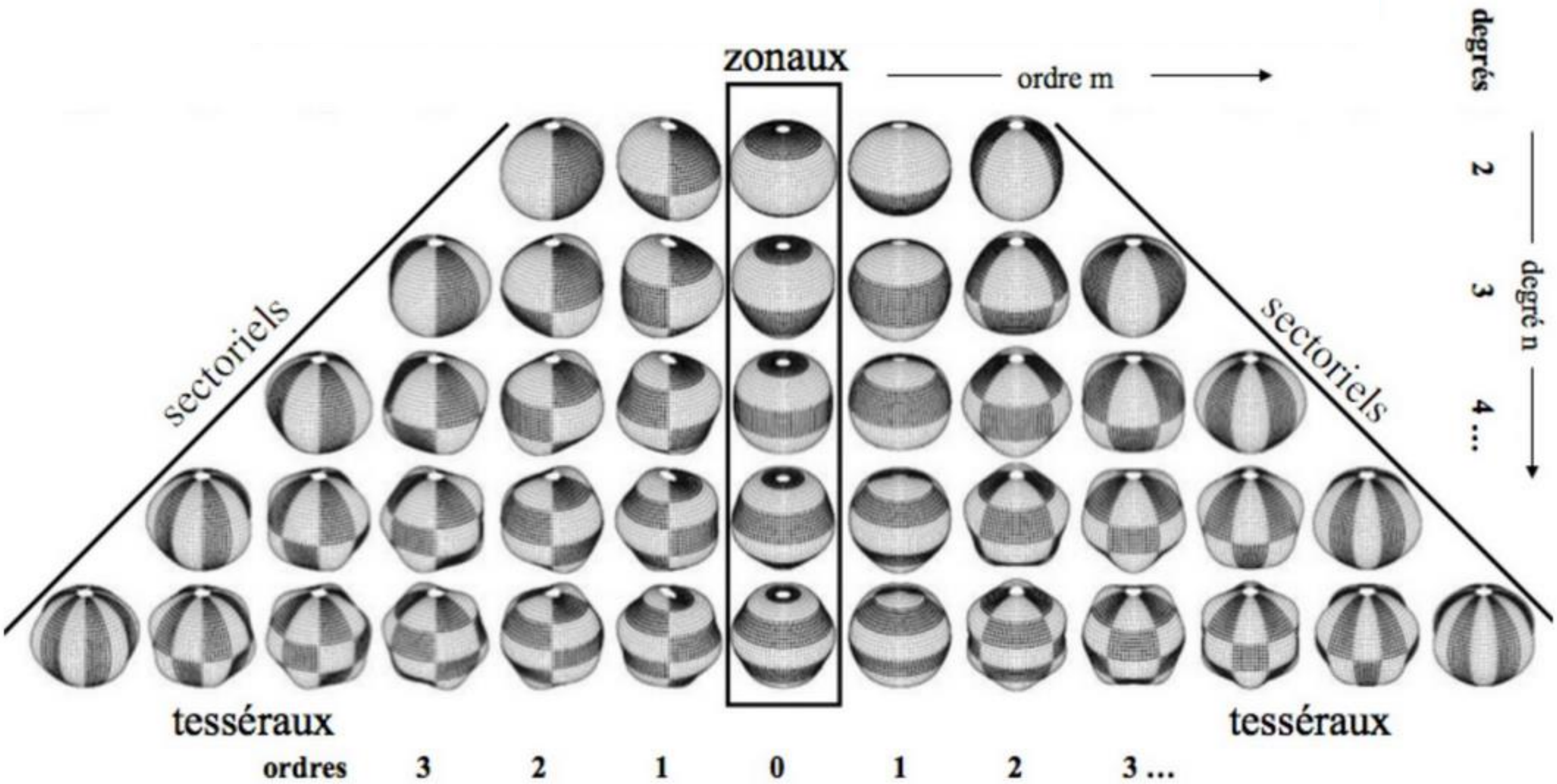
Fourier series make use of the orthogonality of sine and cosine functions.

https://www.youtube.com/watch?v=LznjC4Lo7lE



A square wave defined using 4 Fourier terms

Spherical Harmonics



The degree « n » is the total number of waves. The order « m » is the number of waves in longitude. The number of waves in latitude is thus « n – m ».

Zonal Harmonics (m=0)

Each boundary is a root of the Legendre polynomial.

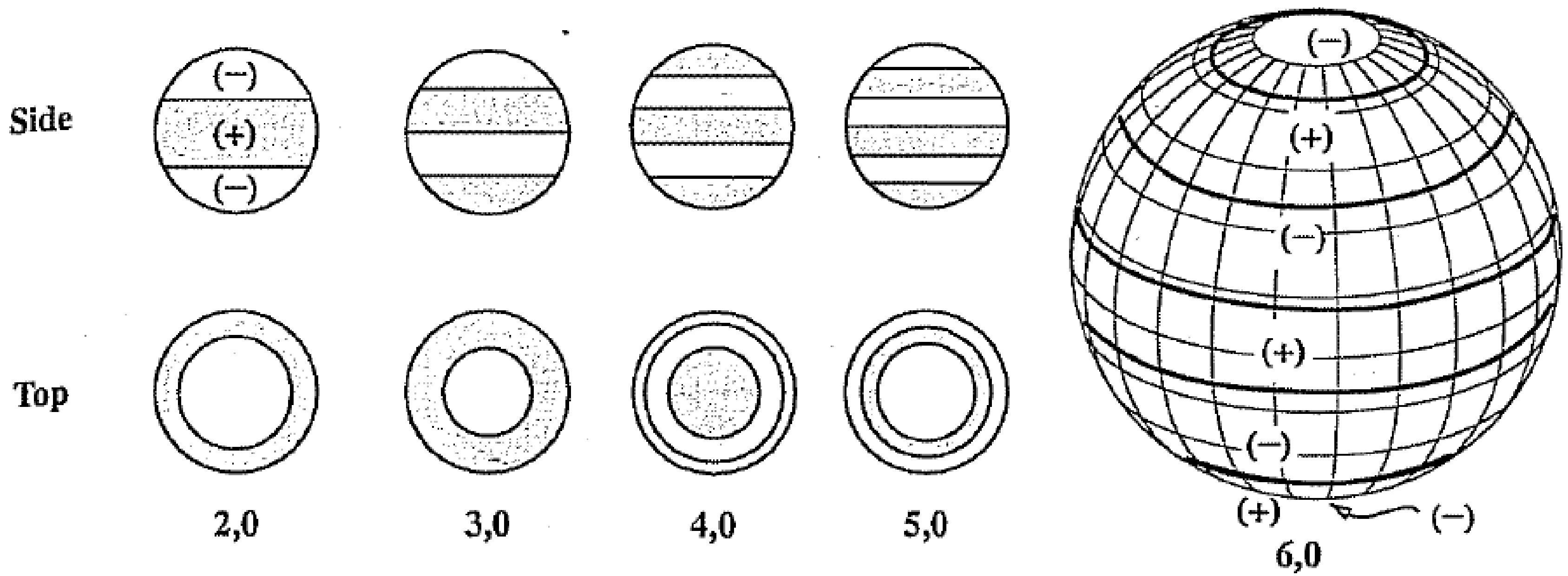


Figure 8-4. **Zonal Harmonics.** J_2 accounts for most of the Earth's gravitational departure from a perfect sphere. This band (and others) reflects the Earth's oblateness. The shading indicates regions of additional mass. The third harmonic appears similar to the J_2 from the top but is reversed for the bottom view.

Vallado, *Fundamental of Astrodynamics and Applications*, Kluwer, 2001.

Zonal Harmonics (m=0)

The zonal coefficients are independent of longitude (symmetry with respect to the rotation axis).

$$U = \frac{\mu}{r}\left\{1+\sum_{l=2}^{\infty}\sum_{m=0}^{l}\left(\frac{R_\oplus}{r}\right)^l \bar{P}_{lm}\left[\sin\phi_{sat}\right]\left[\bar{C}_{l,m}\cos(m\lambda_{sat})+\bar{S}_{l,m}\sin(m\lambda_{sat})\right]\right\}$$

$$J_l = -C_{l,0}$$

$$S_{l,0} = 0 \text{ (definition)}$$

$$U = \frac{\mu}{r}\left\{1+\sum_{l=2}^{\infty}\left(\frac{R_\oplus}{r}\right)^l \left[-J_l P_l\left[\sin\phi_{sat}\right]+\sum_{m=1}^{l}\bar{P}_{lm}\left[\sin\phi_{sat}\right]\left[\bar{C}_{l,m}\cos(m\lambda_{sat})+\bar{S}_{l,m}\sin(m\lambda_{sat})\right]\right]\right\}$$

EGM96

0,1 ?

Degree and Order		Normalized Gravitational Coefficients	
n	m	$\overline{C}_{nm}$	$\overline{S}_{nm}$
2	0	-.484165371736E -03	
2	1	-.186987635955E -09	.119528012031E-08
2	2	.243914352398E -05	-.140016683654E-05
3	0	.957254173792E -06	
3	1	.202998882184E -05	.248513158716E-06
3	2	.904627768605E -06	-.619025944205E-06
3	3	.721072657057E -06	.141435626958E-05
4	0	.539873863789E -06	
4	1	-.536321616971E -06	-.473440265853E-06
4	2	.350694105785E -06	.662671572540E-06
4	3	.990771803829E -06	-.200928369177E-06
4	4	-.188560802735E -06	.308853169333E-06
5	0	.685323475630E -07	
5	1	-.621012128528E -07	-.944226127525E-07
5	2	.652438297612E -06	-.323349612668E-06
5	3	-.451955406071E -06	-.214847190624E-06
5	4	-.295301647654E -06	.496658876769E-07
5	5	.174971983203E -06	-.669384278219E-06

First Zonal Harmonic: J2,0 or J2

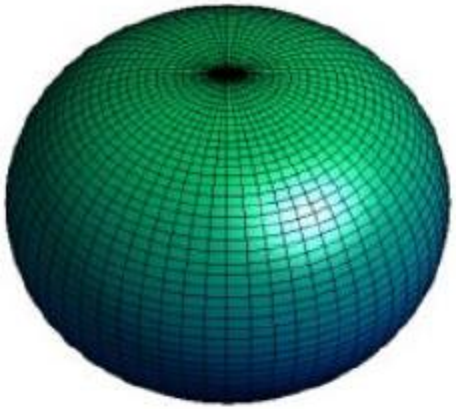
It represents the Earth's equatorial bulge and quantifies the major effects of oblateness on orbits.

It is almost a thousand times as large as any of the other coefficients.

