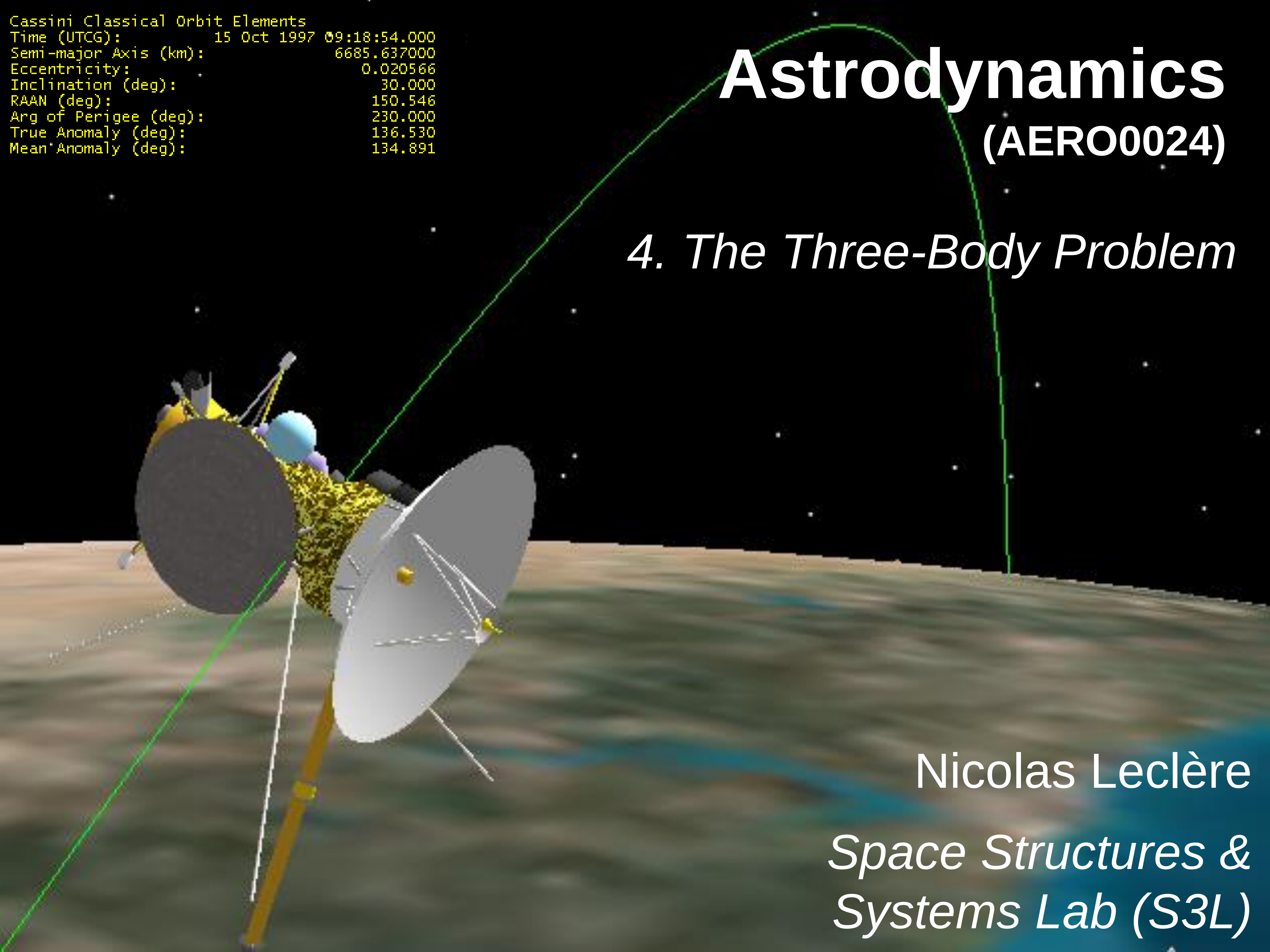
Astrodynamics
(AERO0024)

4. The Three-Body Problem

Nicolas Leclère

Space Structures & Systems Lab (S3L)



4. The Three-Body Problem

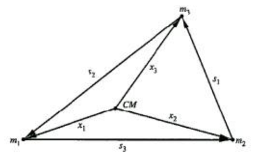
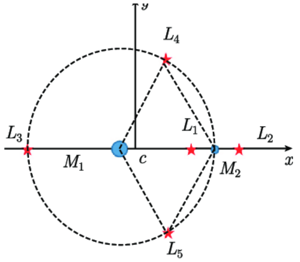
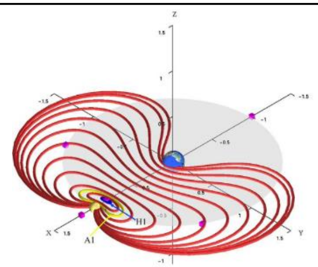


FIGURE 3.27 Position vectors $\mathbf{s}_i = \mathbf{r}_j - \mathbf{r}_k$ for the three-body problem. Adapted from Hestenes, *New Foundations for Classical Mechanics*, 1999, Fig. 5.1.

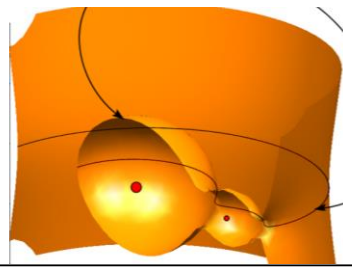
4.1 The general 3-body problem



4.2 The circular restricted 3-body problem

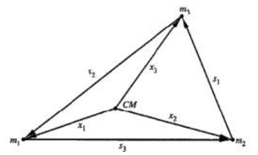


4.3 Orbit families



4.4 Zero velocity curves

4. The Three-Body Problem



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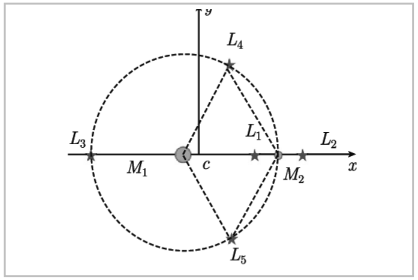

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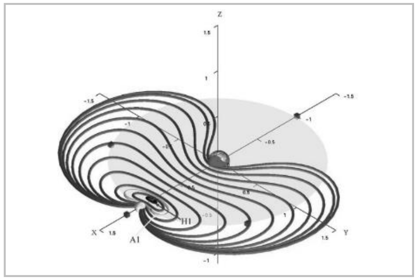
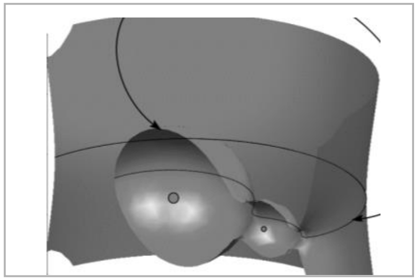


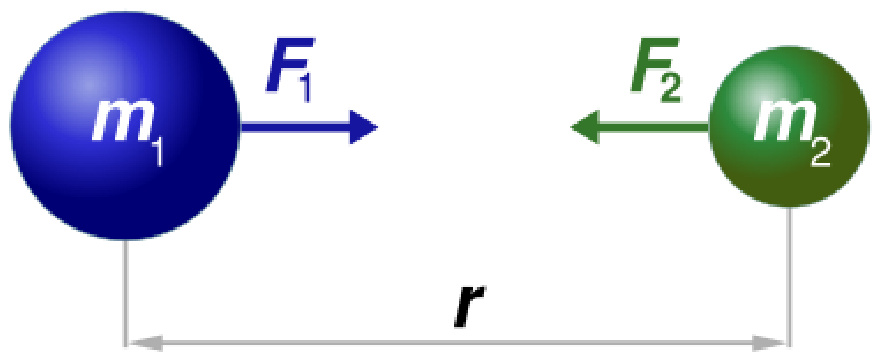
4.3 Orbit Families



4.4 Zero velocity curve

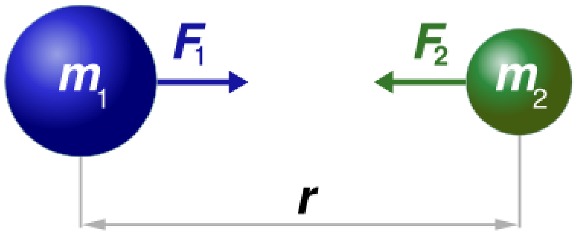
You saw…

Every point mass attracts every other point mass by a force pointing along the line intersecting both points. The force is proportional to the product of the two masses and inversely proportional to the square of the distance between the point masses:



$$F_1 = F_2 = G\frac{m_1 \times m_2}{r^2}$$

What if we had another body ?



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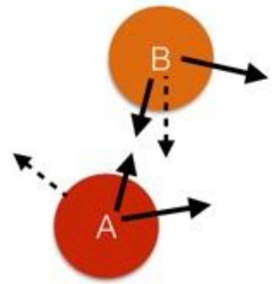
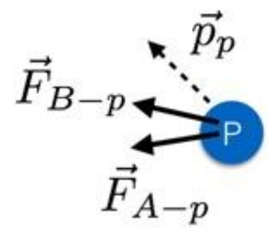




How do we express the interacting forces?

$$\ddot{\boldsymbol{r}}_1 = -Gm_2\frac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^3} - Gm_3\frac{\boldsymbol{r}_1 - \boldsymbol{r}_3}{|\boldsymbol{r}_1 - \boldsymbol{r}_3|^3}$$

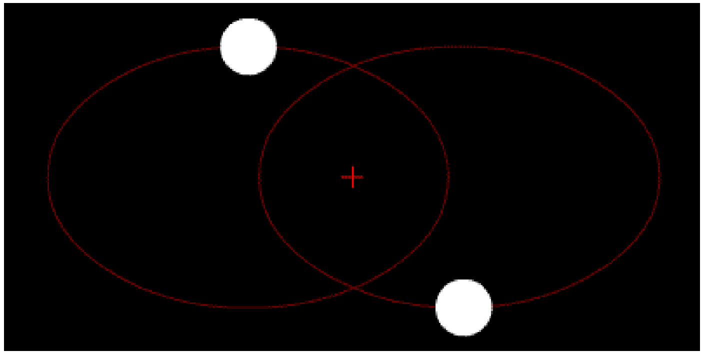
$$\ddot{\boldsymbol{r}}_2 = -Gm_3\frac{\boldsymbol{r}_2 - \boldsymbol{r}_3}{|\boldsymbol{r}_2 - \boldsymbol{r}_3|^3} - Gm_1\frac{\boldsymbol{r}_2 - \boldsymbol{r}_1}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|^3}$$

$$\ddot{\boldsymbol{r}}_3 = -Gm_1\frac{\boldsymbol{r}_3 - \boldsymbol{r}_1}{|\boldsymbol{r}_3 - \boldsymbol{r}_1|^3} - Gm_2\frac{\boldsymbol{r}_3 - \boldsymbol{r}_2}{|\boldsymbol{r}_3 - \boldsymbol{r}_2|^3}$$

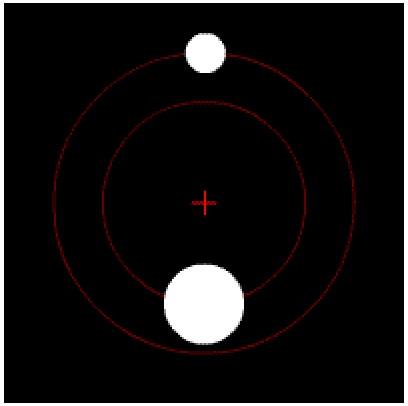
Center of mass

For the 2-body problem:

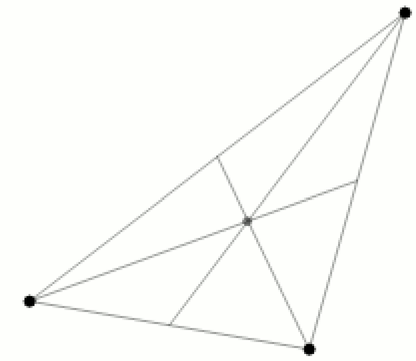
Same masses:



Different masses:



For the 3-body problem:



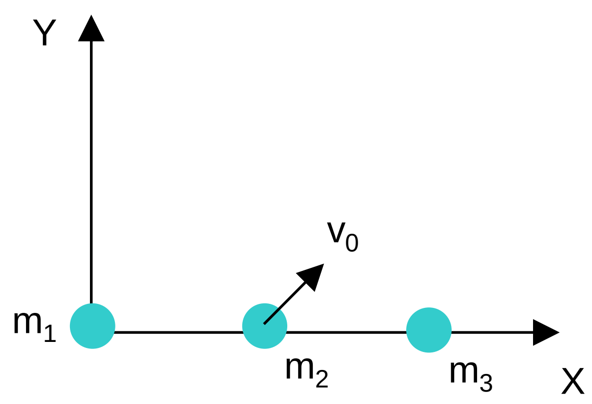
The motion of the three bodies is chaotic for most initial conditions.

