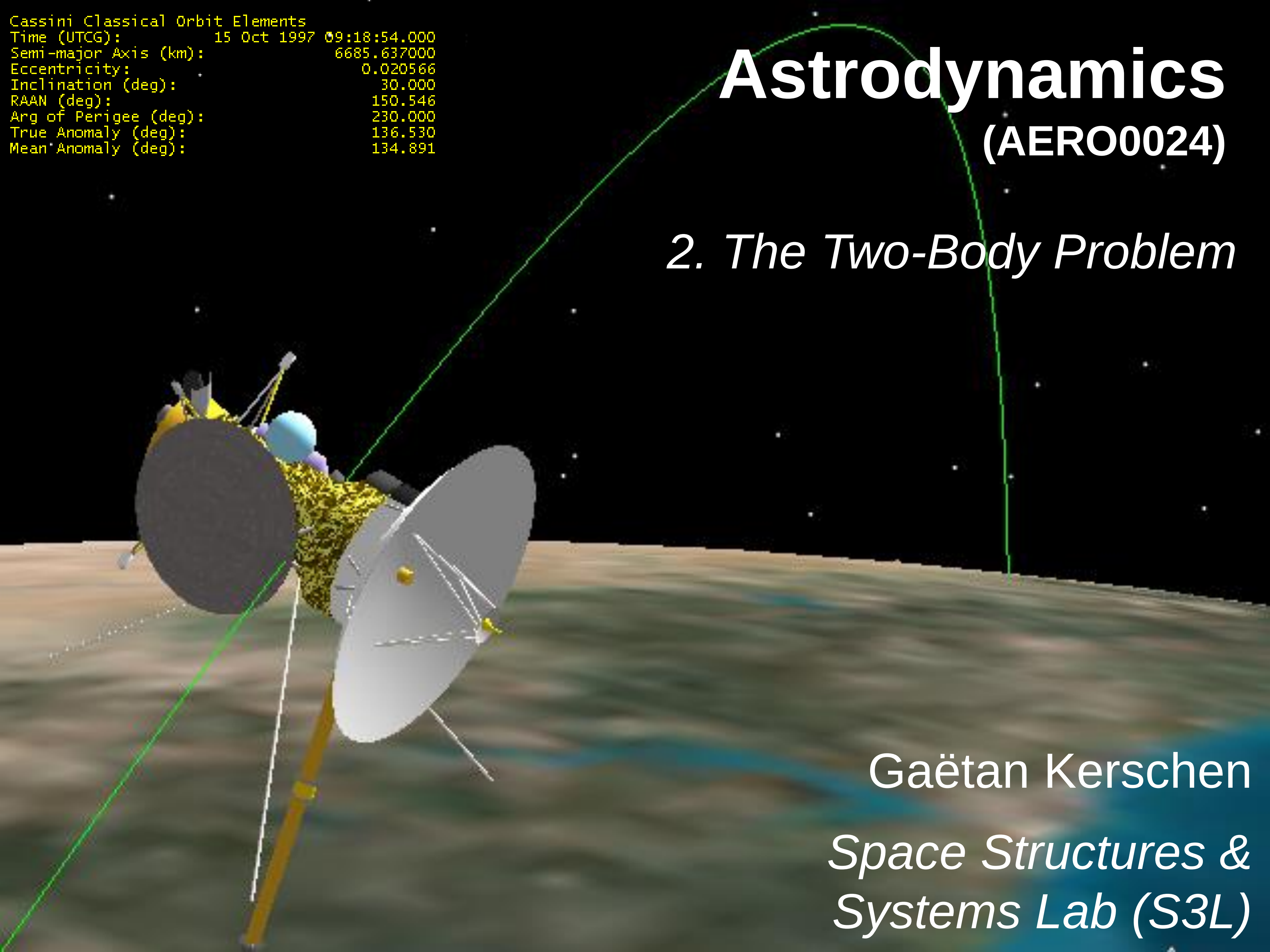


Astrodynamics

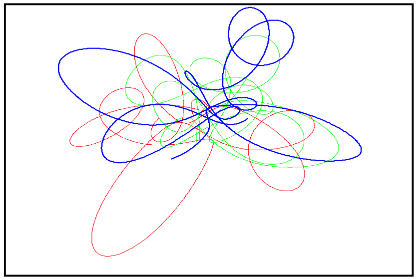
(AERO0024)

2. The Two-Body Problem

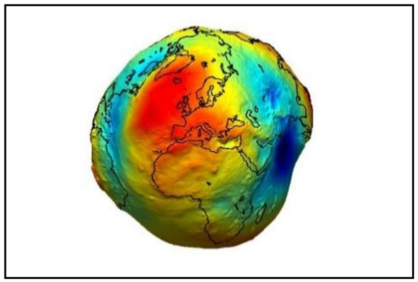
Gaëtan Kerschen

Space Structures & Systems Lab (S3L)