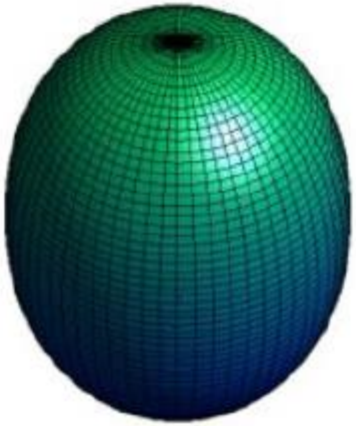
$$J_2 = -C_{2,0} = \sqrt{\frac{2.1.(2.2+1)}{2}}\,0.4841\times10^{-3} = 0.001082$$

Degree and Order		Normalized Gravitat	
n	**m**	$\overline{C}_{nm}$	
2	0	-.484165371736E -03	
2	1	-.186987635955E -09	
2	2	[illegible]	

First Zonal Harmonic: J2,0 or J2



Oblate planet: $J_2 > 0$



Prolate planet: $J_2 < 0$

Calculation of the Rotational Flattening

Equilibrium of a rotating self gravitating fluidlike body (uniform density)

http://farside.ph.utexas.edu/teaching/336k/Newton/node109.html

$$\frac{R_e - R_p}{R} = \frac{5\Omega^2 R^3}{4GM}$$

R is the mean radius

$$\frac{R_e - R_p}{R} = \frac{5(7.27 \times 10^{-5})^2 (6.37 \times 10^6)^3}{4\ 6.67 \times 10^{-11}\ 5.97 \times 10^{24}} = 0.0043$$

$$\frac{R_e - R_p}{R} = 0.0043 \times 6.37 \times 10^6 = 27km\ vs\ 21km$$

First Zonal Harmonic of Other Planets

Planet	J_2
Mercury	60e-6
Venus	4.46e-6
Earth	1.08e-3
Moon	2.03e-4
Jupiter	1.47e-2
Saturn	1.63e-2

$$\frac{R_e - R_p}{R} = \frac{5\Omega^2 R^3}{4GM}$$

Celestial objects	Rotation period
Sun	25.379995 days (Carrington rotation) 35 days (high latitude)
Mercury	58.6462 days[7]
Venus	–243.0187 days[7][8]
Earth	0.99726968 days[7][9]
Moon	27.321661 days[10] (synchronous toward Earth)
Mars	1.02595675 days[7]
Ceres	0.37809 days[11]
Jupiter	0.4135344 days (deep interior)[12] 0.41007 days (equatorial) 0.41369942 days (high latitude)
Saturn	0.44403 days (deep interior)[12] 0.426 days (equatorial) 0.443 days (high latitude)
Uranus	−0.71833 days[7][8][12]
Neptune	0.67125 days[7][12]
Pluto	−6.38718 days[7][8] (synchronous with Charon)

Sectorial Harmonics (l=m)

The sectorial coefficients represent bands of longitude.

The polynomials $P_{l,l}$ are zero only at the poles.

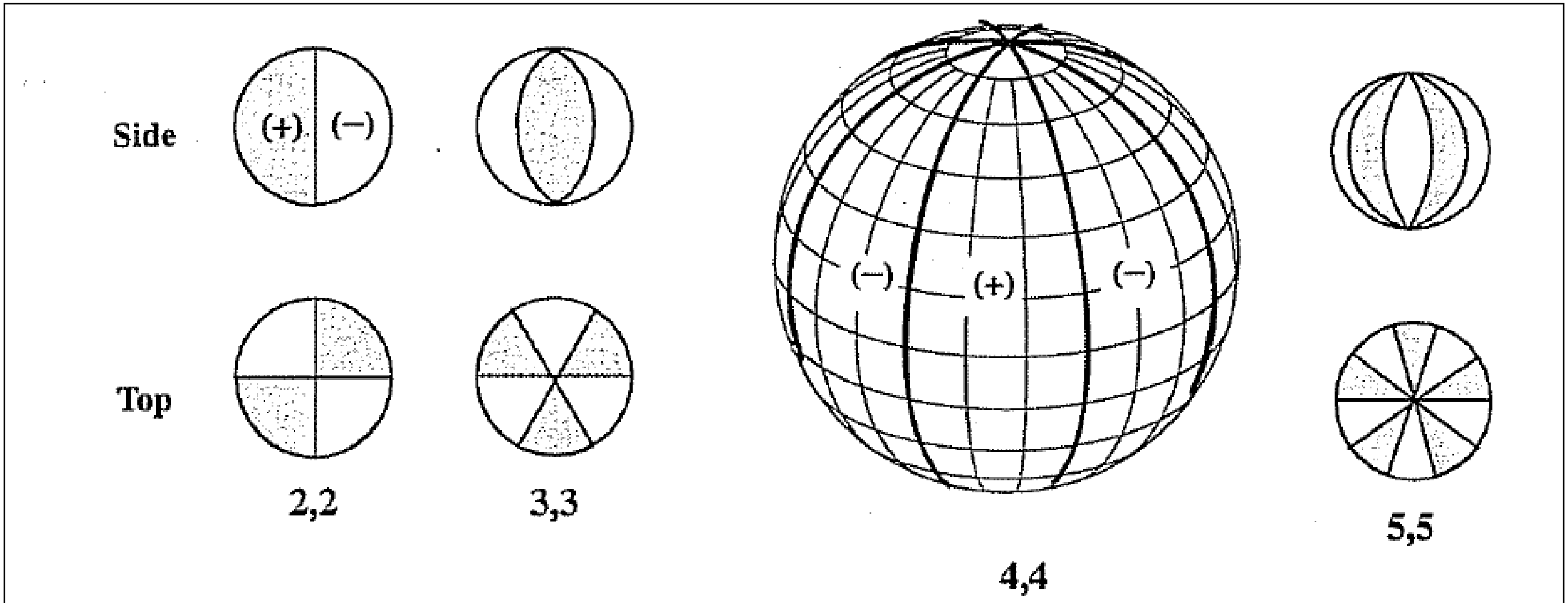
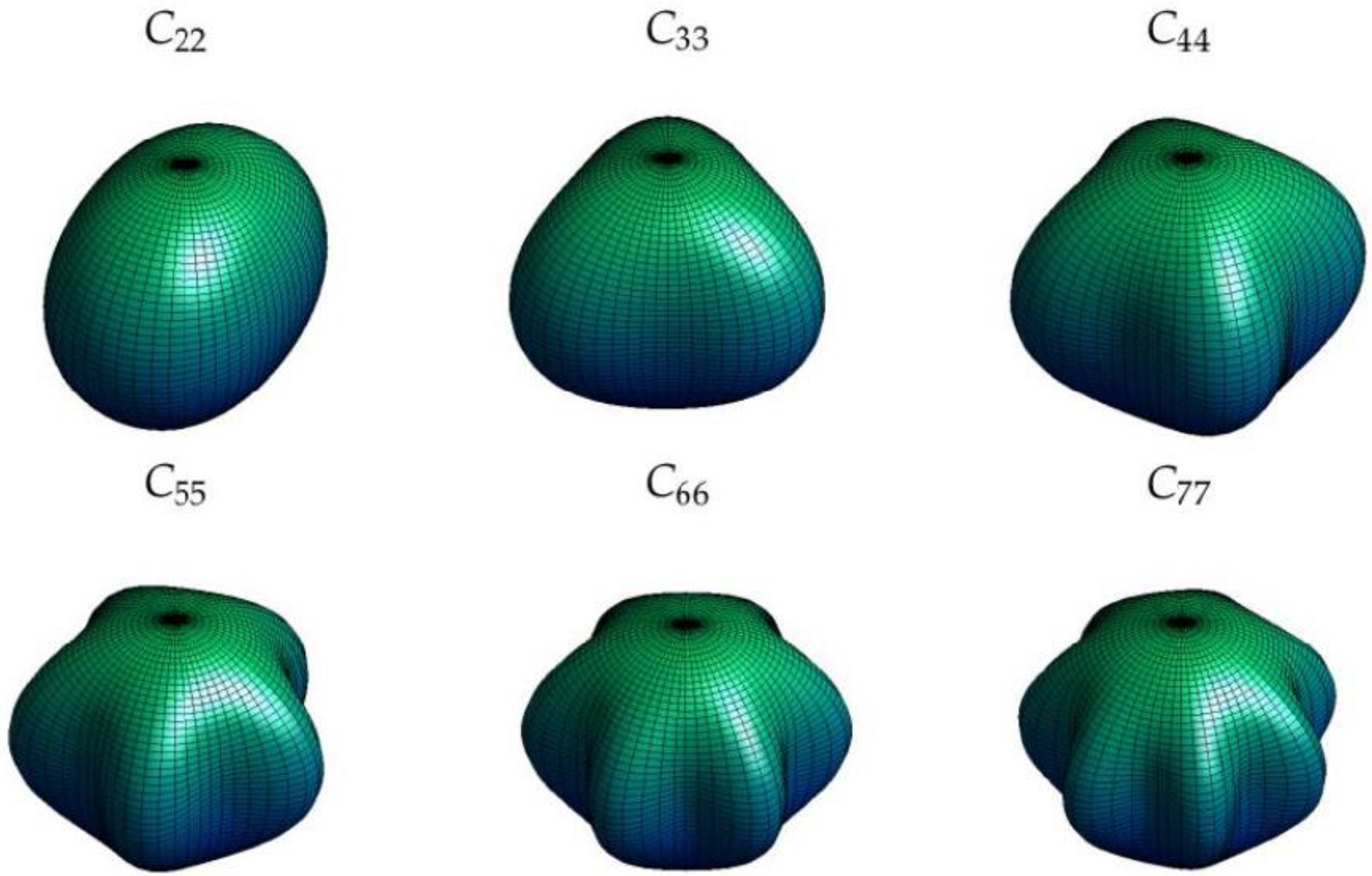


Figure 8-5. Sectorial Harmonics. Sectorial harmonics take into account the extra mass distribution in longitudinal regions.

Vallado, *Fundamental of Astrodynamics and Applications*, Kluwer, 2001.

Sectorial Harmonics ($l=m$)



Sectorial harmonics preserve symmetry with respect to the equatorial plane

Polynomials $P_{l,l}$ are zero only at the poles

Tesseral Harmonics ($l \neq m \neq 0$)