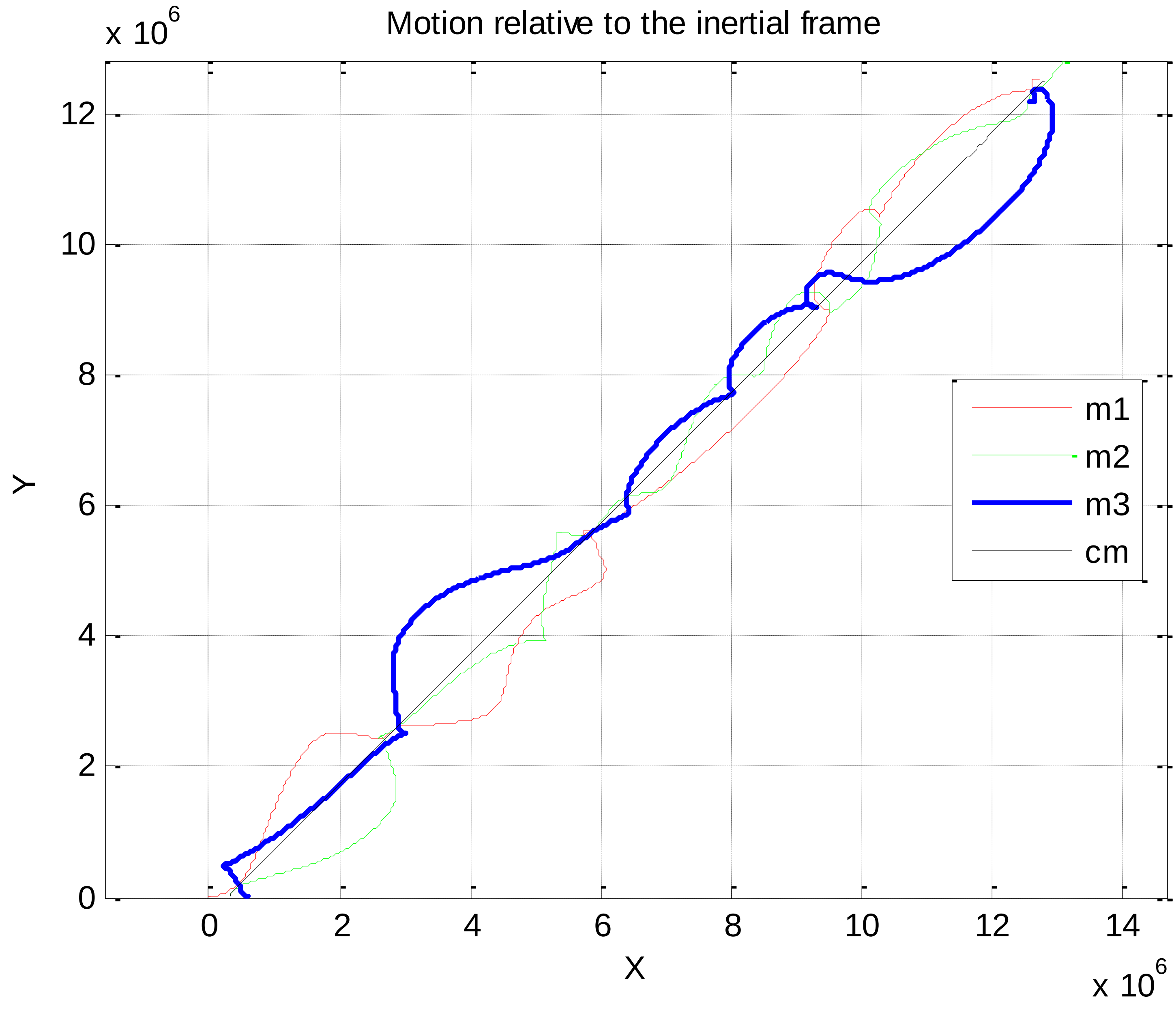
Three-Body Problem: Matlab Example

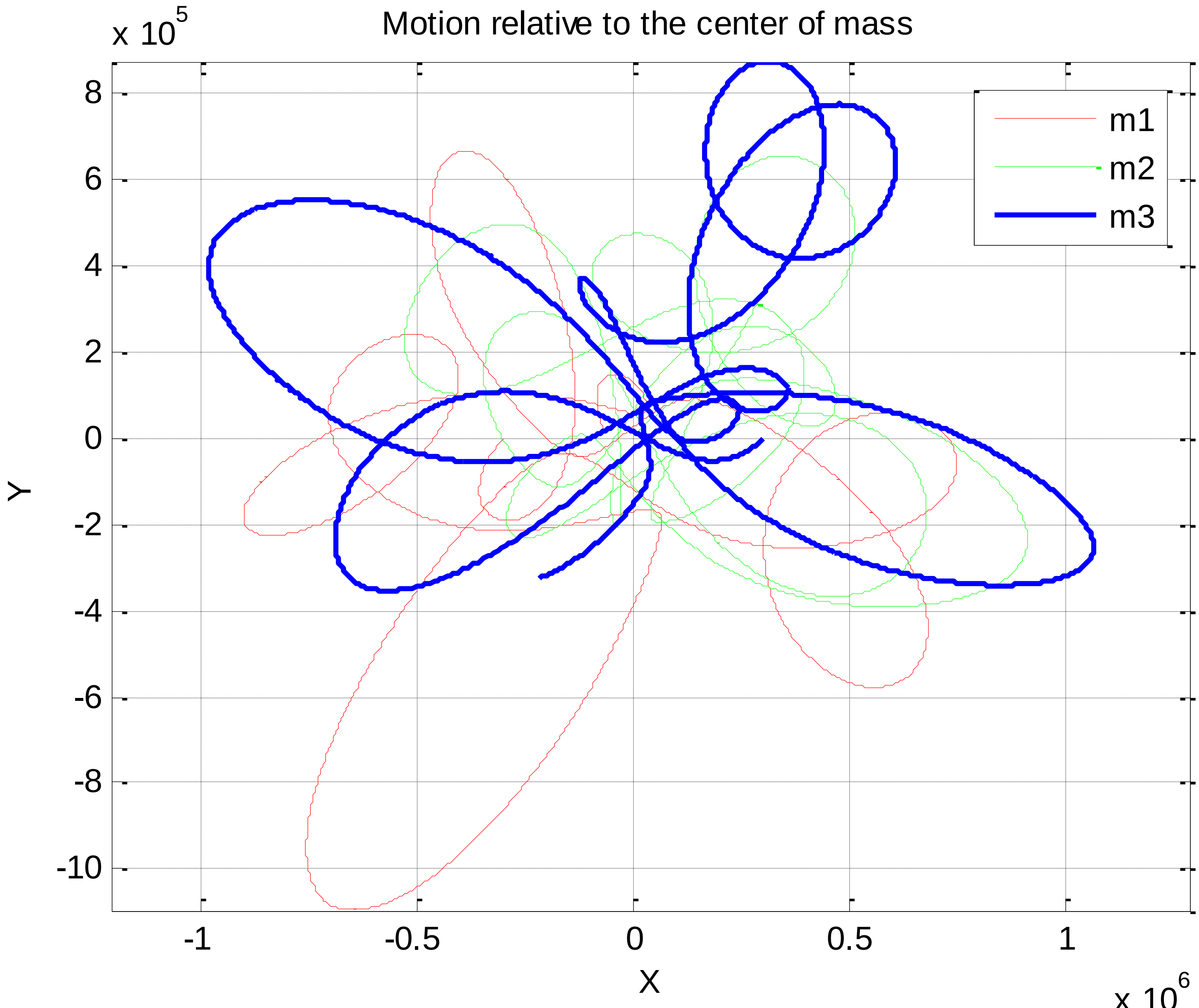
Three identical masses:

⇒ Two are at rest.

⇒ The third one has a velocity directed upward to the right making a 45 degrees angle with the X axis.

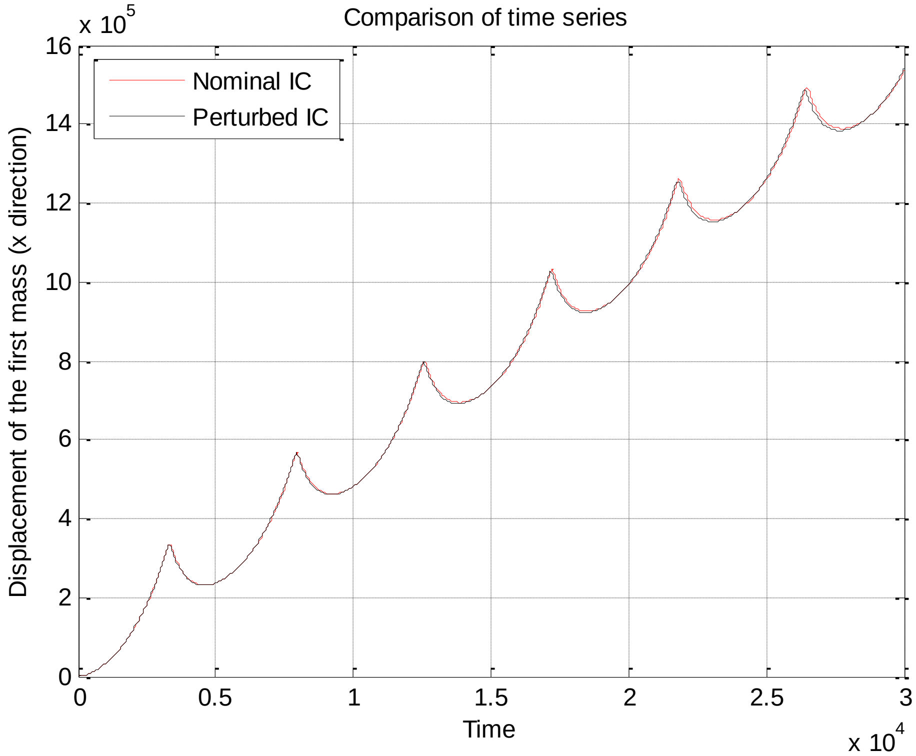



Motion relative to the inertial frame
x 10^6
12
10
8
6
4
2
0
Y
0
2
4
6
8
10
12
14
X
x 10^6
m1
m2
m3
cm



What do you vonclude ?

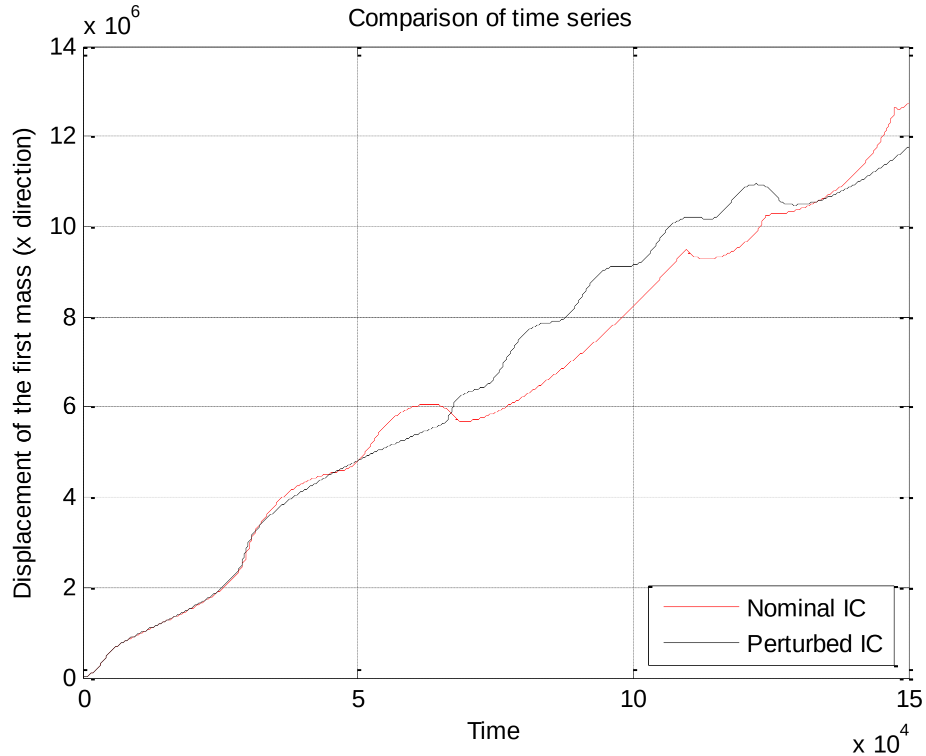
Why Is the 3-Body Problem So Difficult ?



2-body problem
(initial conditions perturbed by 0.1%)

Chaotic by nature !

3-body problem
(initial conditions perturbed by 0.1%)



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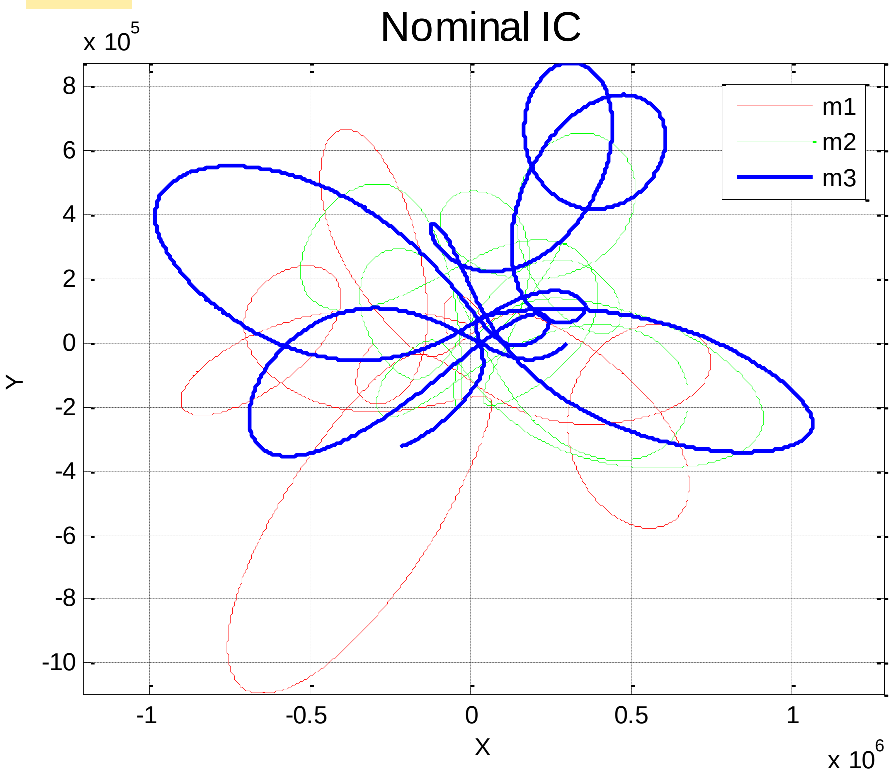

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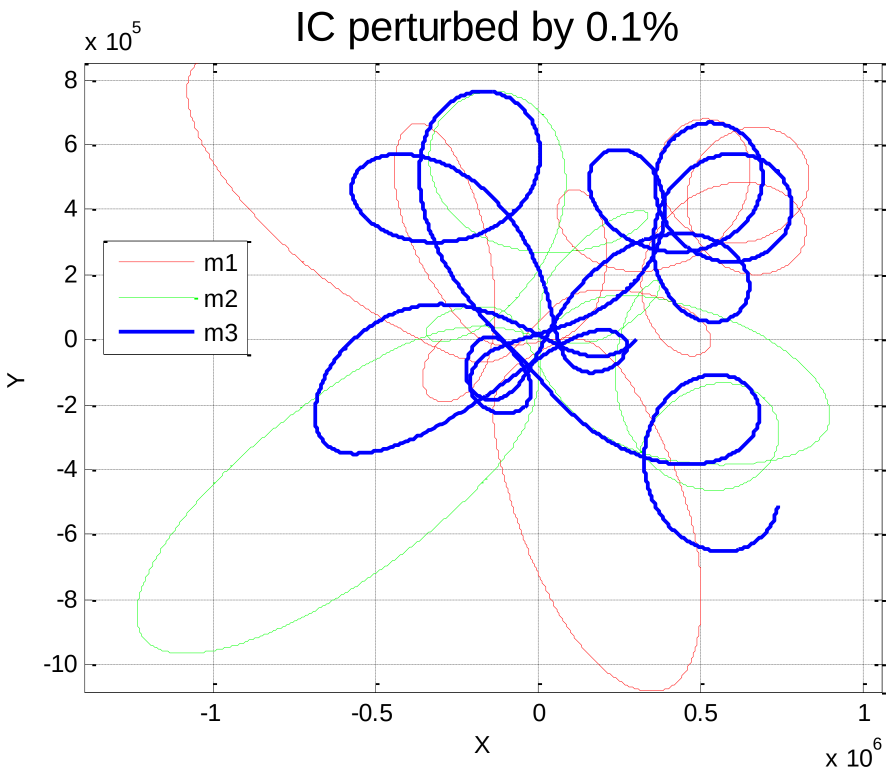