2. The Two-Body Problem



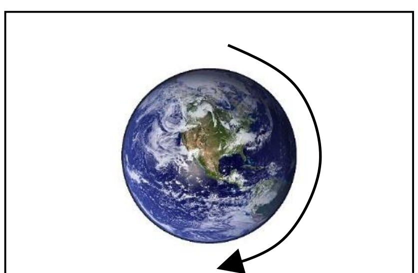
2.1 Justification of the 2-body model



2.2 Gravitational field

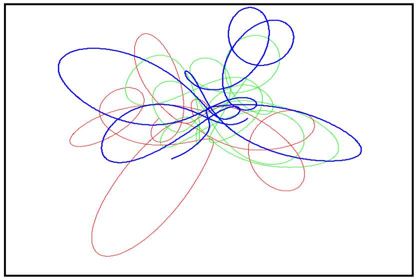
$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r}$$

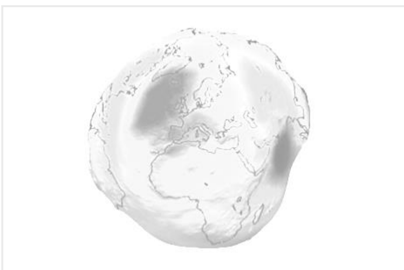
2.3 Relative motion



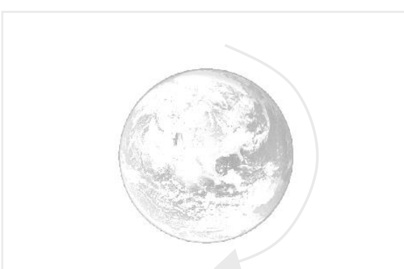