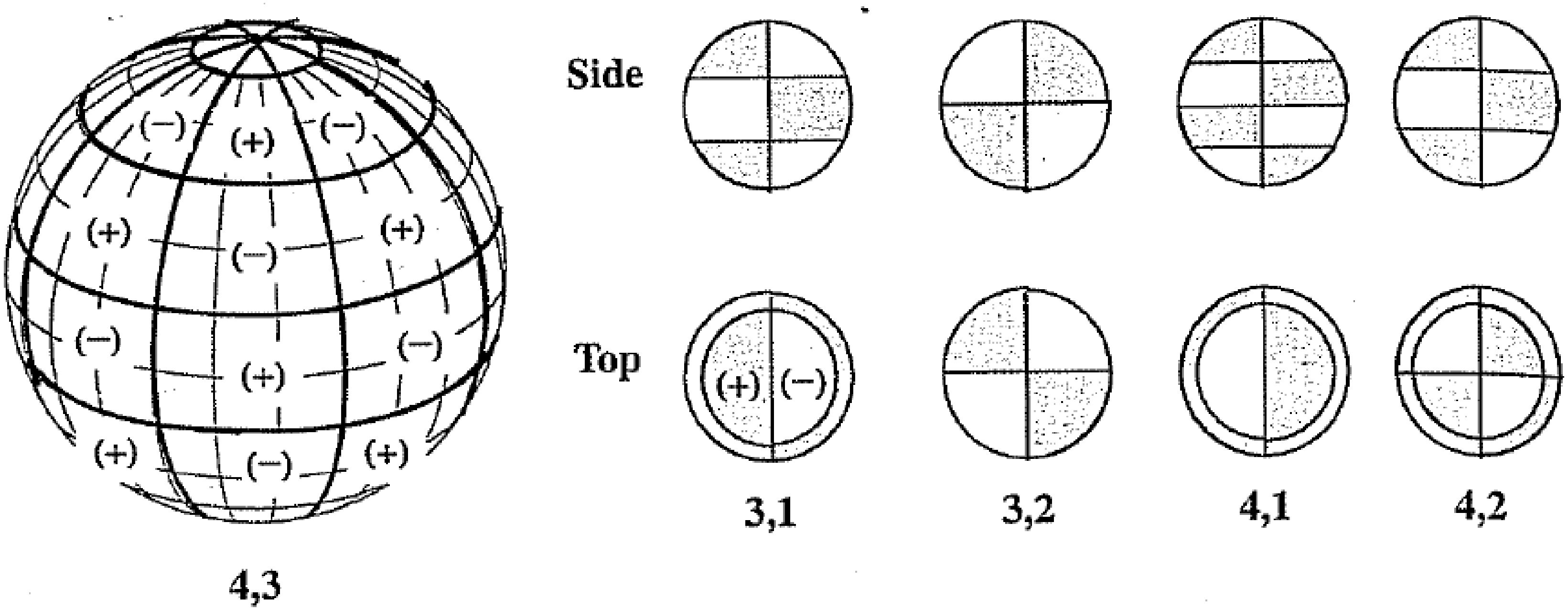
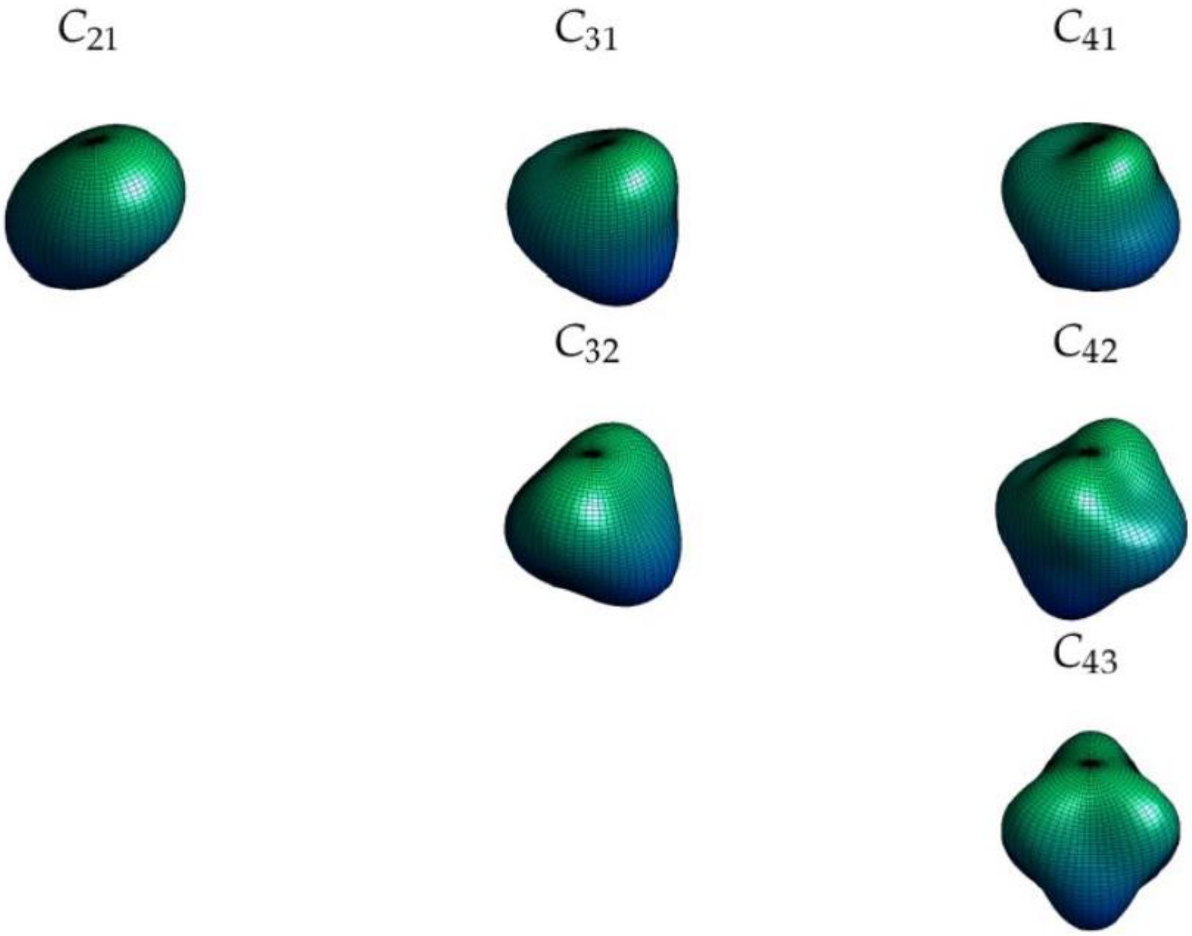


Figure 8-6. **Tesseral Harmonics.** Tesseral harmonics attempt to model specific regions on the Earth which depart from a perfect sphere.

Vallado, *Fundamental of Astrodynamics and Applications*, Kluwer, 2001.

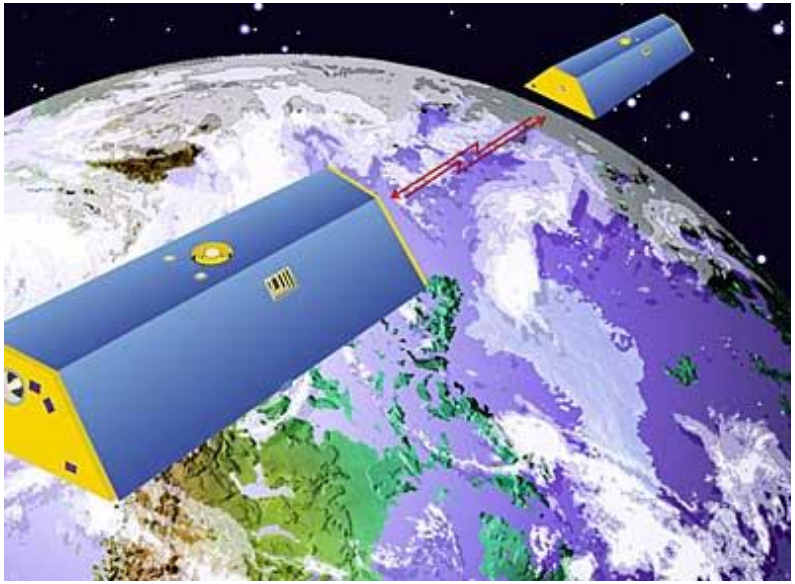
Tesseral Harmonics ($l \neq m \neq 0$)



Determination of Gravitational Coefficients

Because the internal distribution of the Earth is not known, the coefficients cannot be calculated from their definition.

They are determined experimentally; e.g, using satellite tracking.



Satellite-to-satellite tracking: GRACE employs microwave ranging system to measure changes in the distance between two identical satellites as they circle Earth. The ranging system detects changes as small as 10 microns over a distance of 220 km.

Gravitational Coefficients: GRACE

EGM-2008 has been publicly released:

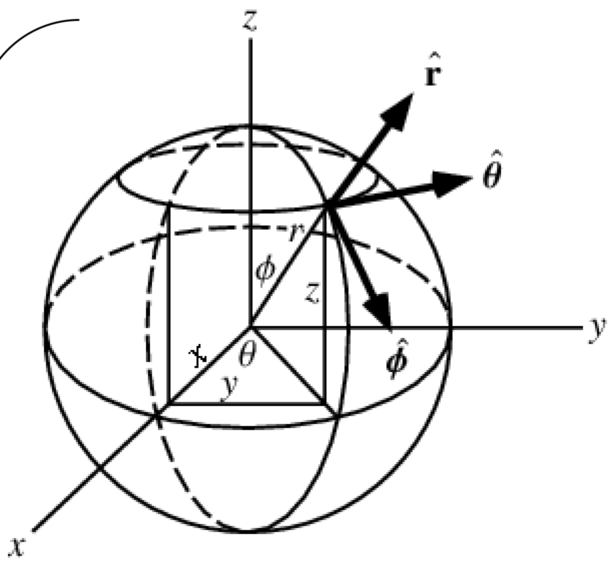
$\Rightarrow$ Extensive use of GRACE twin satellites.

$\Rightarrow$ 4.6 million terms in the spherical expansion (130317 in EGM-96)

$\Rightarrow$ Geoid with a resolution approaching 10 km (5'x5').

Resulting Force

$$\mathbf{F} = \nabla U \ \text{ with } \nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{1}{r}\hat{\boldsymbol{\varphi}}\frac{\partial}{\partial \phi} + \frac{1}{r\cos\phi}\hat{\boldsymbol{\lambda}}\frac{\partial}{\partial \lambda}$$



$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{1}{r}\hat{\boldsymbol{\varphi}}\frac{\partial}{\partial \phi} + \frac{1}{r\sin\phi}\hat{\boldsymbol{\theta}}\frac{\partial}{\partial \theta}$$

Spherical Earth

Gravitational force acts through the Earth's center.

$$U = \frac{\mu}{r}$$

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{1}{r}\hat{\boldsymbol{\varphi}}\frac{\partial}{\partial \phi} + \frac{1}{r\cos\phi}\hat{\boldsymbol{\lambda}}\frac{\partial}{\partial \lambda}$$

Oblate Earth: J2

$$\begin{cases} U = \dfrac{\mu}{r}\left\{1+\sum_{l=2}^{\infty}\left(\dfrac{R_\oplus}{r}\right)^l\left[-J_l P_l\left[\sin\phi_{sat}\right]+\sum_{m=1}^{l}\bar{P}_{lm}\left[\sin\phi_{sat}\right]\left[\bar{C}_{l,m}\cos(m\lambda_{sat})+\bar{S}_{l,m}\sin(m\lambda_{sat})\right]\right]\right\} \\ P_2[\gamma]=\dfrac{1}{2}\left(3\gamma^2-1\right) \end{cases}$$