The n-body problem

The system can be extended to a n-body problem, the equations thus take the form:

$$F_{ij} = \frac{G m_i m_j (\boldsymbol{q_i} - \boldsymbol{q_j})}{\left\| \boldsymbol{q_j} - \boldsymbol{q_i} \right\|^3}$$

$$m_i \frac{d^2 q_i}{dt^2} = \sum_{j=1, j \neq i}^{n} \frac{G m_i m_j (\boldsymbol{q_i} - \boldsymbol{q_j})}{\left\| \boldsymbol{q_j} - \boldsymbol{q_i} \right\|^3}$$

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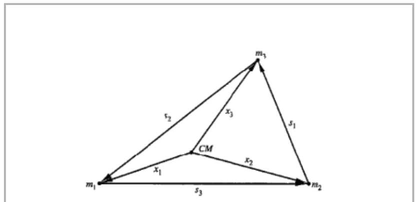

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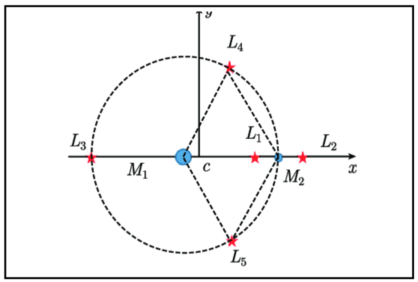

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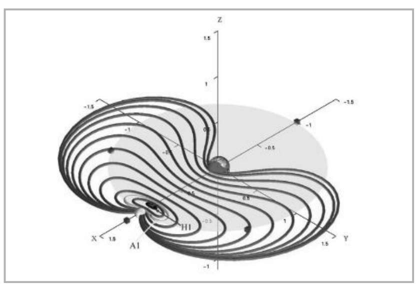
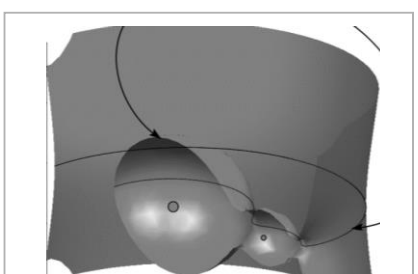


4.3 Orbit Families



4.4 Zero velocity curve

A particular situation

The Circular restricted 3-body problem is a particular case of the classical 3-body problem.

It consists of two large masses, m_1 and m_2 that have a circular orbit around their center of mass.

A third mass significantly smaller than the two others orbits with respect to the primaries gravity.

We have $m_1 > m_2 \gg m_3$

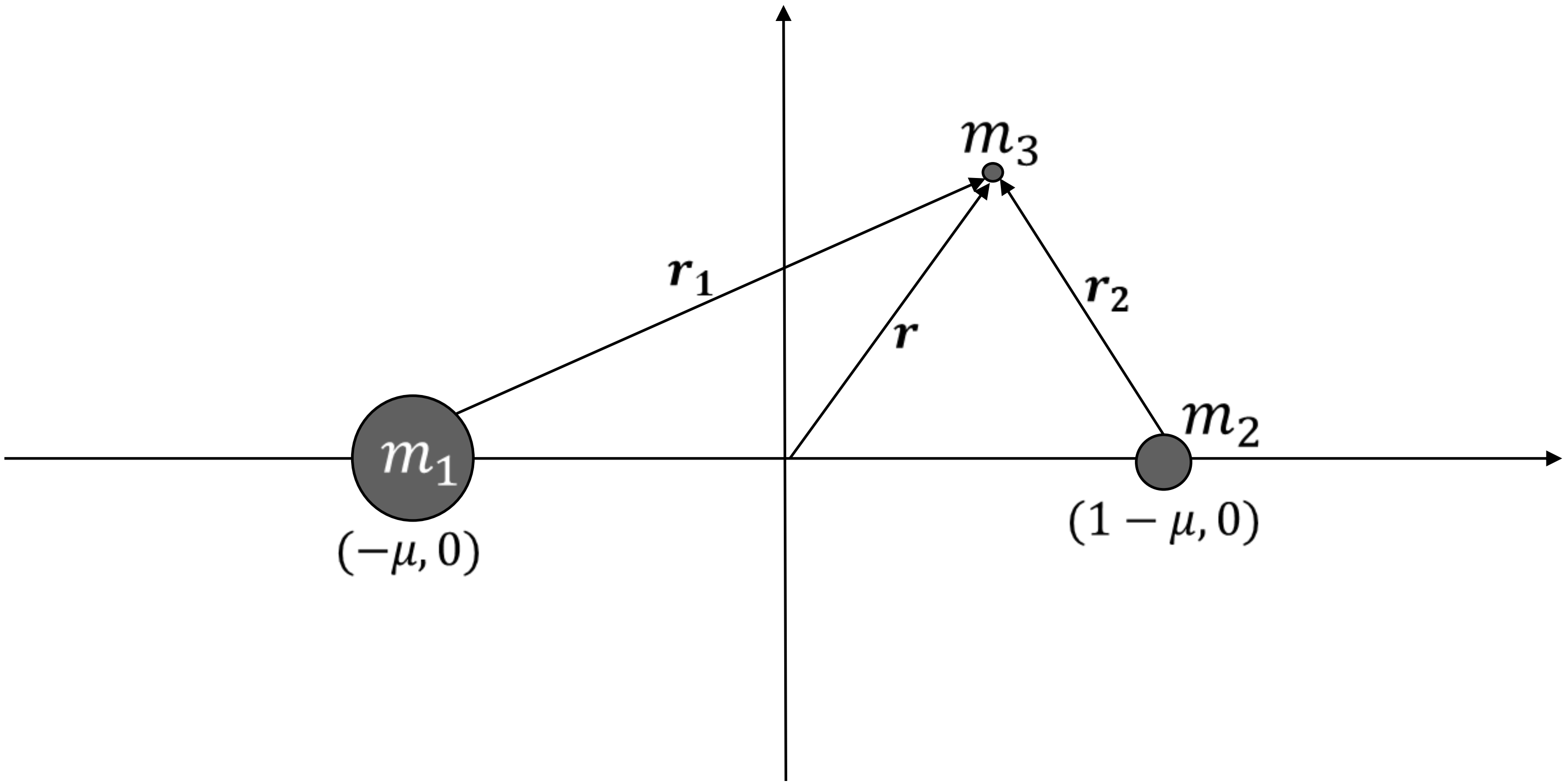
Low eccentricity

It may seem a bit restrictive but these hypotheses are actually describing useful situations:

The Earth-Moon system with a satellite can be described as a CRTBP. The eccentricity of the moon orbit around the Earth is 0.054 (low enough).

Similarly, most of the planets with respect to the Sun are well suited for the CRTBP approximation.

The CRTBP



μ is an adimensional parameter that only depends on the two main masses $\mu = \frac{m_2}{m_1+m_2}$

The CRTBP

If R is the distance between the two masses, and since the distance from the first mass and the smaller one is $r_1 = \frac{Rm_2}{m_1+m_2}$ and from the second mass to the smaller one is $r_2 = \frac{Rm_1}{m_1+m_2}$

If this distance is set to one, $R = 1$, the position of the two masses depends on μ

Newton's second law

What is interesting in such a problem is the motion of the tertiary mass.

According to Newton's second law, the equation of motion can be written:

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F_1} + \boldsymbol{F_2}$$

where $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ are the forces from mass 1 and mass 2.

Newton's second law

Similarly to the 2 body problem, the forces take the form:

$$\boldsymbol{F_1} = -\boldsymbol{G}\frac{m_1 m}{r_1^2}\boldsymbol{\hat{u}_{r1}} = -\boldsymbol{G}\frac{m_1 m}{r_1^3}\boldsymbol{r_1}$$

$$\boldsymbol{F_2} = -\boldsymbol{G}\frac{m_2 m}{r_2^2}\boldsymbol{\hat{u}_{r2}} = -\boldsymbol{G}\frac{m_2 m}{r_2^3}\boldsymbol{r_2}$$

$$\cancel{m}\ddot{\boldsymbol{r}} = -\boldsymbol{G}\frac{m_1 \cancel{m}}{r_1^3}\boldsymbol{r_1} - \boldsymbol{G}\frac{m_2 \cancel{m}}{r_2^3}\boldsymbol{r_2}$$

Cartesian form

But how do we express the position of the body in space ? From the center of mass, the distance can be expressed in Cartesian form

$$\boldsymbol{r} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\widehat{\boldsymbol{k}}$$

$$\boldsymbol{r_1} = (x - x_1)\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\widehat{\boldsymbol{k}} = (x + \mu)\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\widehat{\boldsymbol{k}}$$

$$\boldsymbol{r_2} = (x + x_2)\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\widehat{\boldsymbol{k}} = (x - 1 + \mu)\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\widehat{\boldsymbol{k}}$$

Rotating frame

We still need to get an expression for the acceleration $\ddot{\boldsymbol{r}}$. Let start with the velocity $\dot{\boldsymbol{r}}$. Since we are working in a rotating frame attached to the main masses we must take the rotation into account. In a rotating frame the time derivative can be written as:

$$\left(\frac{d}{dt}\right)_i = \left(\frac{d}{dt}\right)_r + \boldsymbol{\Omega} \times$$

Thus

$$\dot{\boldsymbol{r}} = \boldsymbol{v_{rel}} + \boldsymbol{\Omega} \times \boldsymbol{r}$$

$$\boldsymbol{v_{rel}} = \dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}} + \dot{z}\widehat{\boldsymbol{k}}$$

Different forces

The acceleration $\ddot{\boldsymbol{r}}$ can be easily obtained knowing the velocity $\dot{\boldsymbol{r}}$

$$\left(\frac{d^2}{dt^2}\right)_i = \left[\left(\frac{d}{dt}\right)_r + \boldsymbol{\Omega}\times\ \right]\left[\left(\frac{d}{dt}\right)_r + \boldsymbol{\Omega}\times\ \right]$$

Thus

Coriolis Centrifugal

$$\ddot{\boldsymbol{r}} = \boldsymbol{a}_{rel} + \boxed{2\boldsymbol{\Omega}\times\boldsymbol{v}_{rel}} + \boxed{\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})} + \dot{\boldsymbol{\Omega}}\times\boldsymbol{r}$$

$$\boldsymbol{a}_{rel} = \ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}} + \ddot{z}\hat{\boldsymbol{k}}$$

Equation of motion

$$\ddot{\boldsymbol{r}} = -G\frac{m_1}{r_1^3}\boldsymbol{r_1} - G\frac{m_2}{r_2^3}\boldsymbol{r_2}$$

The equations of motion can be re-written as:

$$\ddot{x} - 2\Omega\dot{y} - \Omega^2 x = -G\frac{m_1}{r_1^3}(x+\mu) - G\frac{m_2}{r_2^3}(x-1+\mu)$$

$$\ddot{y} + 2\Omega\dot{x} - \Omega^2 y = -G\frac{m_1}{r_1^3}y - G\frac{m_2}{r_2^3}y$$

$$\ddot{z} = -G\frac{m_1}{r_1^3}z - G\frac{m_2}{r_2^3}z$$

Small note about the cross product

$$\hat{\imath} \times \hat{\imath} = 0 \qquad \hat{\imath} \times \hat{\jmath} = \hat{k} \qquad \hat{\imath} \times \hat{k} = -\hat{\jmath}$$

$$\hat{\jmath} \times \hat{\jmath} = 0 \qquad \hat{\jmath} \times \hat{k} = \hat{\imath} \qquad \hat{\jmath} \times \hat{\imath} = -\hat{k}$$

$$\hat{k} \times \hat{k} = 0 \qquad \hat{k} \times \hat{\imath} = \hat{\jmath} \qquad \hat{k} \times \hat{\jmath} = -\hat{\imath}$$

$$\boldsymbol{\Omega} \times \boldsymbol{v_{rel}} = \begin{bmatrix} 0 \\ 0 \\ \Omega \end{bmatrix} \times \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\dot{y} \\ \dot{x} \\ 0 \end{bmatrix}$$

$$\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = \begin{bmatrix} 0 \\ 0 \\ \Omega \end{bmatrix} \times \left(\begin{bmatrix} 0 \\ 0 \\ \Omega \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} -x \\ -y \\ 0 \end{bmatrix}$$

Adimensionalized equations

Let us adimensionalize these equations with respect to mass, time and length:

$$\ddot{x} = x + 2\dot{y} - \frac{1-\mu}{r_1^3}(x+\mu) - \frac{\mu}{r_2^3}(x-1+\mu)$$

$$\ddot{y} = y - 2\dot{x} - \frac{1-\mu}{r_1^3}y - \frac{\mu}{r_2^3}y$$

$$\ddot{z} = -\frac{1-\mu}{r_1^3}z - \frac{\mu}{r_2^3}z$$

Where $r_1 = \sqrt{(x+\mu)^2 + y^2 + z^2}$ and $r_2 = \sqrt{(x-(1-\mu))^2 + y^2 + z^2}$

Lagrange Points

From these equations, five particular equilibrium points can be found. These points are called **Lagrange points.**

The conditions for equilibrium are that both velocities and accelerations are equal to 0

$$\dot{x} = \dot{y} = \dot{z} = 0$$

$$\ddot{x} = \ddot{y} = \ddot{z} = 0$$

$$0 = x - \frac{1-\mu}{r_1^3}(x+\mu) - \frac{\mu}{r_2^3}(x-1+\mu)$$

$$0 = y - \frac{1-\mu}{r_1^3}y - \frac{\mu}{r_2^3}y$$

$$0 = -\frac{1-\mu}{r_1^3}z - \frac{\mu}{r_2^3}z$$

Lagrange Points

From the last equation, one can easily observe that $z = 0$. It means that the five points lie in the orbital plane.

$$0 = \left(-\frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3} \right) z$$

Two cases can be considered, if $y = 0$ we have **collinear Lagrange points** otherwise if $y \neq 0$ we have **equilateral Lagrange points**.