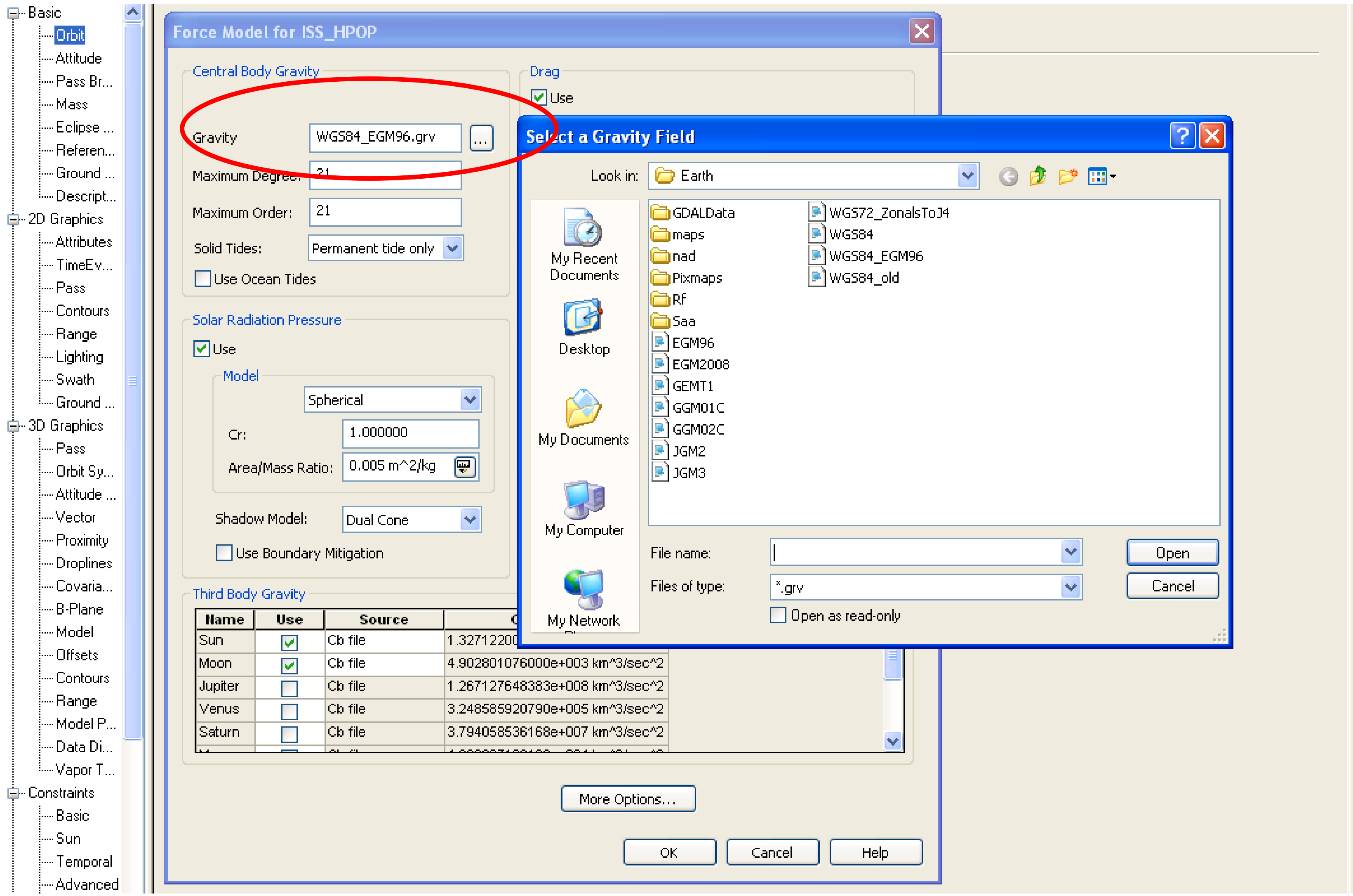
$$U = \frac{\mu}{r}\left\{1-J_2\left(\frac{R_\oplus}{r}\right)^2\frac{3\sin^2\phi_{sat}-1}{2}\right\}$$

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r}+\frac{1}{r}\hat{\boldsymbol{\varphi}}\frac{\partial}{\partial\phi}+\frac{1}{r\cos\phi}\hat{\boldsymbol{\lambda}}\frac{\partial}{\partial\lambda}$$

Perturbation of the radial acceleration

Longitudinal acceleration that can be decomposed into an azimuth and normal accelerations

STK: Gravity Models (HPOP)



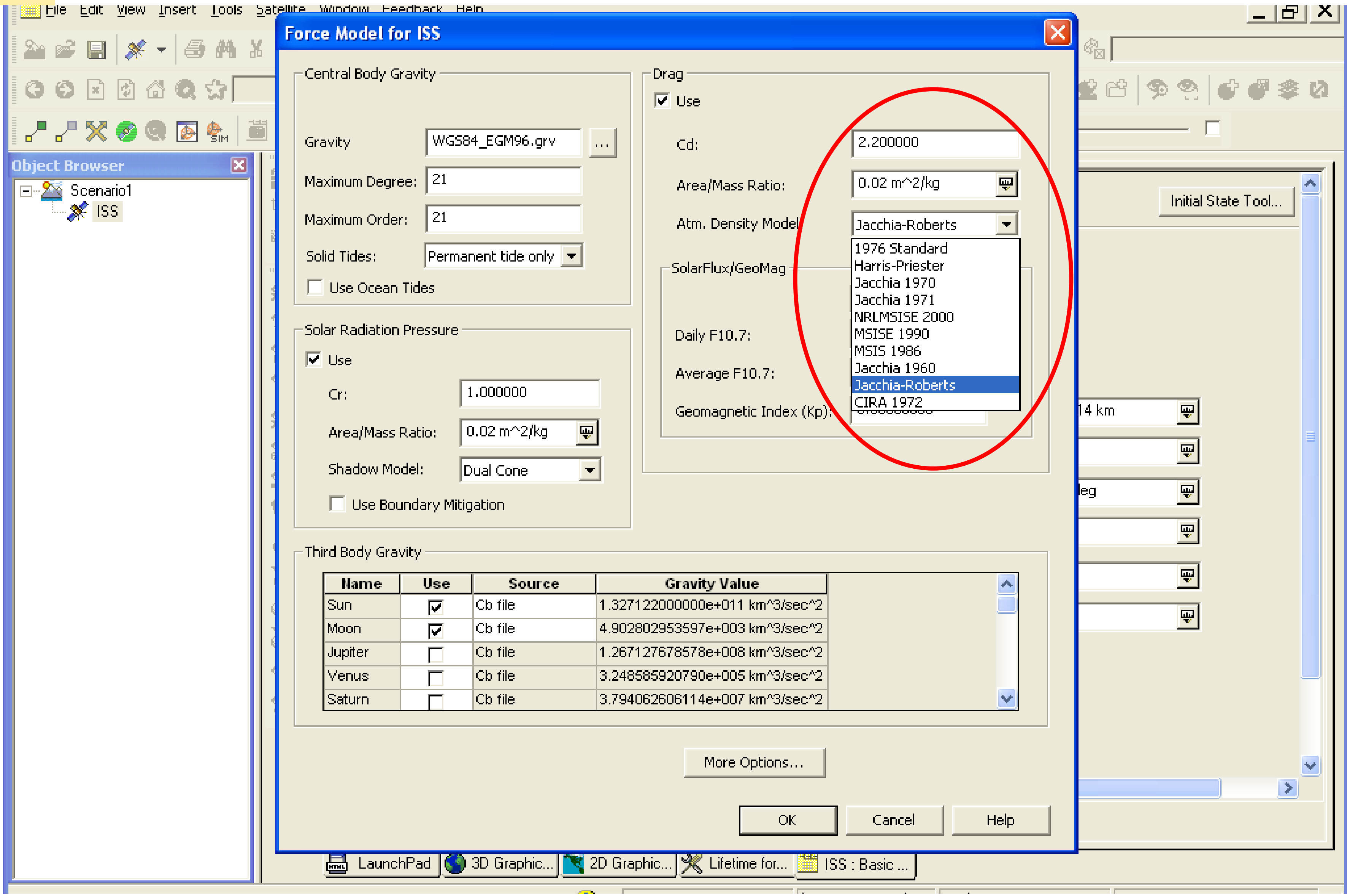
Atmospheric Drag

Atmospheric forces represent the largest nonconservative perturbations acting on low-altitude satellites.

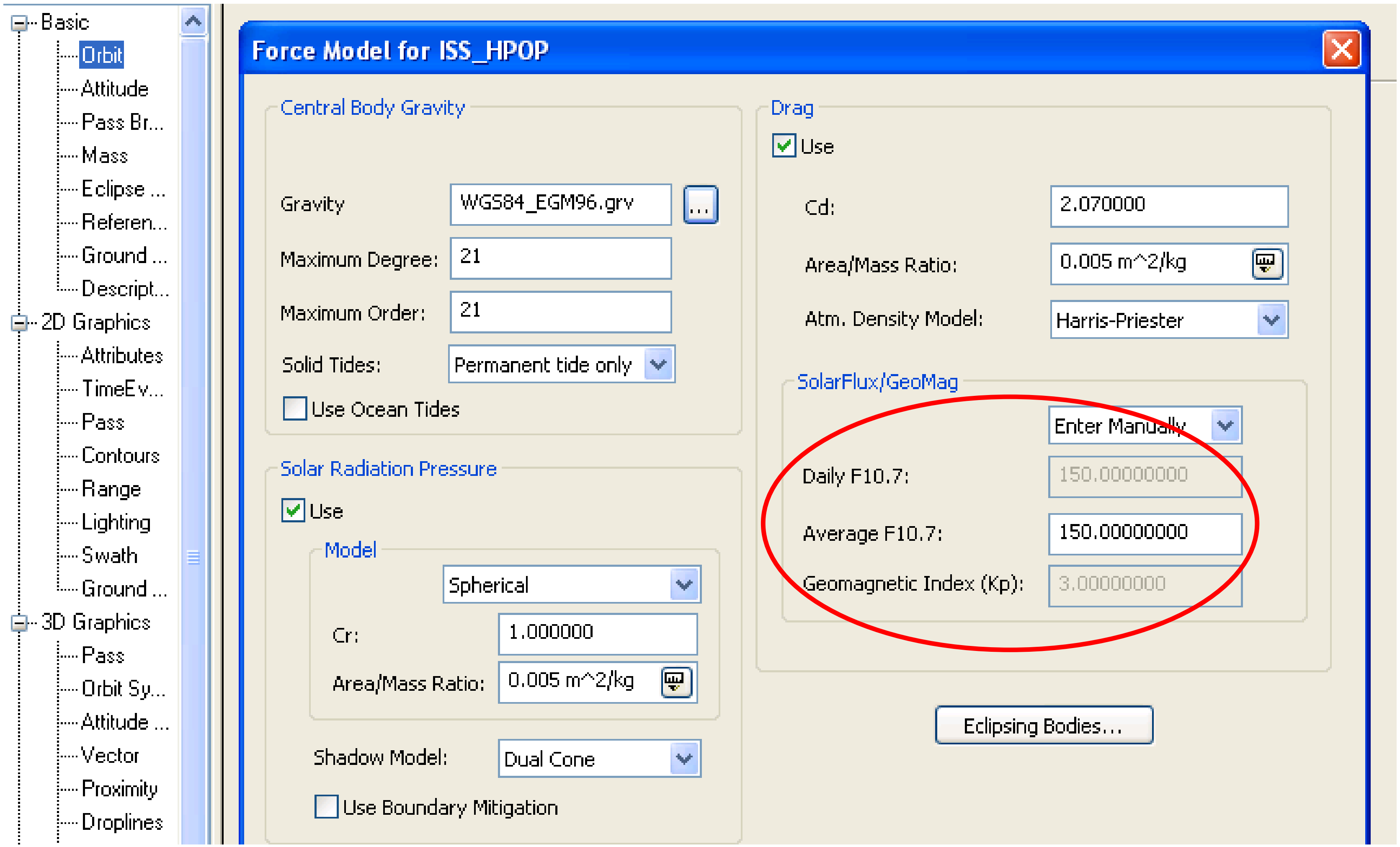
The drag is directly opposite to the velocity of the satellite, hence decelerating the satellite.

The lift force can be neglected in most cases.

STK – Atmospheric Models (HPOP)



STK – Solar Activity (HPOP)



Mathematical Modeling

The atmosphere co-rotates with the Earth.

$$\mathbf{v}_r = \mathbf{v} - \boldsymbol{\omega}_\oplus \times \mathbf{r}$$

Inertial velocity

Earth's angular velocity

Knowledge of attitude

Velocity with respect to the atmosphere

$$\ddot{\mathbf{r}}_{\oplus sat} = -\frac{1}{2} C_D \frac{A}{m} \rho v_r^2 \frac{\mathbf{v}_r}{v_r}$$

[1.5-3]

Atmospheric density

All these parameters are difficult to estimate !