Equilateral Lagrange Points

If $y \neq 0$, the second equation can be simplified to:

$$1 = \frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}$$

It can be injected in the first equation

$$x = \left(1 - \frac{\mu}{r_2^3}\right)(x+\mu) + \frac{\mu}{r_2^3}(x-1+\mu)$$

which directly gives that r_2^3 must be equal to 1 and thus r_1^3 too.

Equilateral Lagrange Points

Considering the original expression of r_1 and r_2

$$r_1 = \sqrt{(x+\mu)^2 + y^2 + z^2}$$
$$r_2 = \sqrt{(x-(1-\mu))^2 + y^2 + z^2}$$

$$\downarrow$$

$$1 = (x+\mu)^2 + y^2$$
$$1 = (x-(1-\mu))^2 + y^2$$

These two simple second degree equations can easily be solved to give two points named L_4 and L_5

$$L_4 = \left(0.5 - \mu; {}^{+\sqrt{3}}/_{2}; 0\right)$$
$$L_5 = \left(0.5 - \mu; {}^{-\sqrt{3}}/_{2}; 0\right)$$

Collinear Lagrange Points

If $y = 0$, the first equation is the only one remaining:

$$0 = x - \frac{1-\mu}{r_1^3}(x+\mu) - \frac{\mu}{r_2^3}(x-1+\mu)$$

with the expression of r_1 and r_2:

$$r_1 = \sqrt{(x+\mu)^2 + y^2 + z^2}$$
$$r_2 = \sqrt{(x-(1-\mu))^2 + y^2 + z^2}$$

$$\downarrow$$

$$r_1 = x + \mu$$
$$r_2 = x - (1-\mu)$$

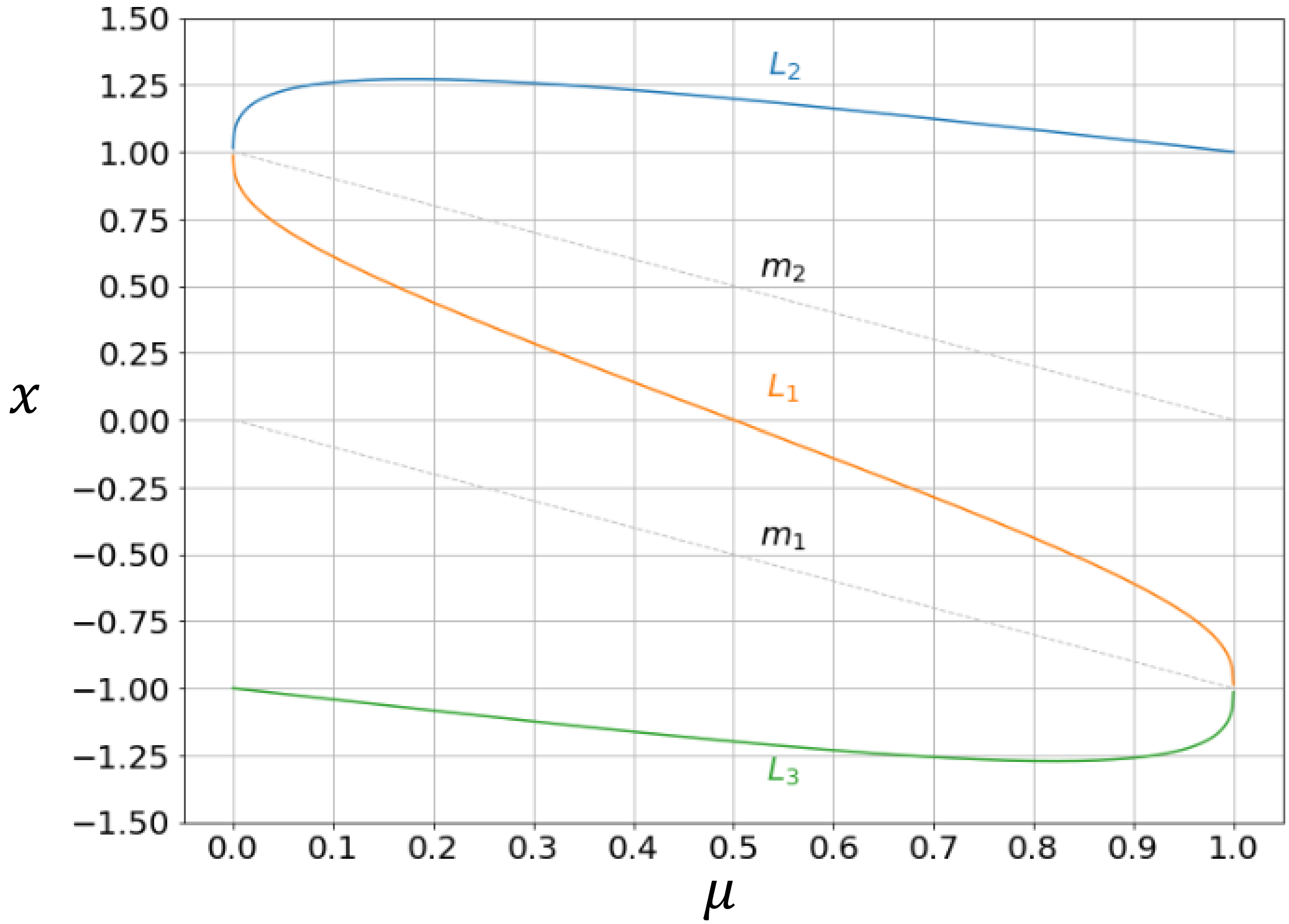
Collinear Lagrange Points

Once injected in the first equation:

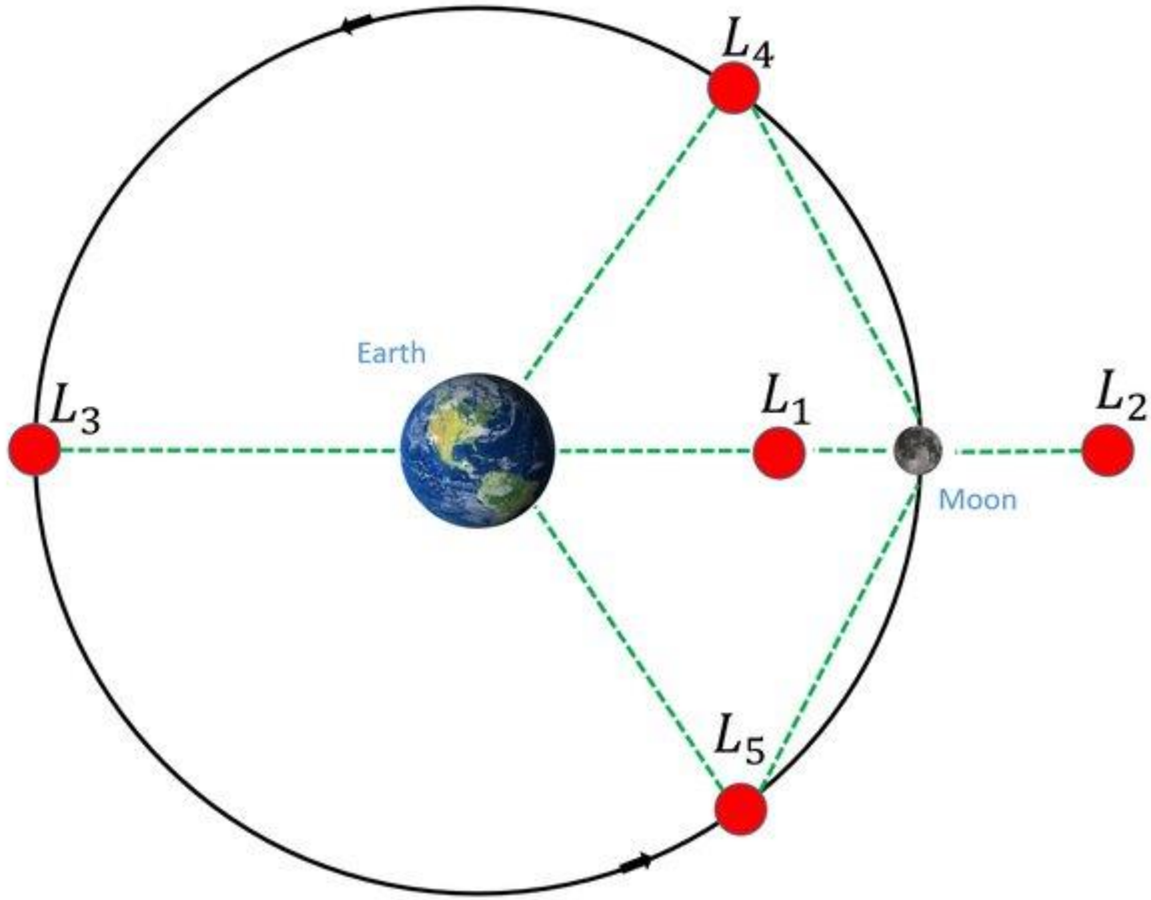
$$0 = x - \frac{1-\mu}{|(x+\mu)|^3}(x+\mu) - \frac{\mu}{\left|\left(x-(1-\mu)\right)\right|^{\mathbf{3}}}(x-1+\mu)$$

Due to the cubic value, the sign is uncertain, we can not simplify the expression. Three solution exist. It can be solved numerically for value between $\mu = 0$ and $\mu = 1$ corresponding to m_1 or m_2 equal to 0 which loses its interest.

Collinear Lagrange Points



Lagrange Points



Stability of the Lagrange Points

Equilibria can be stable or unstable, it is the same for the Lagrange points.

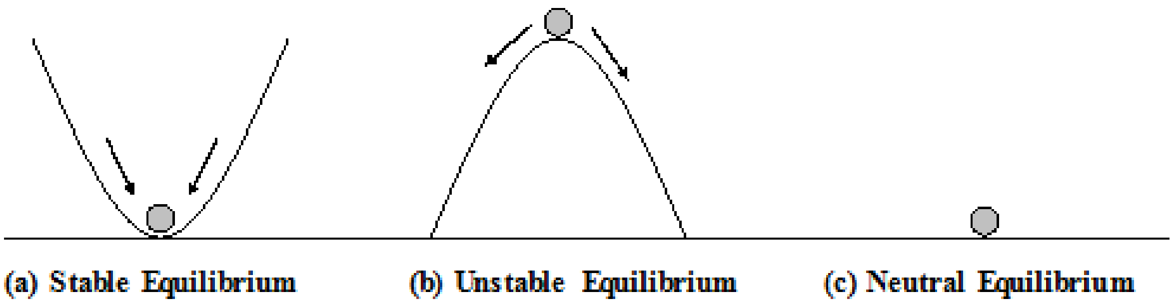
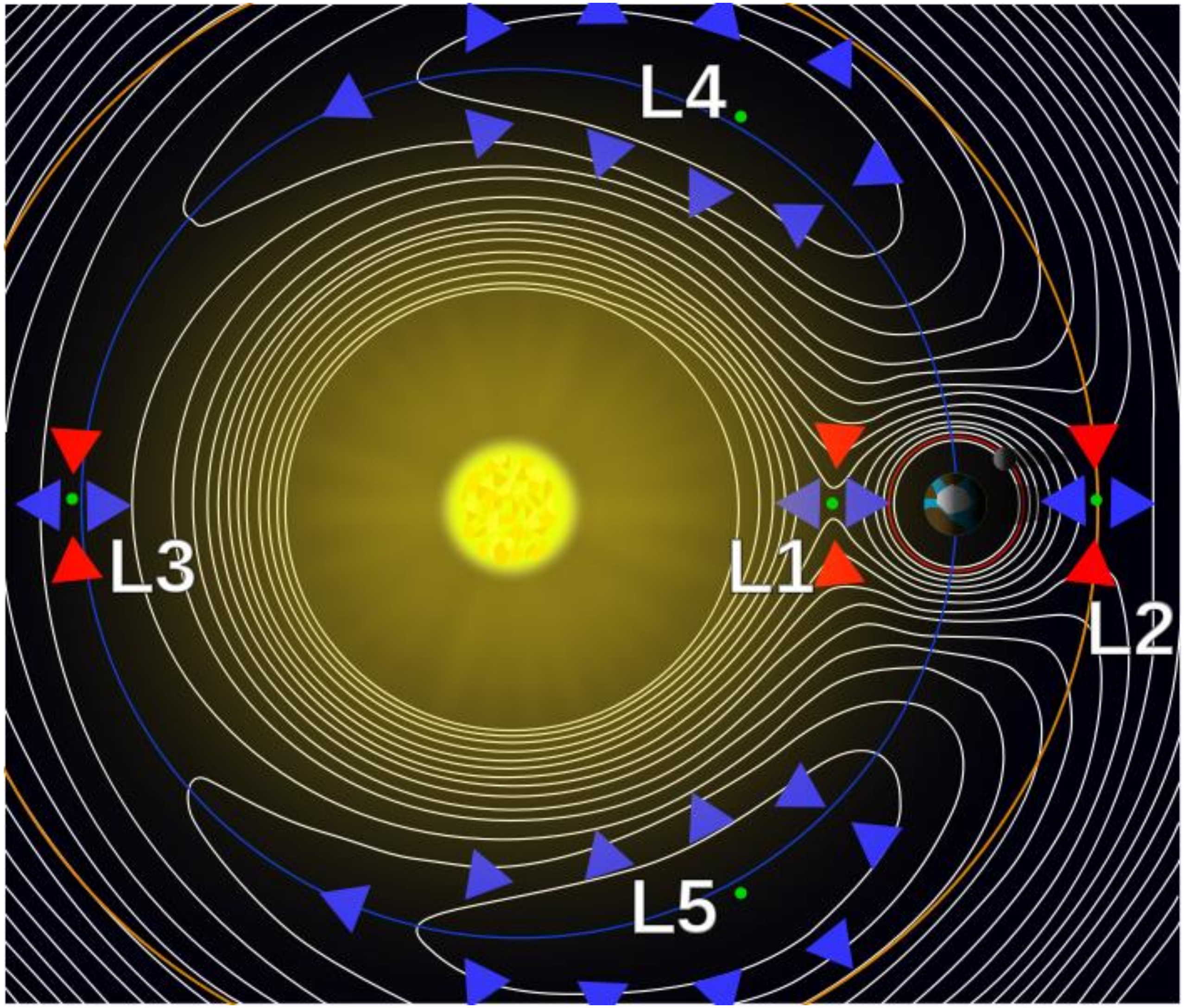


Figure 1 – Three Types of Equilibria

In the CRTBP, the 3 collinear Lagrange points are **unstable** and the 2 equilateral Lagrange points are **stable**.