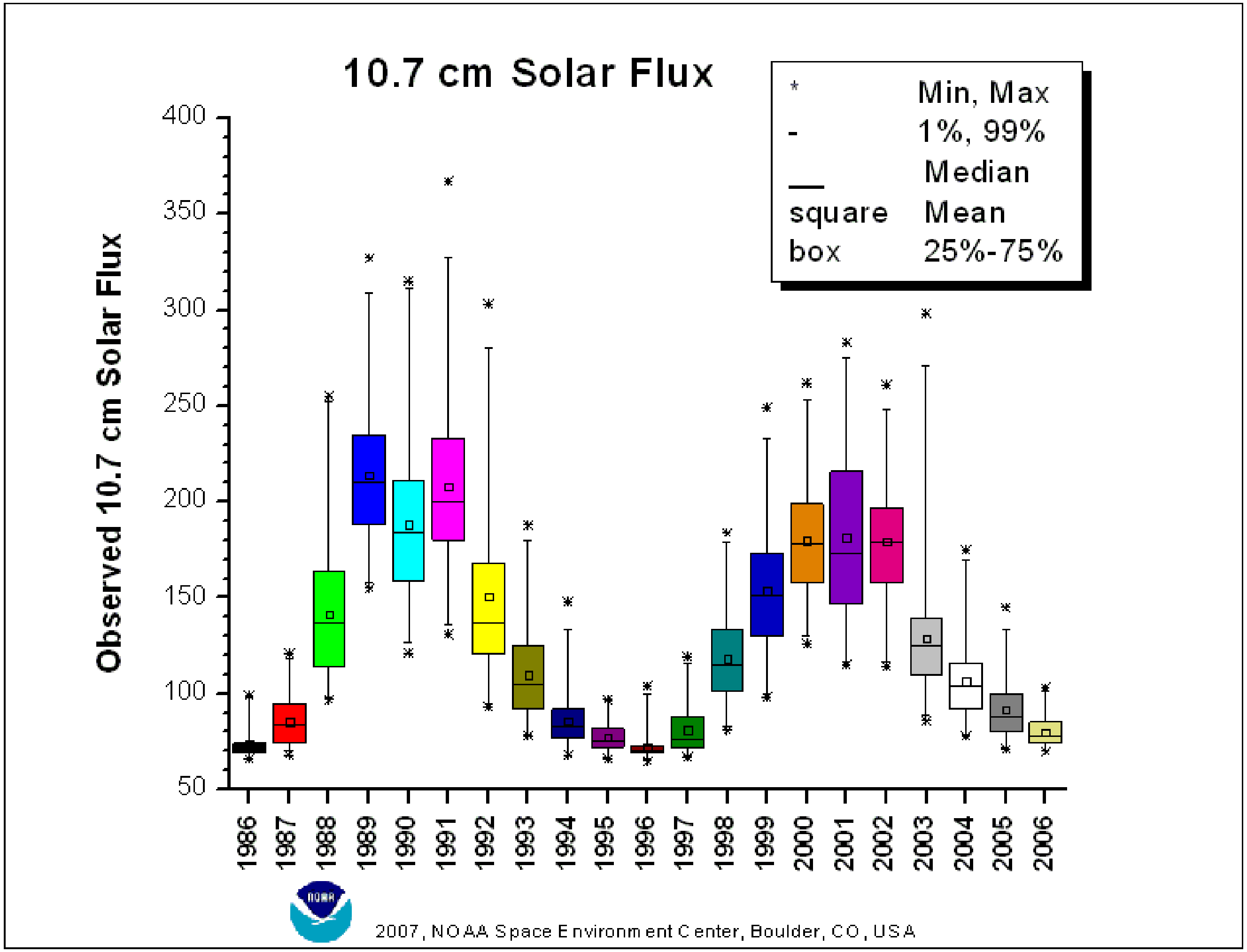
Atmospheric Density

The gross behavior of the atmospheric density is well established, but it is still this factor which makes the determination of satellite lifetimes so uncertain.

There exist several models (e.g., Jacchia-Roberts, Harris-Priester).

Dependence on temperature, molecular weight, altitude, solar activity, etc.

Solar Activity

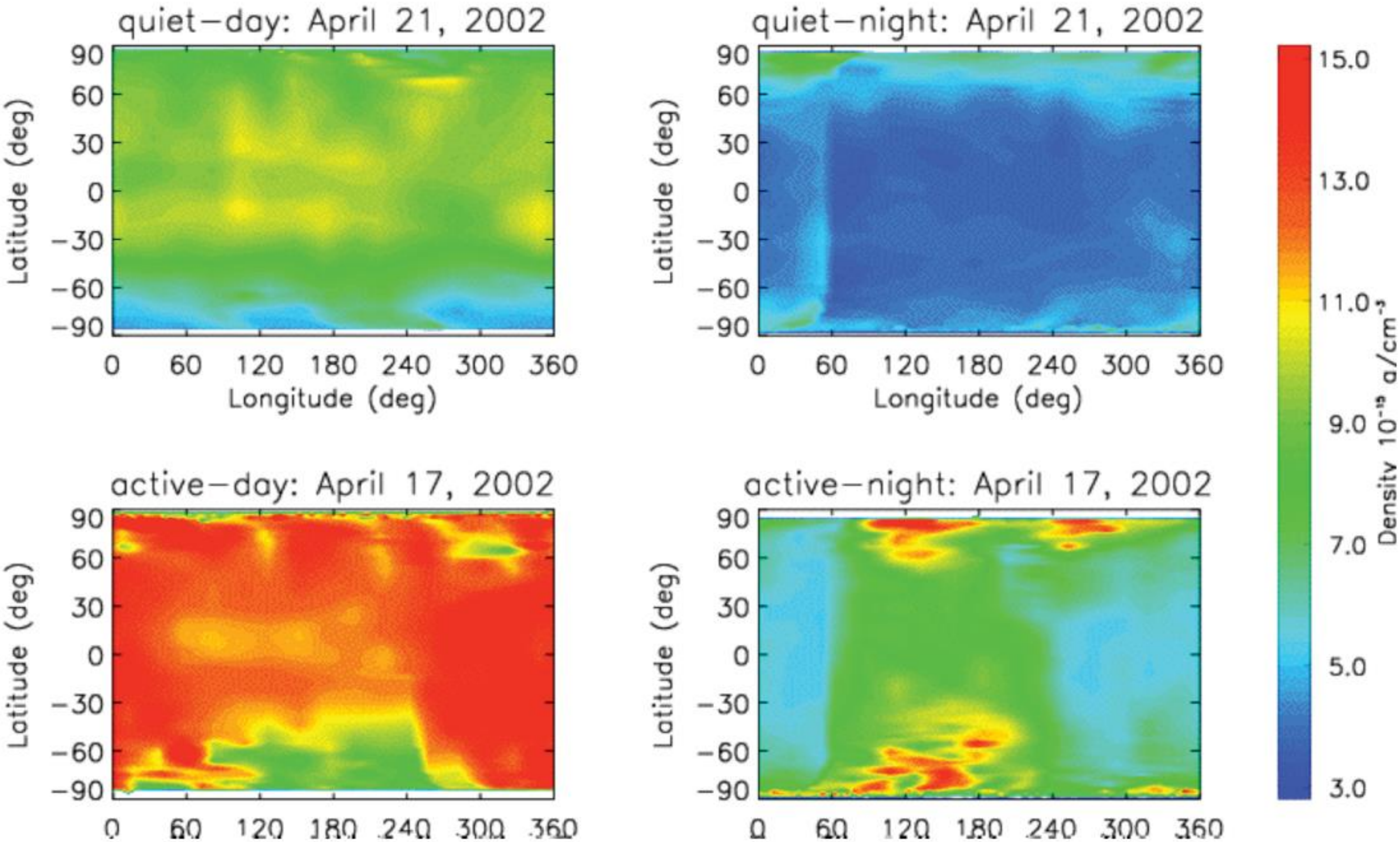


Atmospheric Density using CHAMP

An accelerometer measures the non-gravitational accelerations in three components, of which the along-track component mainly represents the atmospheric drag.

By subtracting modeled accelerations for SRP and Earth Albedo, the drag acceleration is isolated and is proportional to the atmospheric density.

CHAMP Density at 410 kms



Further Reading

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Atmospheric densities derived from CHAMP/STAR accelerometer observations

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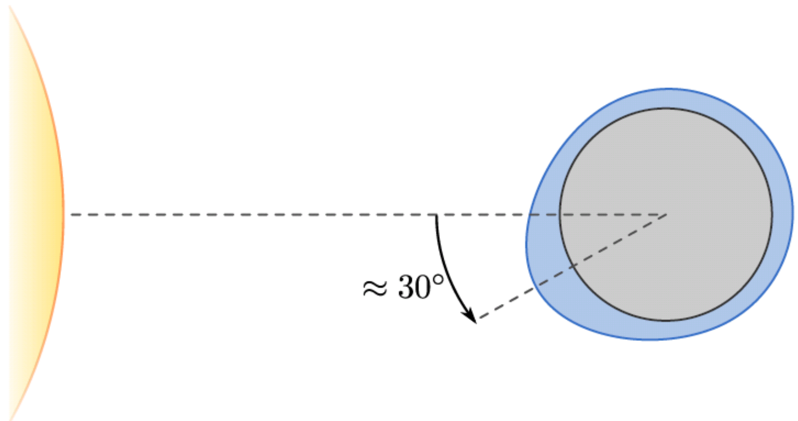
Harris-Priester (120-2000km)

Static model (e.g., no variation with the 27-day solar rotation).

Interpolation determines the density at a particular time.

Simple, computationally efficient and fairly accurate.

Atmospheric Bulge



The high atmosphere bulges toward a point in the sky some 15º to 30º east of the sun (density peak at 2pm local solar time).

The observed accelerations of Vanguard satellite (1958) indicated that the air density at 665 km is about 10 times as great when perigee passage occur one hour after noon as when it occurs during the night !

Harris-Priester (120-2000km)

The modified Harris-Priester (HP) model may be considered as a middle ground between the two extremes (Harris and Priester, 1962; Long et al., 1989; Hatten and Russell, 2016). Like the Standard Atmosphere, HP relies on exponential interpolation of density between values tabulated at discrete altitudes. However, HP also uses functional dependencies to model latitudinal and diurnal effects. Further, HP may be revised to take into account varying levels of solar activity. This effect has been achieved by including a set of 10 tables, each of which corresponds to a different value of the 81-day centered average 10.7 cm solar flux index $\overline{F}_{10.7}$. Given a value of $\overline{F}_{10.7}$, an interpolation scheme, such as nearest-neighbor (Dowd and Tapley, 1979) or linear (Tolman et al., 2004), is used to calculate density values.

Thus, HP may produce significantly more accurate density values than a simple exponential atmospheric model while executing in a fraction of the time of more complex models (Montenbruck and Gill, 2001). Such balance makes HP a suitable candidate for use in preliminary studies in which a combination of high speed and reasonable accuracy is paramount. However, even in this context, the HP model is not without its deficiencies. This work addresses

Harris-Priester (120-2000km)

Account for diurnal density bulge due to solar radiation

$$\rho(h) = \rho_m(h) + \underbrace{[\rho_M(h) - \rho_m(h)]\cos^n\left(\frac{\psi}{2}\right)}$$

Height above the Earth's reference ellipsoid

Angle between satellite position vector and the apex of the diurnal bulge

Interpolation Between Altitudes

$$\rho_m(h) = \rho_m(h_i) \exp\left(\frac{h_i - h}{H_{m_i}}\right), \quad h_i \leqslant h \leqslant h_{i+1}$$

$$\rho_M(h) = \rho_M(h_i) \exp\left(\frac{h_i - h}{H_{M_i}}\right), \quad h_i \leqslant h \leqslant h_{i+1},$$

$$H_{m_i} = \frac{h_i - h_{i+1}}{\ln\left(\frac{\rho_m(h_{i+1})}{\rho_m(h_i)}\right)}$$

$$H_{M_i} = \frac{h_i - h_{i+1}}{\ln\left(\frac{\rho_M(h_{i+1})}{\rho_M(h_i)}\right)}$$

h [km]	ρ_m [g/km^3]	ρ_M [g/km^3]	h [km]	ρ_m [g/km^3]	ρ_M [g/km^3]
100	497400.0	497400.0	420	1.558	5.684
120	24900.0	24900.0	440	1.091	4.355
130	8377.0	8710.0	460	0.7701	3.362
140	3899.0	4059.0	480	0.5474	2.612
150	2122.0	2215.0	500	0.3916	2.042
160	1263.0	1344.0	520	0.2819	1.605
170	800.8	875.8	540	0.2042	1.267
180	528.3	601.0	560	0.1488	1.005
190	361.7	429.7	580	0.1092	0.7997
200	255.7	316.2	600	0.08070	0.6390
210	183.9	239.6	620	0.06012	0.5123
220	134.1	185.3	640	0.04519	0.4121
230	99.49	145.5	660	0.03430	0.3325
240	74.88	115.7	680	0.02632	0.2691
250	57.09	93.08	700	0.02043	0.2185
260	44.03	75.55	720	0.01607	0.1779
270	34.30	61.82	740	0.01281	0.1452
280	26.97	50.95	760	0.01036	0.1190
290	21.39	42.26	780	0.008496	0.09776
300	17.08	35.26	800	0.007069	0.08059
320	10.99	25.11	840	0.004680	0.05741
340	7.214	18.19	880	0.003200	0.04210
360	4.824	13.37	920	0.002210	0.03130
380	3.274	9.955	960	0.001560	0.02360
400	2.249	7.492	1000	0.001150	0.01810

Mean solar activity

Atmospheric Bulge Position

Spacecraft position (ECI)

Unit vector toward the apex of the diurnal bulge in ECI coordinates

$$\cos^n\left(\frac{\psi}{2}\right) = \left(\frac{1}{2} + \frac{\boldsymbol{r}^T\boldsymbol{u}_b}{2r}\right)^{\frac{n}{2}}$$

$$\boldsymbol{u}_b = \begin{pmatrix} \cos(\delta_s)\cos(\alpha_s + \lambda_{lag}) \\ \cos(\delta_s)\sin(\alpha_s + \lambda_{lag}) \\ \sin(\delta_s) \end{pmatrix}$$

Lag

Sun declination

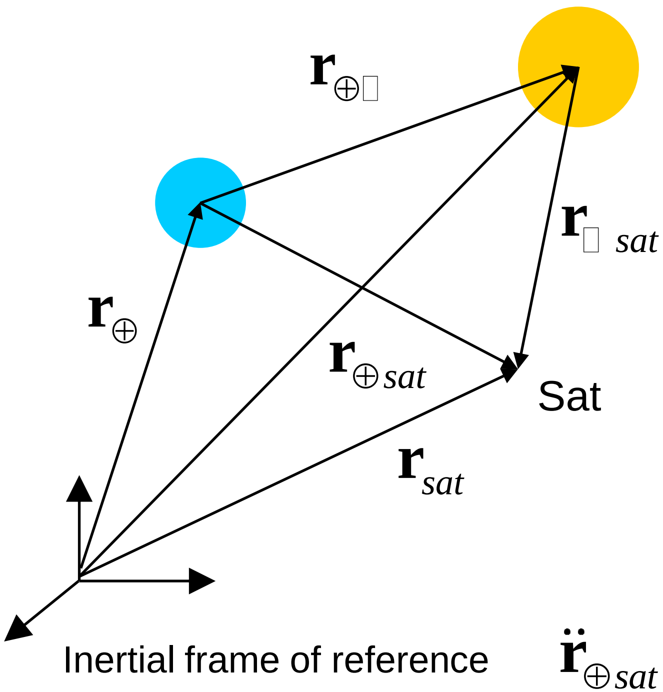
Sun right ascension

Third-Body Perturbations

For an Earth-orbiting satellite, the Sun and the Moon should be modeled for accurate predictions.

Their effects become noticeable when the effects of drag begin to diminish.

Mathematical Modeling (Sun Example)



$$m_{sat}\ddot{\mathbf{r}}_{sat} = -\frac{G m_{\oplus} m_{sat} \mathbf{r}_{\oplus sat}}{r_{\oplus sat}^{3}} - \frac{G m_{\odot} m_{sat} \mathbf{r}_{\odot sat}}{r_{\odot sat}^{3}}$$

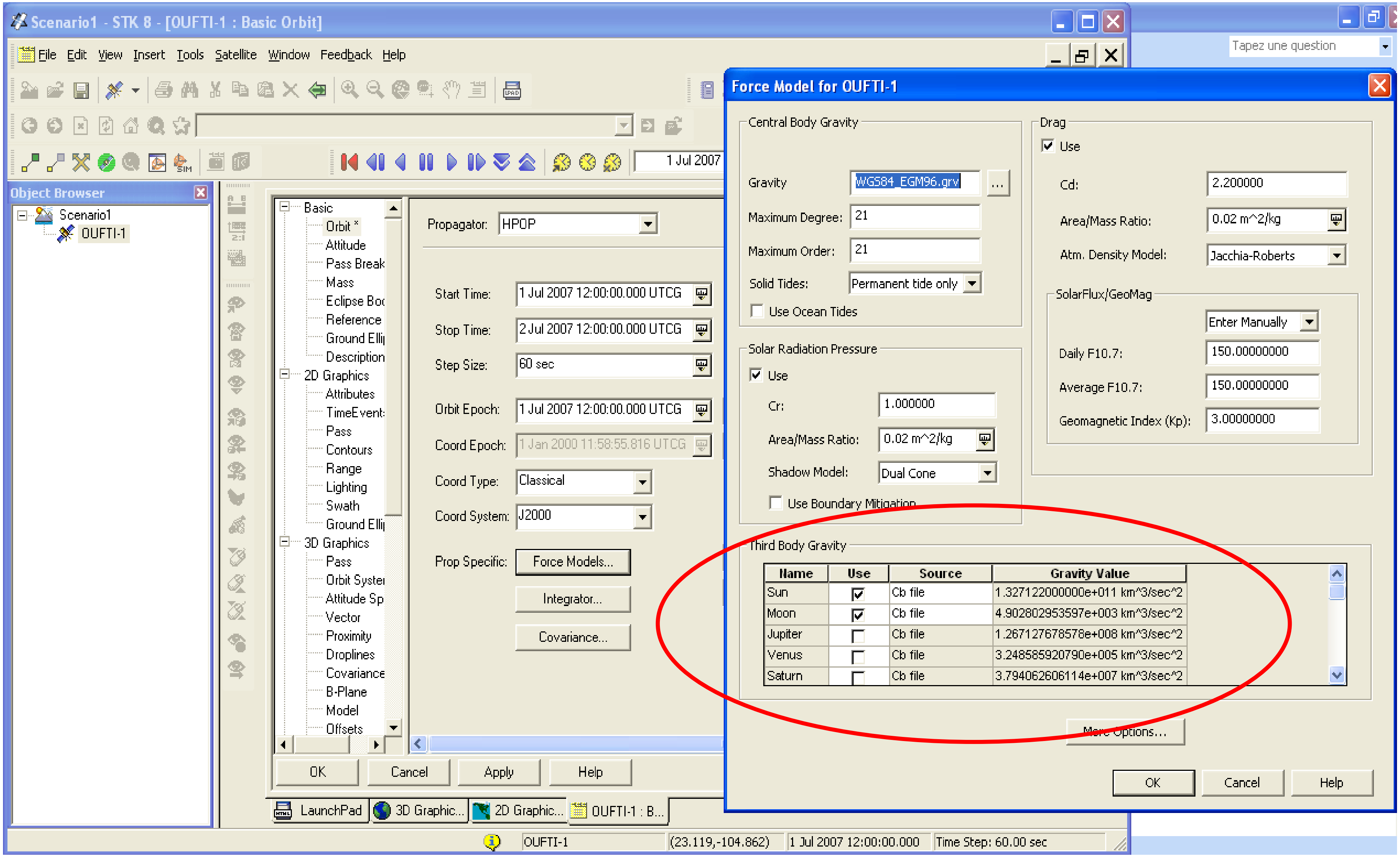
Relative motion
is of interest

$$\ddot{\mathbf{r}}_{\oplus sat} = -\frac{G m_{\oplus} \mathbf{r}_{\oplus sat}}{r_{\oplus sat}^{3}} - \frac{G m_{\odot} \mathbf{r}_{\odot sat}}{r_{\odot sat}^{3}} - \frac{G m_{sat} \mathbf{r}_{\oplus sat}}{r_{\oplus sat}^{3}} - \frac{G m_{\odot} \mathbf{r}_{\oplus\odot}}{r_{\oplus\odot}^{3}}$$

direct indirect

$$\ddot{\mathbf{r}}_{\oplus sat} = -\frac{G\left(m_{\oplus} + m_{sat}\right)\mathbf{r}_{\oplus sat}}{r_{\oplus sat}^{3}} + G m_{\odot}\left(\frac{\mathbf{r}_{sat\odot}}{r_{sat\odot}^{3}} - \frac{\mathbf{r}_{\oplus\odot}}{r_{\oplus\odot}^{3}}\right)$$

STK: Third-Body Gravity (HPOP)



Solar Radiation Pressure

It produces a nonconservative perturbation on the spacecraft, which depends upon the distance from the sun.

It is usually very difficult to determine precisely.

It is NOT related to solar wind, which is a continuous stream of particles emanating from the sun.

Solar radiation (photons)	$\neq$	Solar wind (particles)

800km is regarded as a transition altitude between drag and SRP.