Stability of the Lagrange Points



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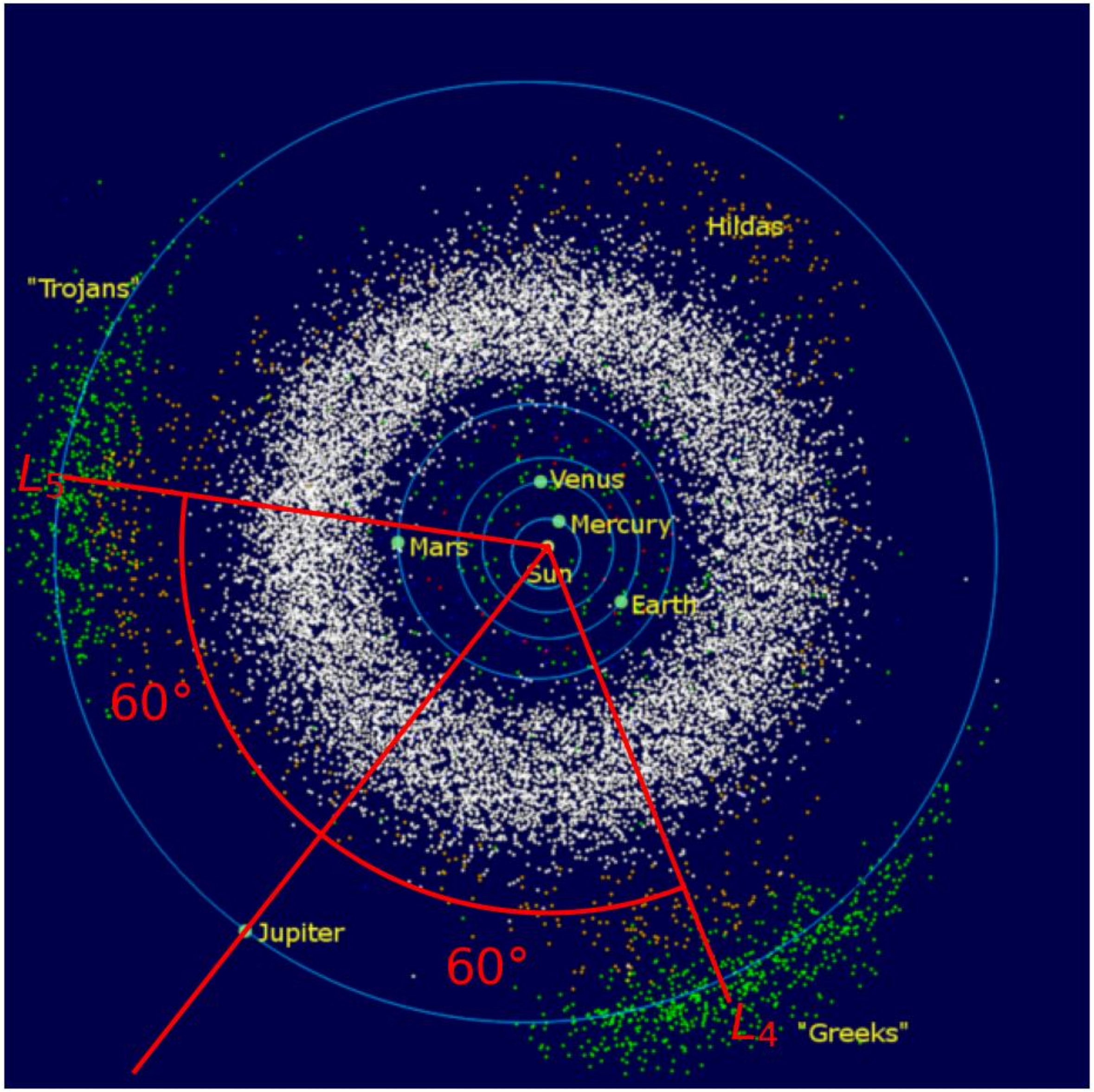

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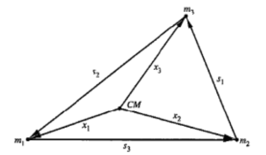

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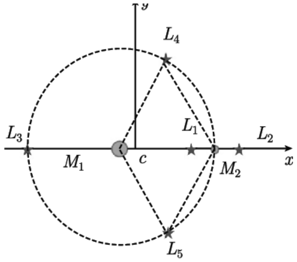

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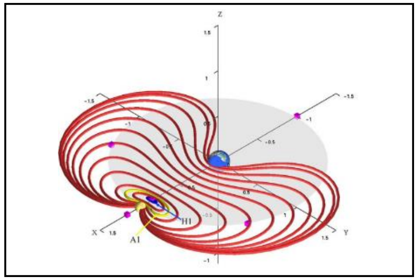
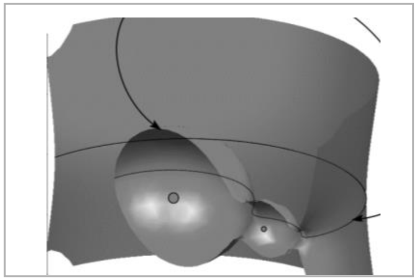


4.3 Orbit families

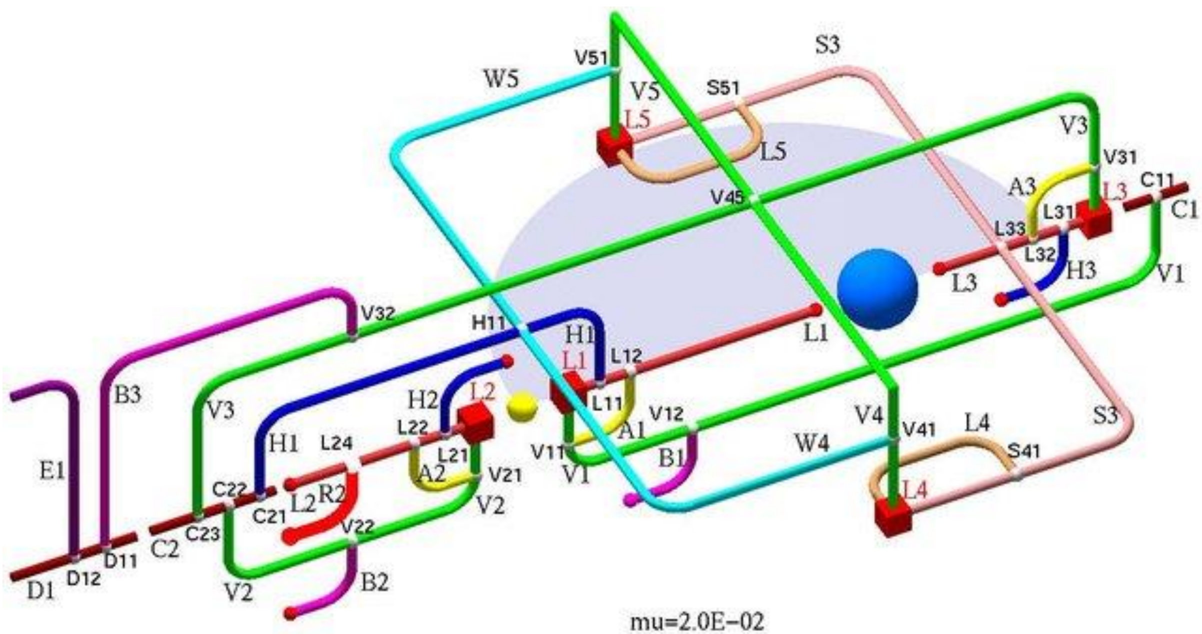


4.4 Zero velocity curve

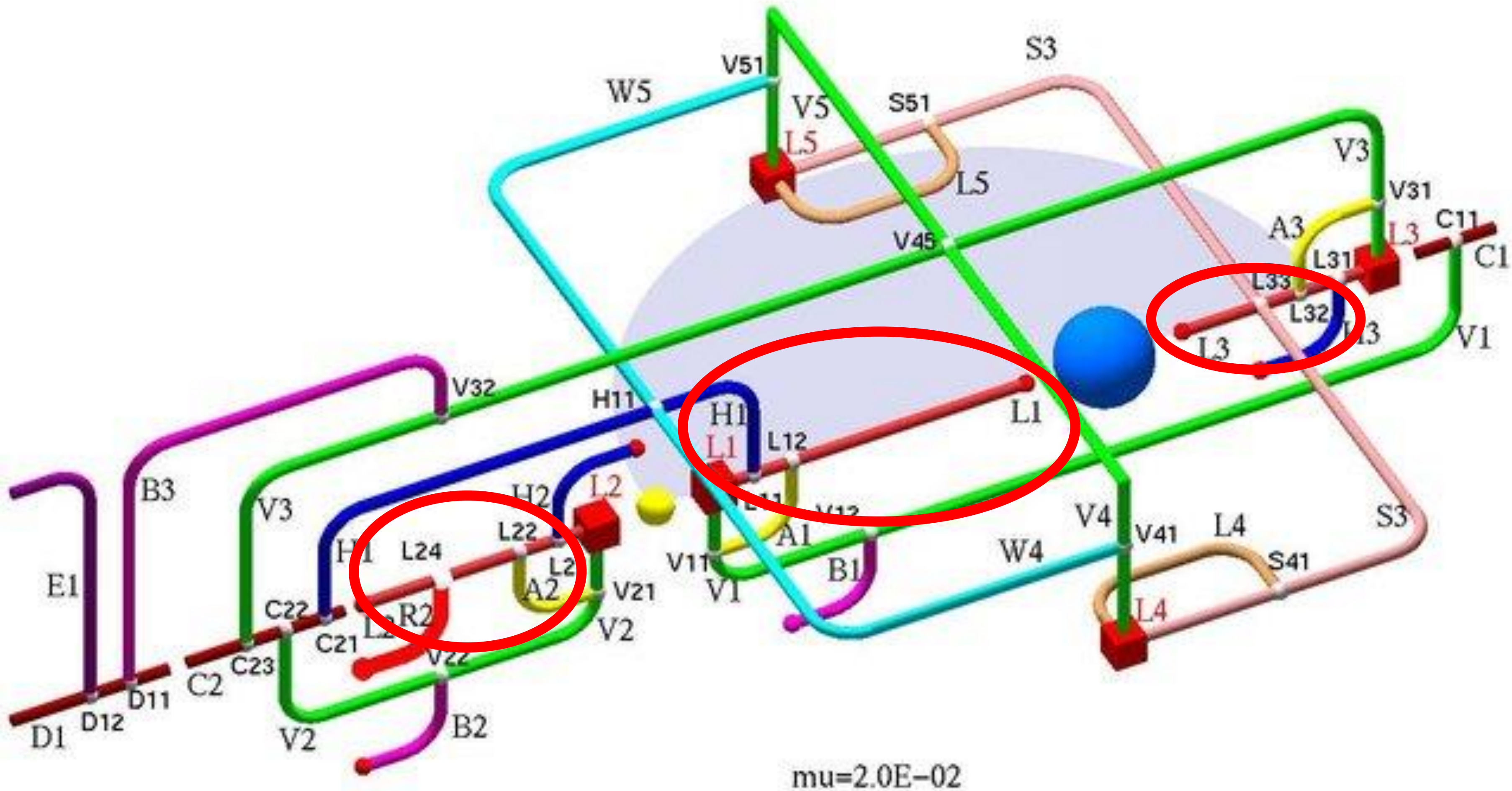
Orbit Families

The existence of the Lagrange points leads to certain particular families around those equilibrium points.

Those orbits are periodic and are connected together with bifurcations.



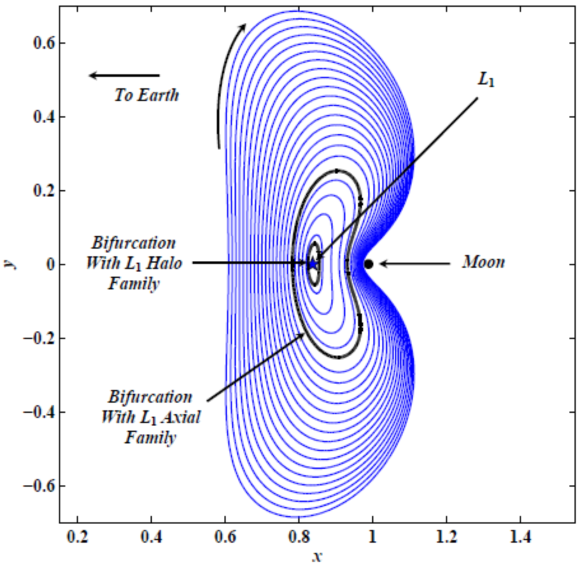
Orbit Families



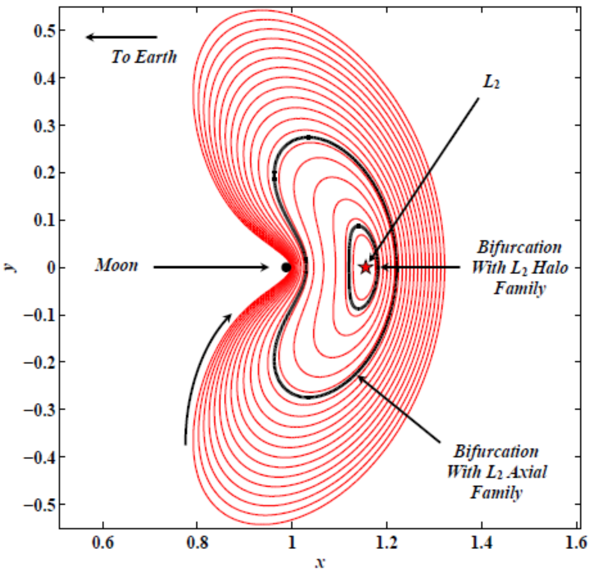
Lyapunov Family

The Lyapunov family comprises planar families of orbits around the 3 collinear Lagrange points.