Mathematical Modeling

$$\mathbf{F}_{\mathrm{SR}} = -p_{SR} c_R A \mathbf{e}_{sc/sun}$$

$$\mathrm{p}_{\mathrm{SR}} = \frac{1350\ \mathrm{W/m^2}}{3e8\ \mathrm{m/s}} = 4.51 \times 10^{-6}\ \mathrm{N/m^2}$$

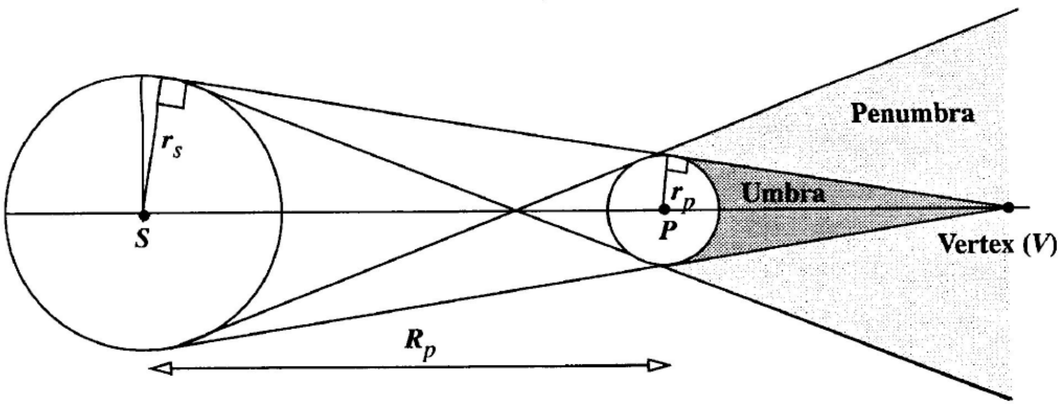
The reflectivity c_R is a value between 0 and 2:

- 0: translucent to incoming radiation.
- 1: all radiation is absorbed (black body).
- 2: all radiation is reflected.

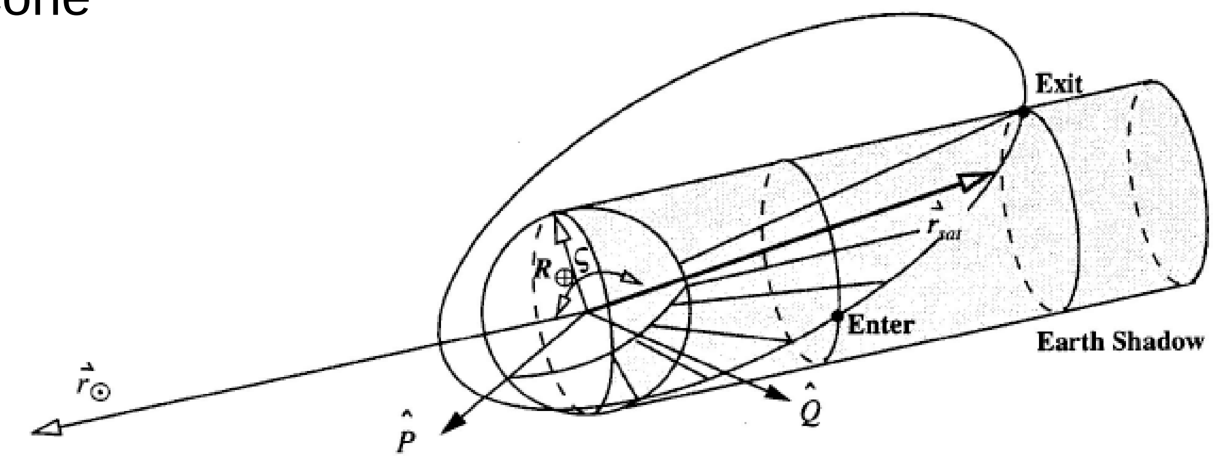
The incident area exposed to the sun must be known. The normals to the surfaces are assumed to point in the direction of the sun (e.g., solar arrays).