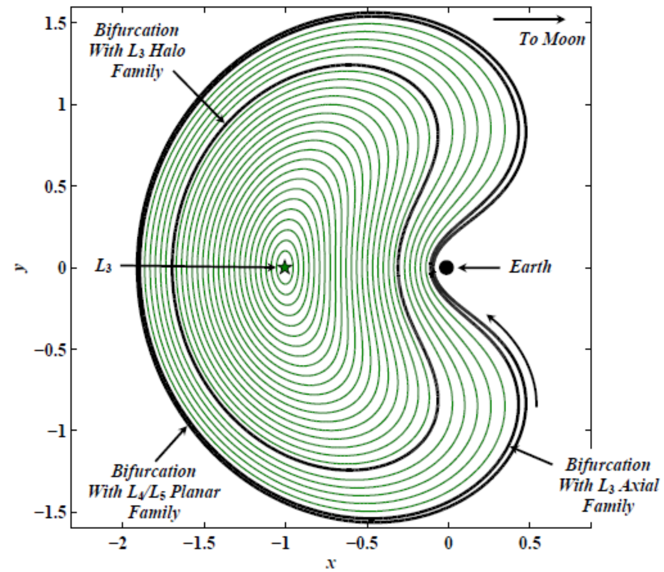
L1



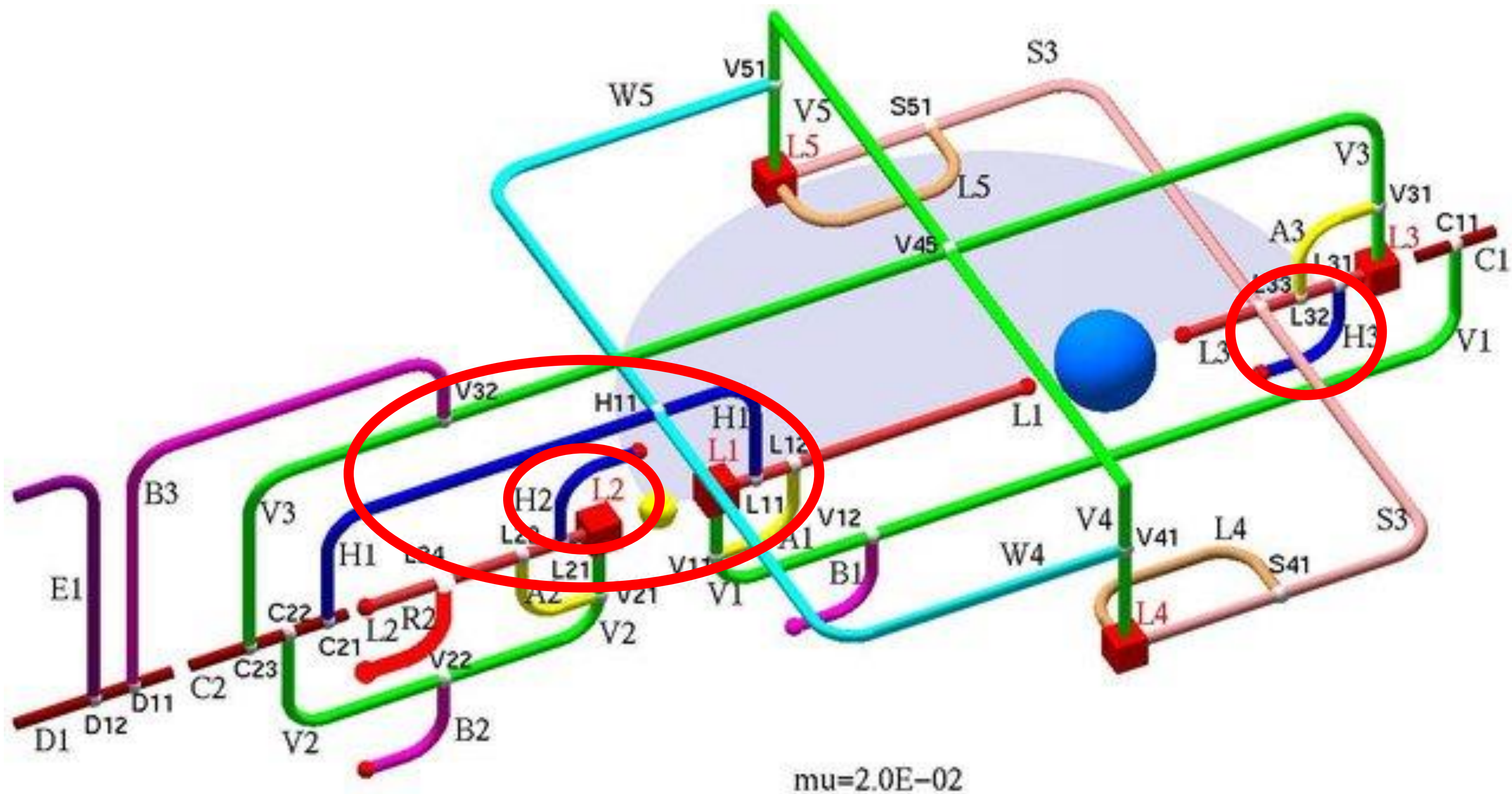
L2



L3

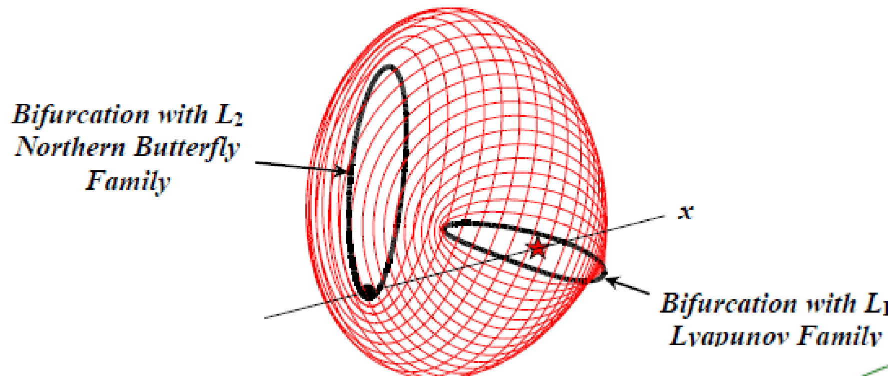


Orbit Families

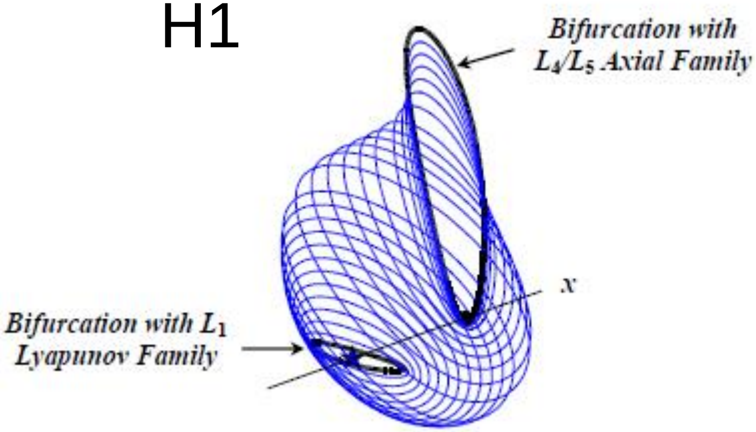


Halo Family

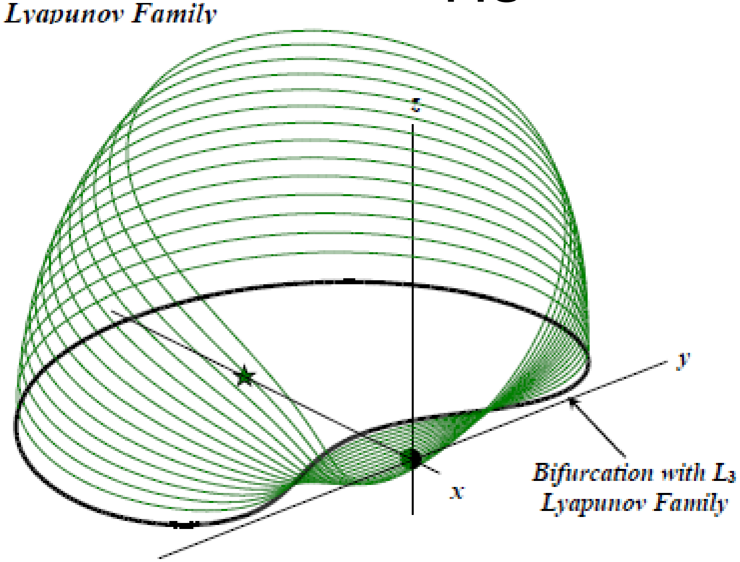
The Halo family comprises out of plane families of orbits around the 3 collinear Lagrange points.