Mathematical Modeling: Eclipses

Use of shadow functions: $\mathbf{F}_{\mathrm{SR}} = 0$

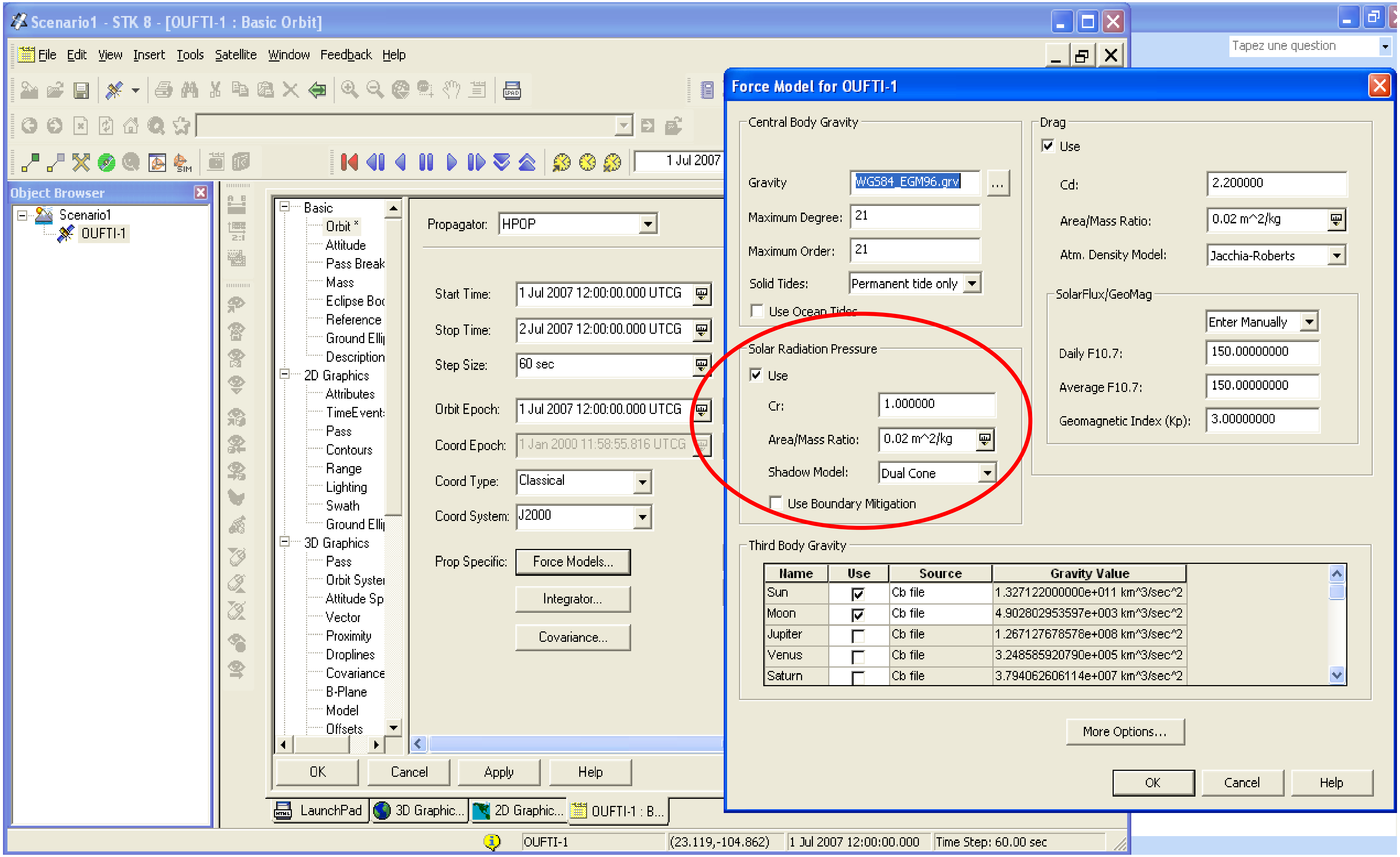


Dual cone

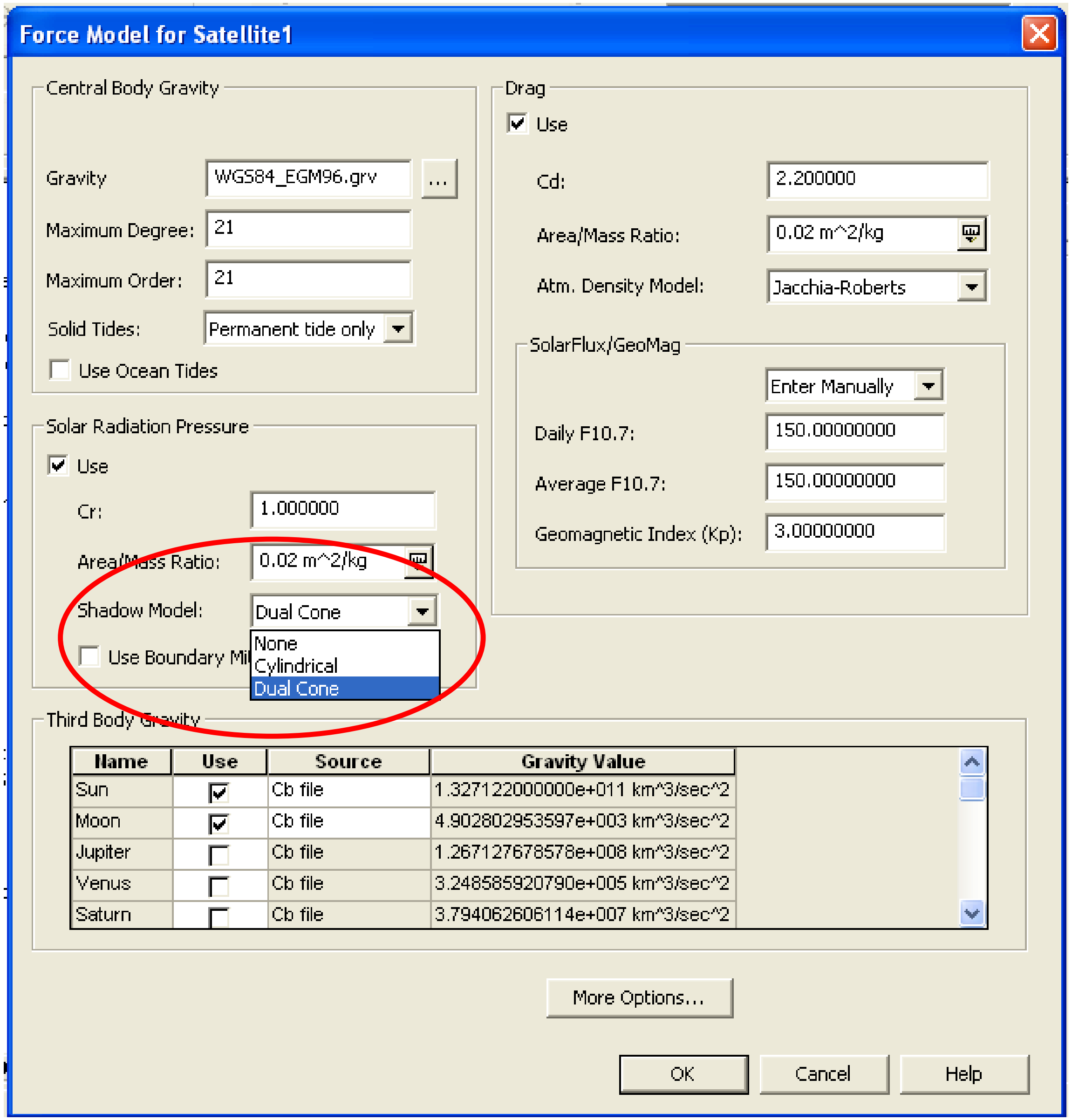


Cylindrical

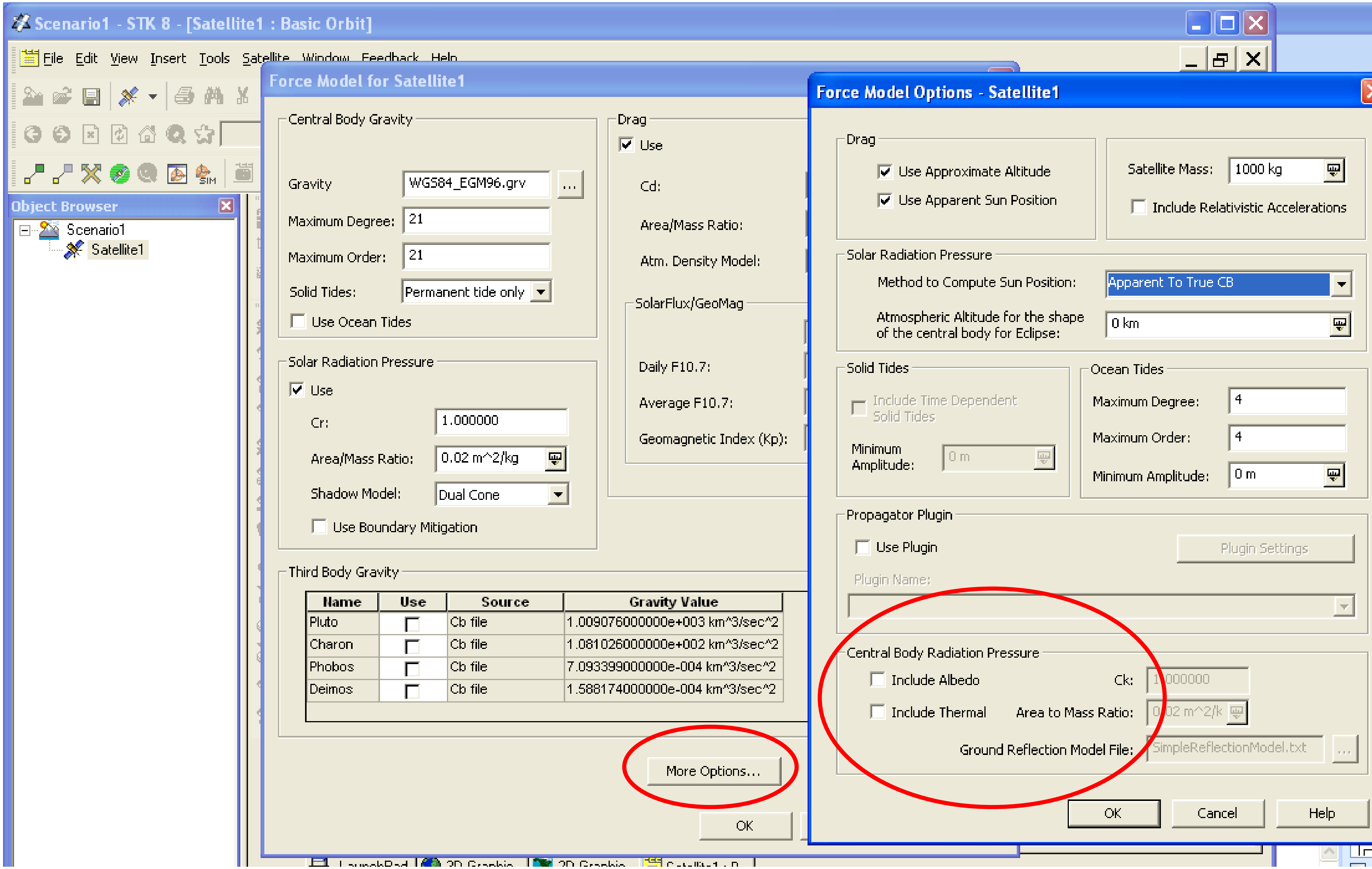
STK: Solar Radiation Pressure (HPOP)



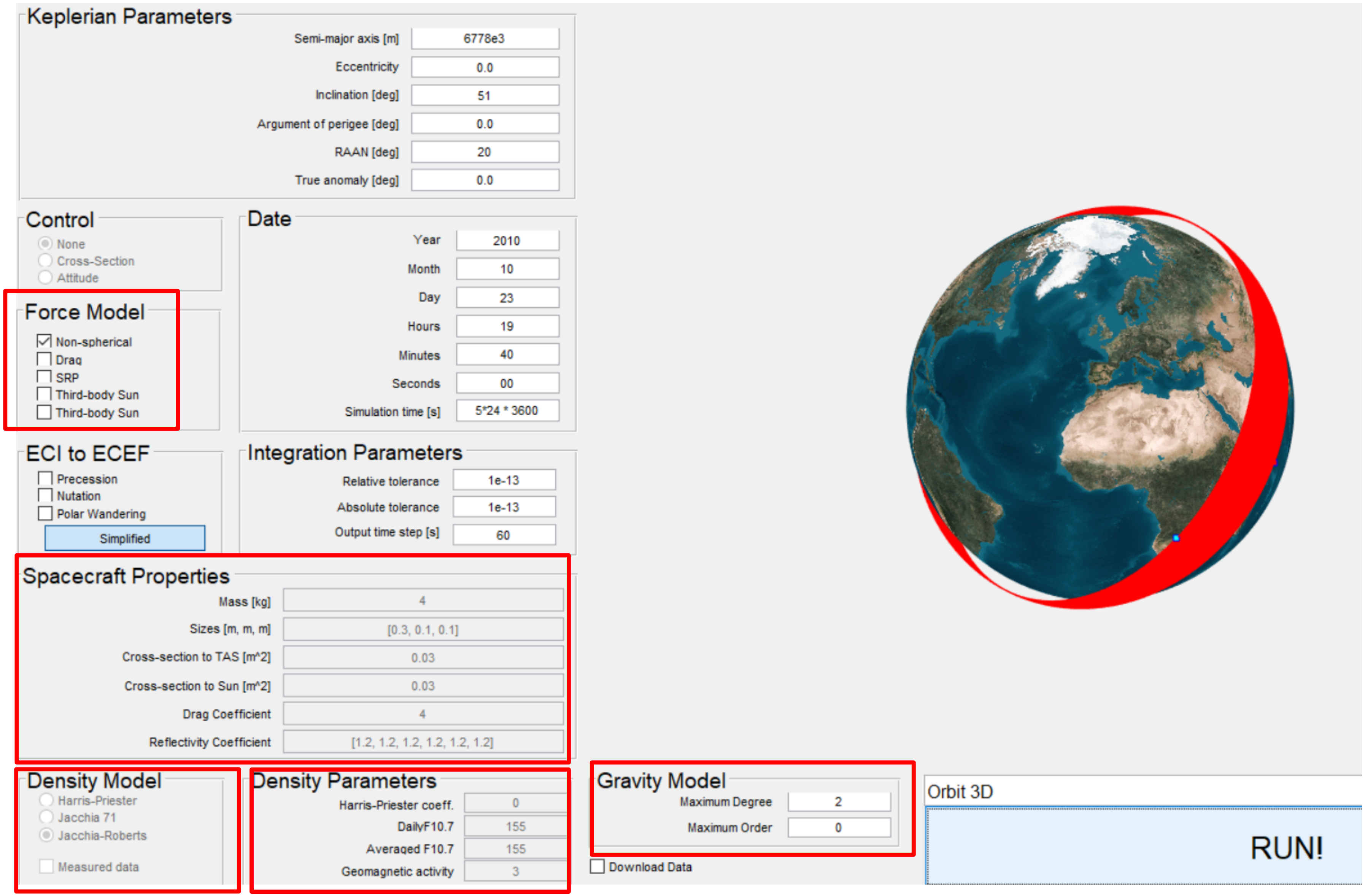
STK: Shadow Models (HPOP)



STK: Central Body Pressure (HPOP)

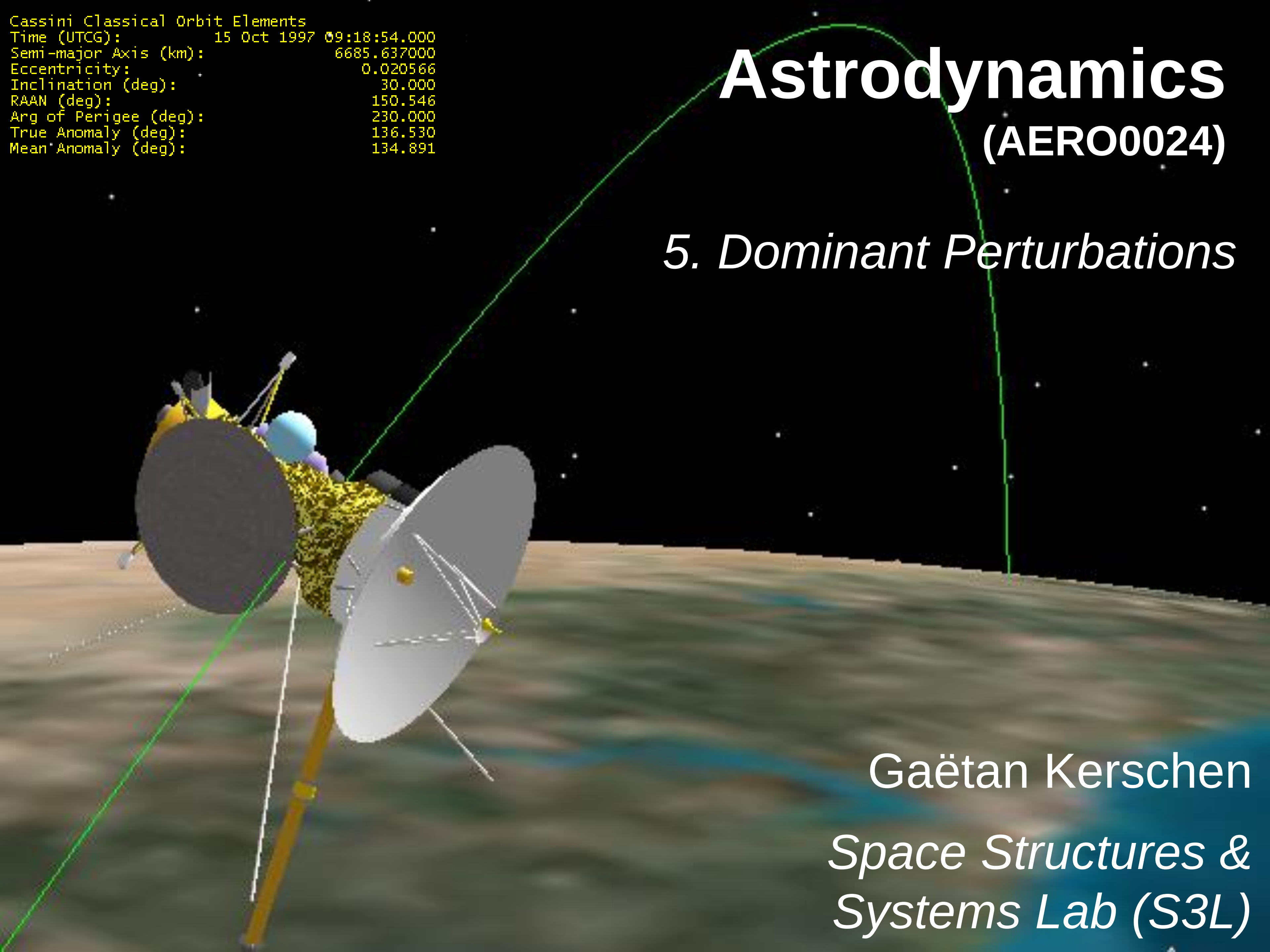


S3L Propagator



Astrodynamics
(AERO0024)

5. Dominant Perturbations



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