H2

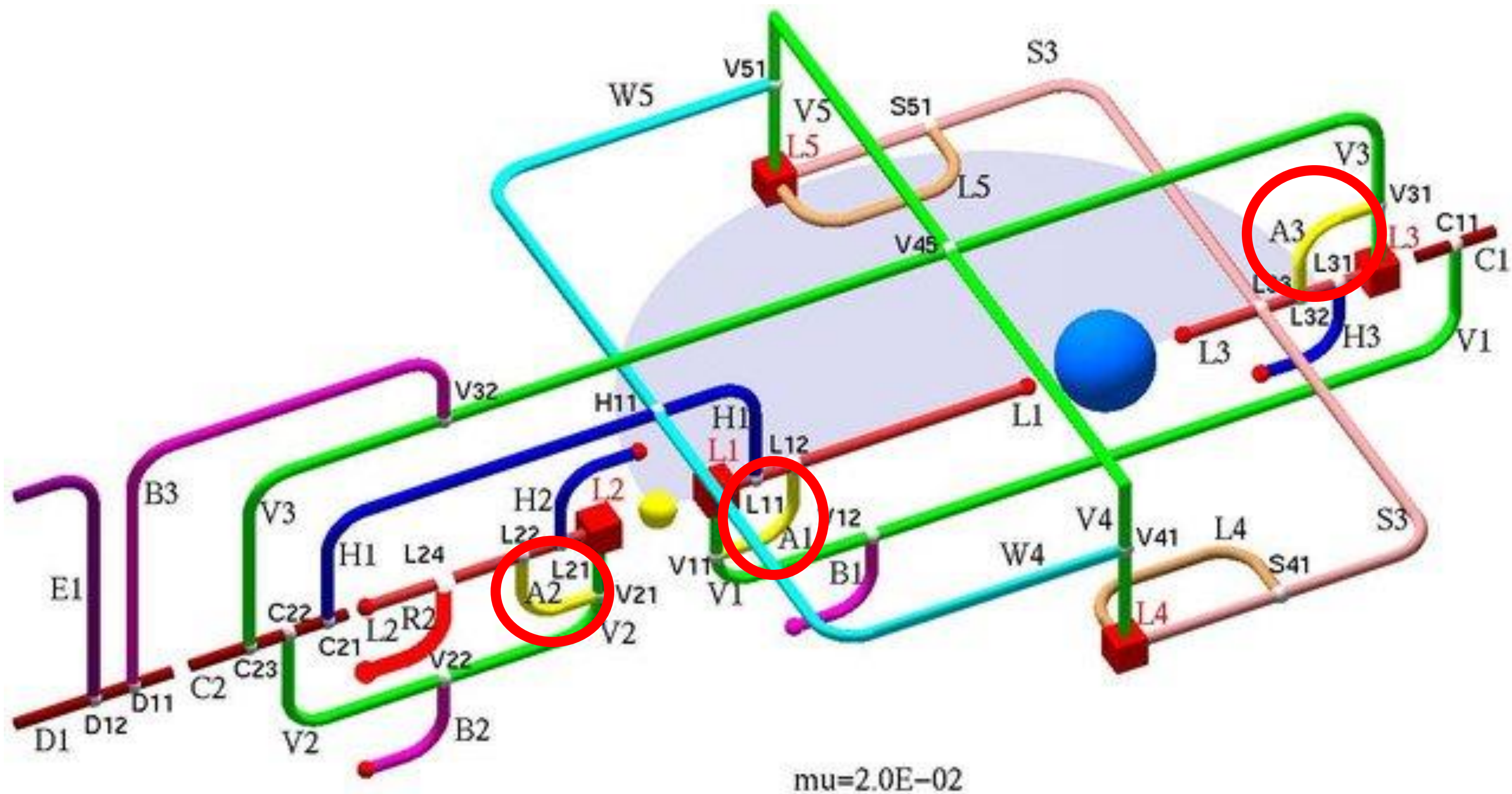


H1



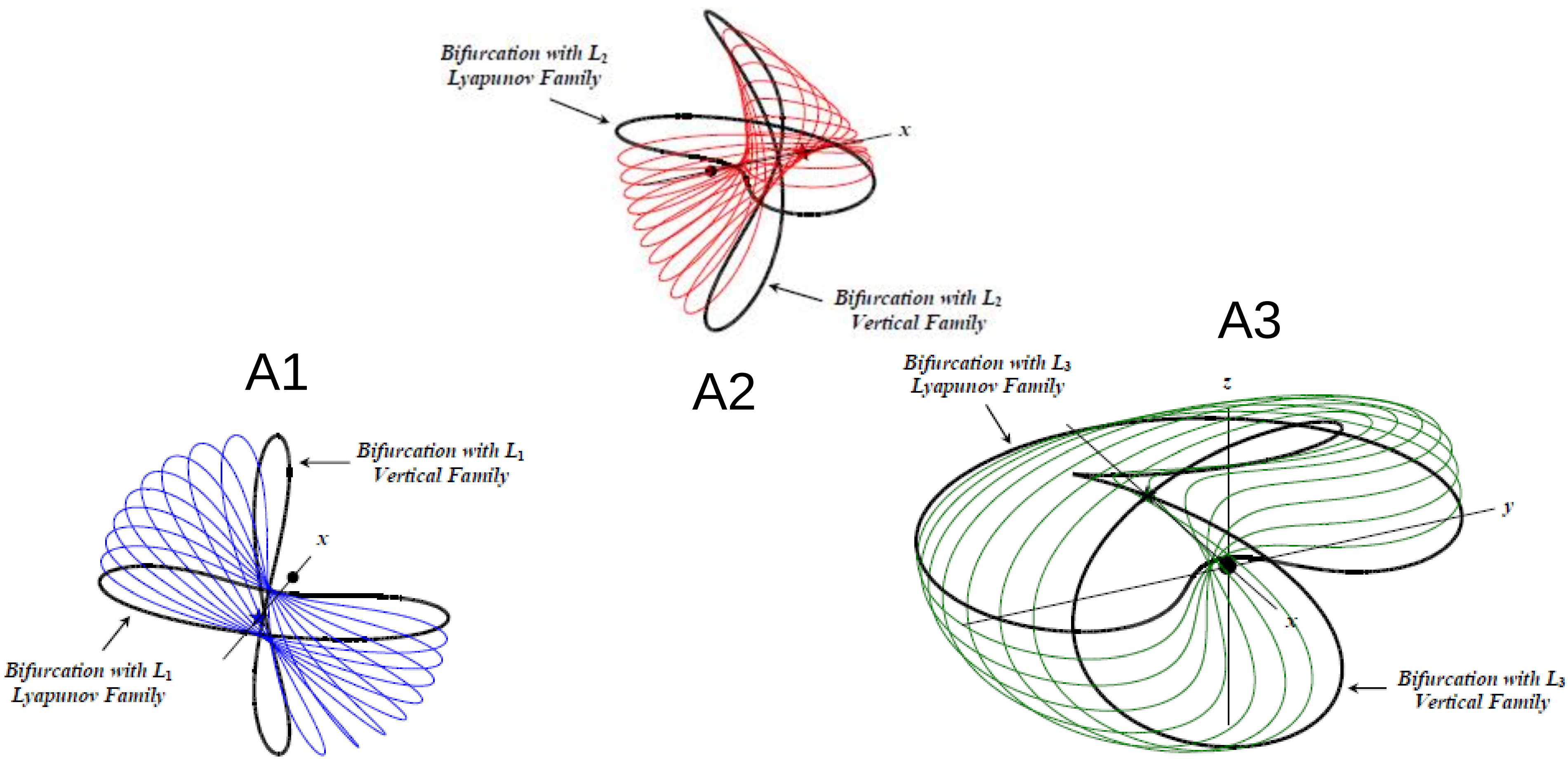
H3

Orbit Families

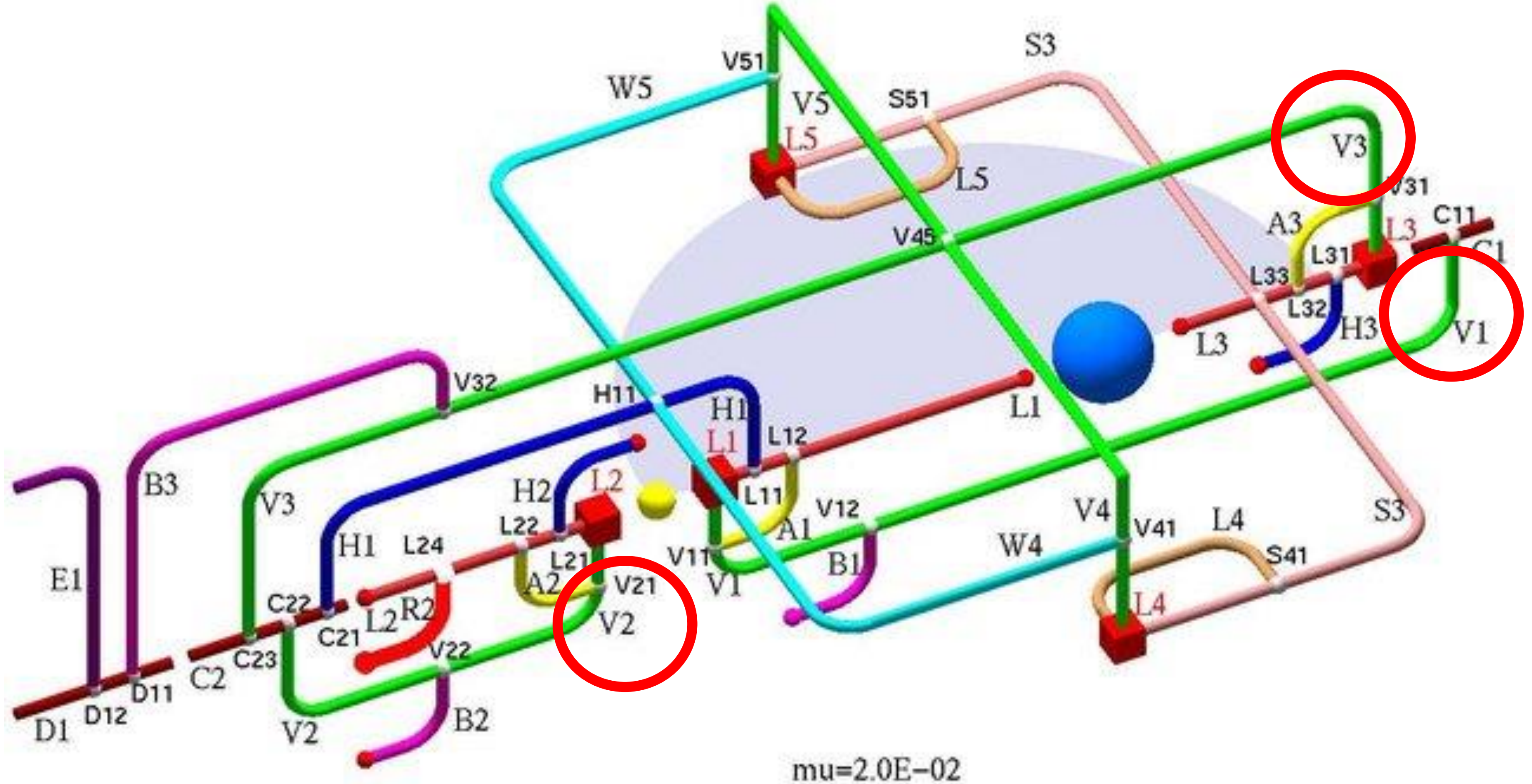


Axial Family

The axial family comprises out of plane families of orbits around the 3 collinear Lagrange points.

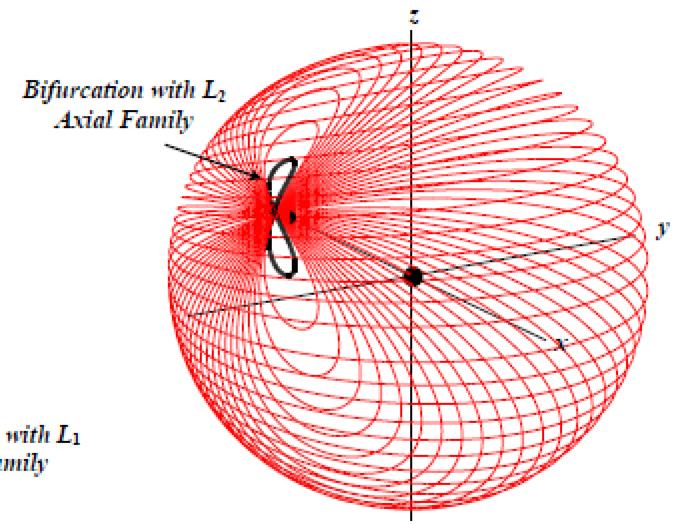


Orbit Families



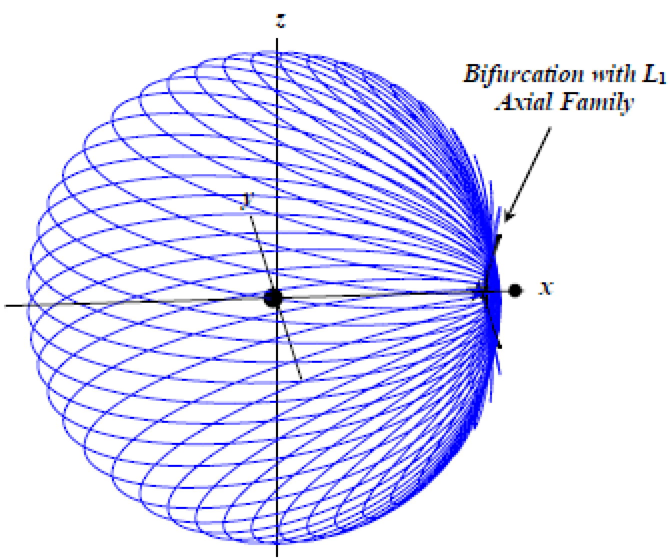
Vertical Family

The vertical family comprises out of plane families of orbits around the 3 collinear Lagrange points.

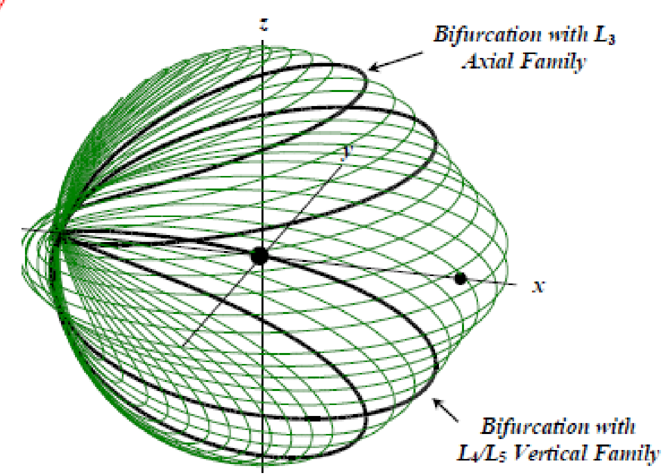


V2

V1



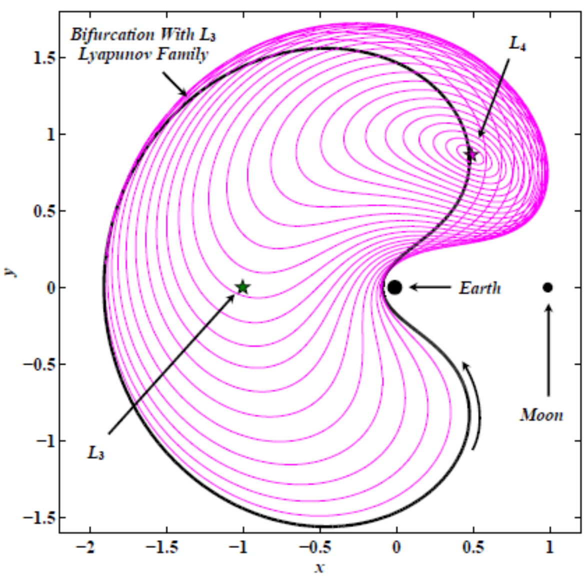
V3

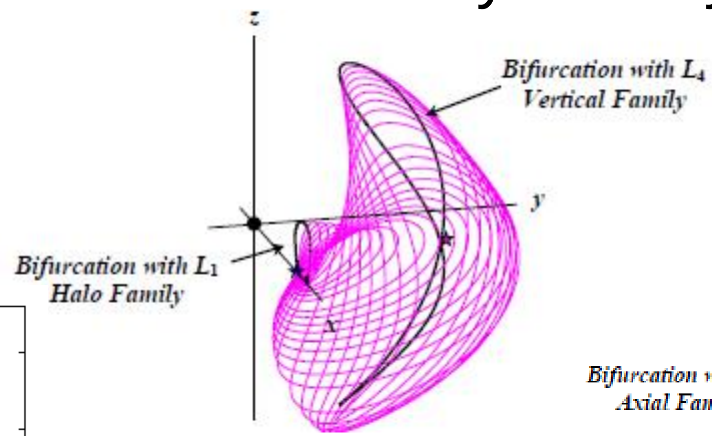


What about L4 and L5 ?

Families also exist around the equilateral Lagrange points but due to the position of the Lagrange points with respect to the masses they are asymmetric orbits.

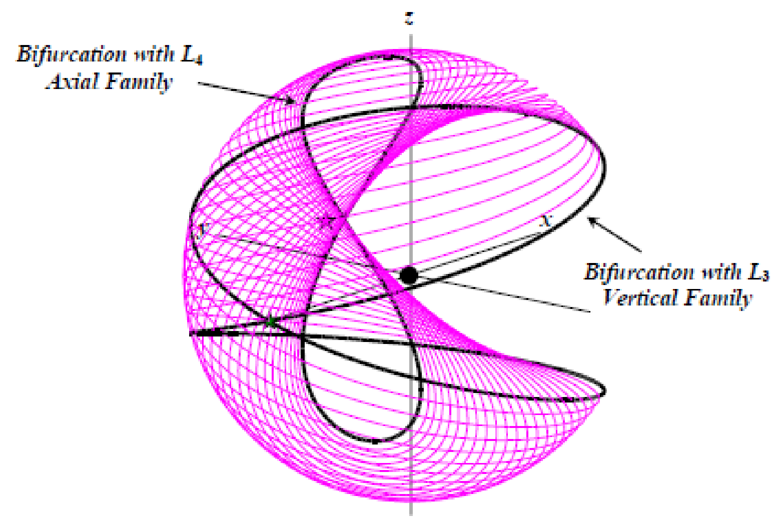
L4





A4

V4



4. The Three-Body Problem

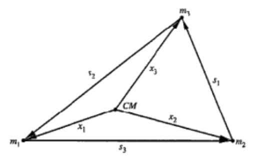
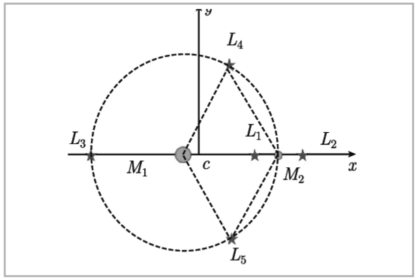
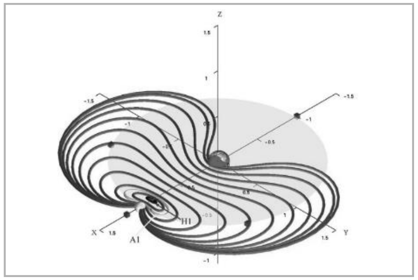


FIGURE 3.27 Position vectors $s_i = r_j - r_k$ for the three-body problem. Adapted from Hestenes, *New Foundations for Classical Mechanics*, 1999, Fig. 5.1.

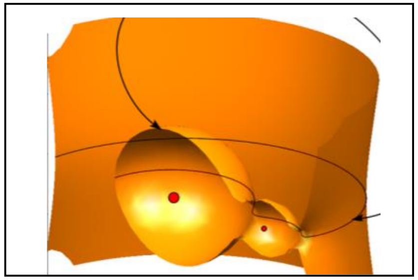
4.1 The general 3-body problem



4.2 The circular restricted 3-body problem



4.3 Orbit Families



4.4 Zero velocity curve

Conservation of energy

The law of conservation of energy states that the sum of the potential and kinetic energies of the third mass is constant. It means that given its initial condition some region of space cannot be reached by the third mass.

The potential energy is :

$$U = -\frac{1-\mu}{x+\mu} - \frac{\mu}{x-(1-\mu)} - \frac{1}{2}\left[(1-\mu)(x+\mu)^2 + \mu\big(x-(1-\mu)\big)^2\right]$$

The kinetic energy is :

$$\frac{1}{2}v^2 = \frac{1}{2}[(\dot{x})^2 + (\dot{y})^2]$$

Jacobi constant

The Jacobi constant corresponds to the total energy of the third mass relative to the reference frame and is defined as:

$$V + U = J$$

$$\frac{1}{2}v^2 - \frac{1-\mu}{x+\mu} - \frac{\mu}{x-(1-\mu)} - \frac{1}{2}\left[(1-\mu)(x+\mu)^2 + \mu\big(x-(1-\mu)\big)^2\right] = J$$

A tradeoff between the two types of energy is possible.

Jacobi constant

It means that if the mass is at a certain location associated to a certain potential energy U_1 and a zero velocity $v_1 = 0$, then the Jacobi constant is equal to the potential. The mass cannot reach certain parts of space because it does not have enough energy.

If, on the other hand, the velocity $v_1 > 0$, the mass can “climb” higher up to the point where $v_1 = 0$.

A simple conclusion from this is that certain parts of space can be called **forbidden regions**.

Forbidden Regions

Those regions can be drawn for given Jacobi constants. Knowing that the limit of the regions is given by $v_1 = 0$, we have

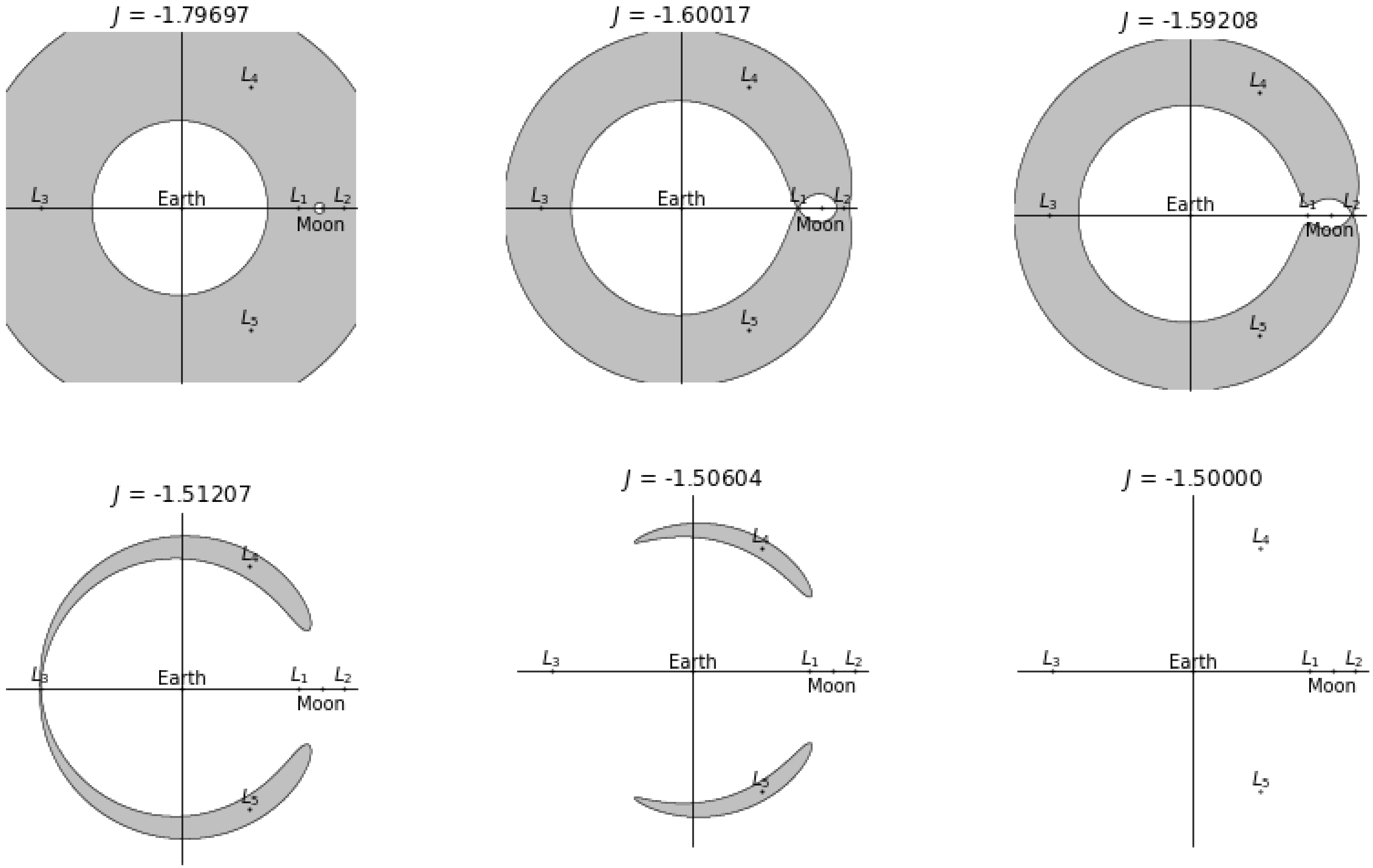
$$\frac{1}{2}v^2 - \frac{1-\mu}{x+\mu} - \frac{\mu}{x-(1-\mu)} - \frac{1}{2}\left[(1-\mu)(x+\mu)^2 + \mu\big(x-(1-\mu)\big)^2\right] = J$$

$$\boxed{\frac{1-\mu}{x+\mu}} + \boxed{\frac{\mu}{x-(1-\mu)}} + \boxed{\frac{1}{2}\left[(1-\mu)(x+\mu)^2 + \mu\big(x-(1-\mu)\big)^2\right]} + J = 0$$

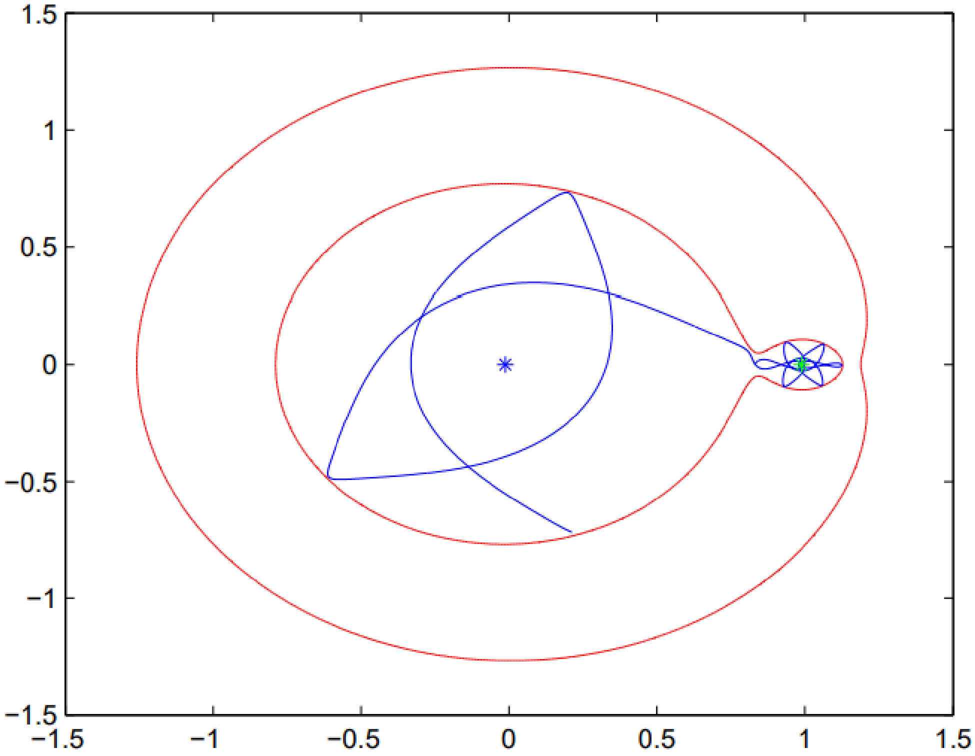
> 0 > 0 > 0

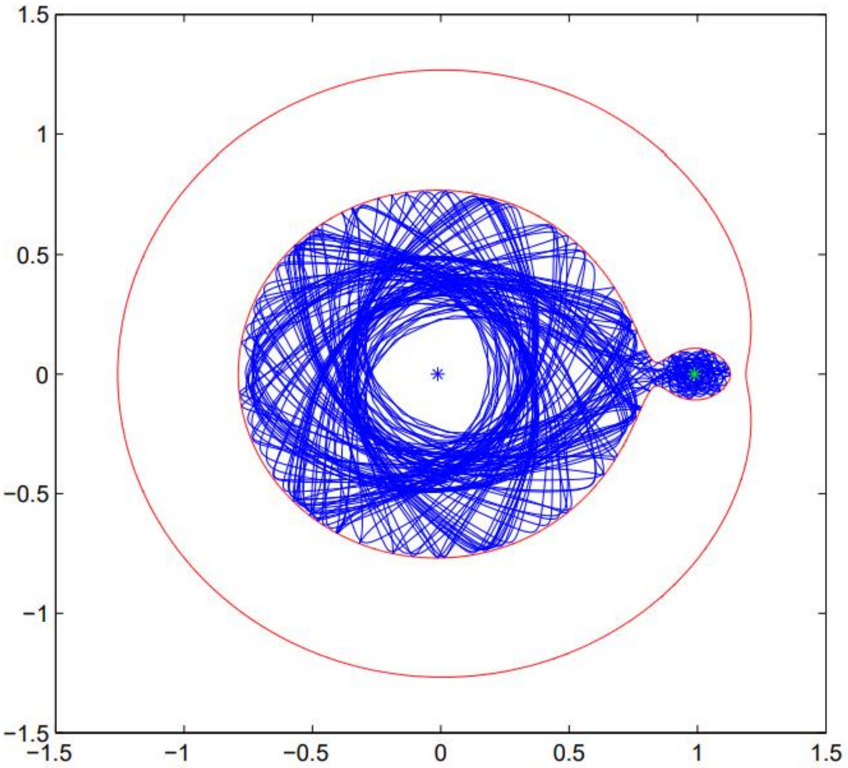
Since all the three terms are positive the Jacobi constant must be negative.

Forbidden Regions



Forbidden Regions



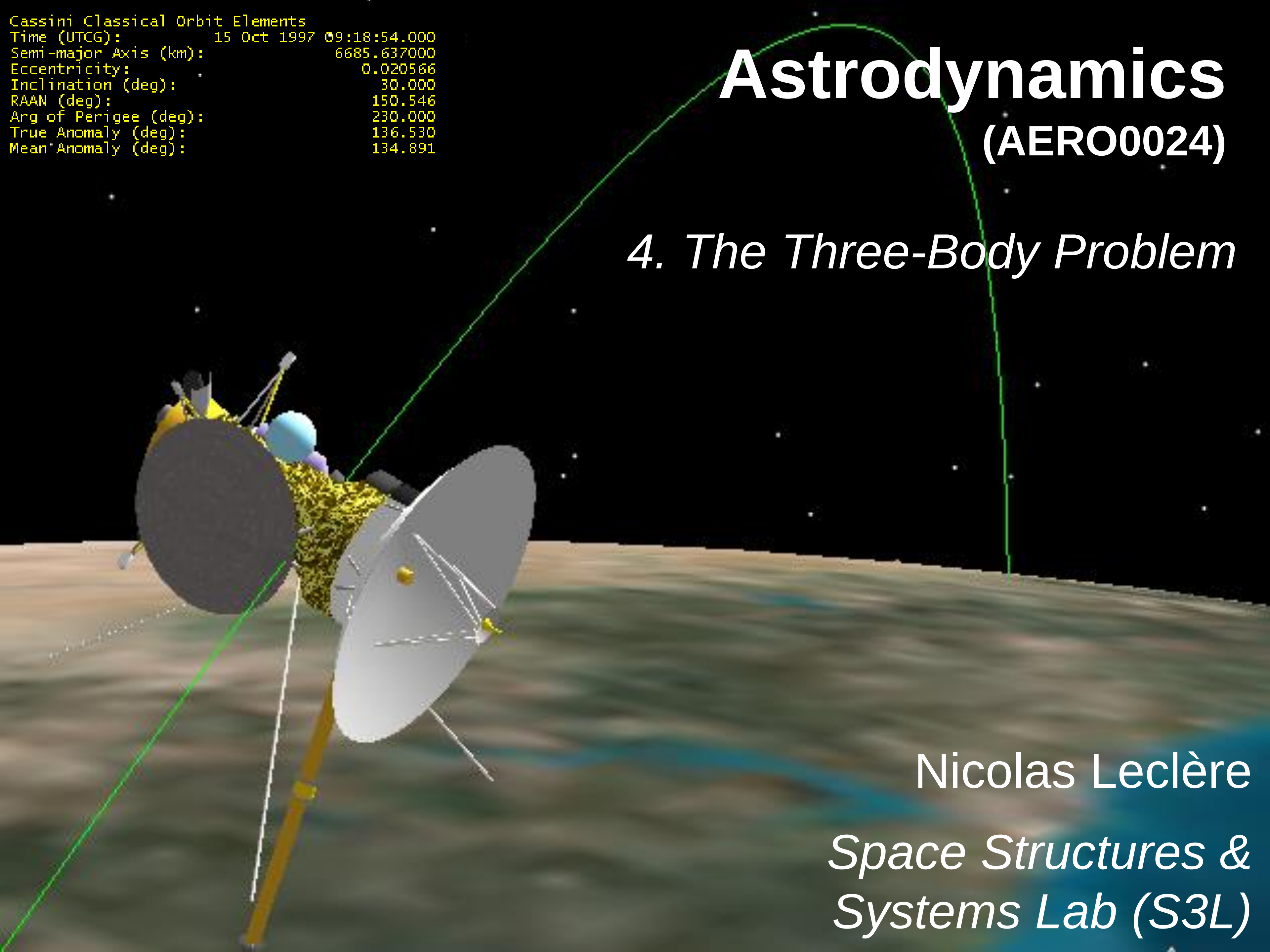


Conclusions

The 3-body problem is inherently chaotic.

Need of (possibly over-simplifying) assumptions to understand dynamics.

CR3BP: rich dynamical environment with several families of periodic orbits of interest for space mission design



Astrodynamics
(AERO0024)

4. The Three-Body Problem

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