2.4 Resulting orbits

2. The Two-Body Problem

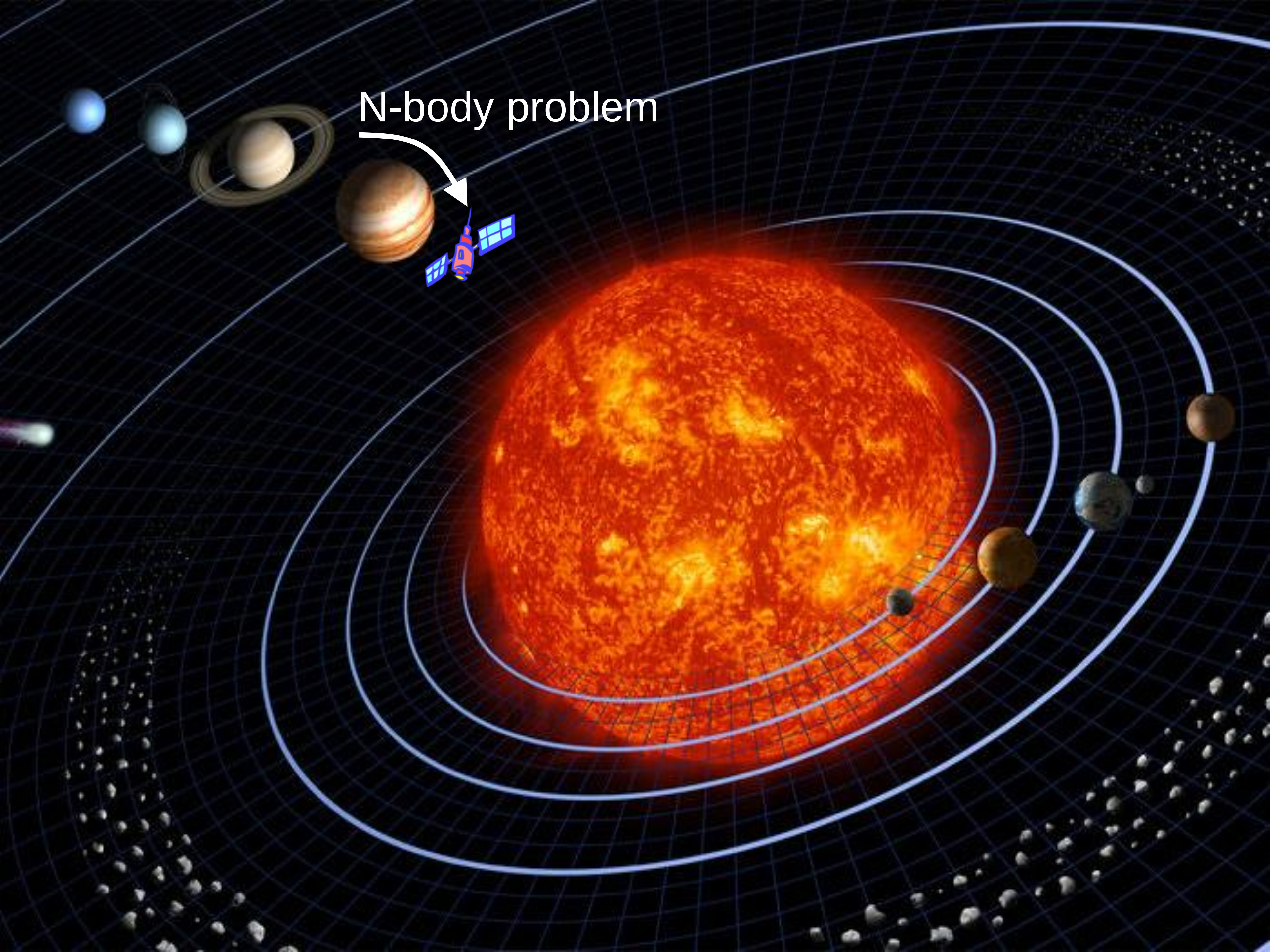




$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r}$$



2.1 Justification of the 2-body model

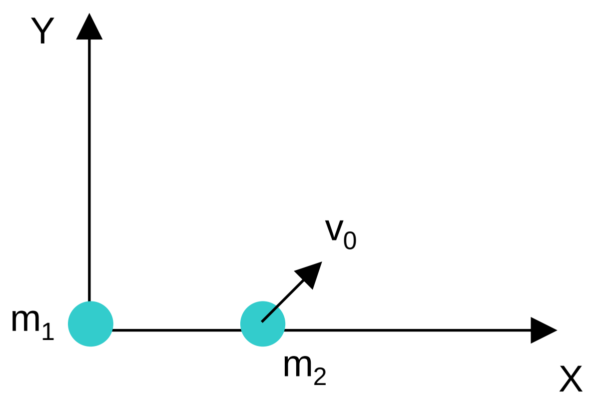

N-body problem

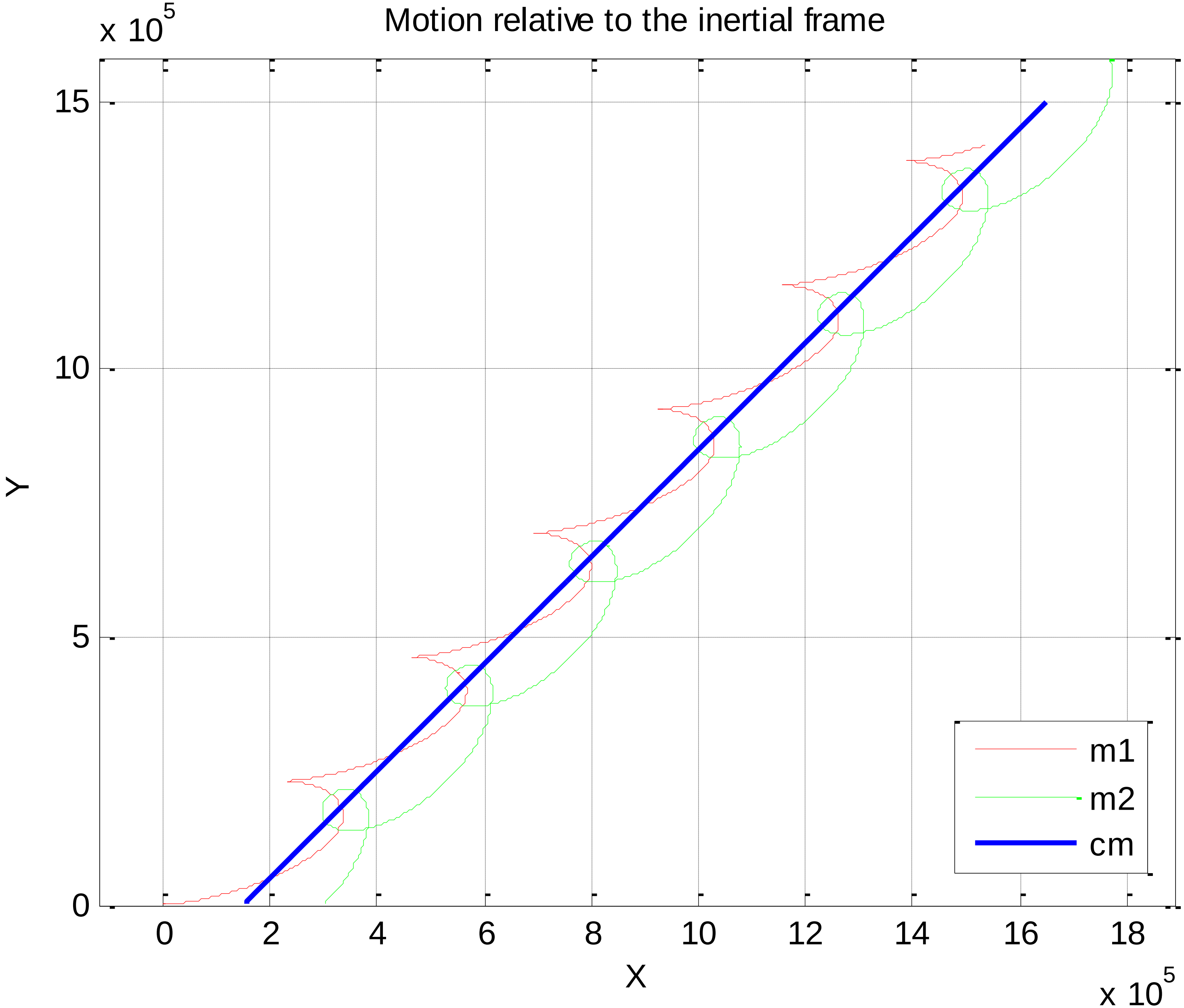
Two-Body Problem: Matlab Example

Two identical masses:

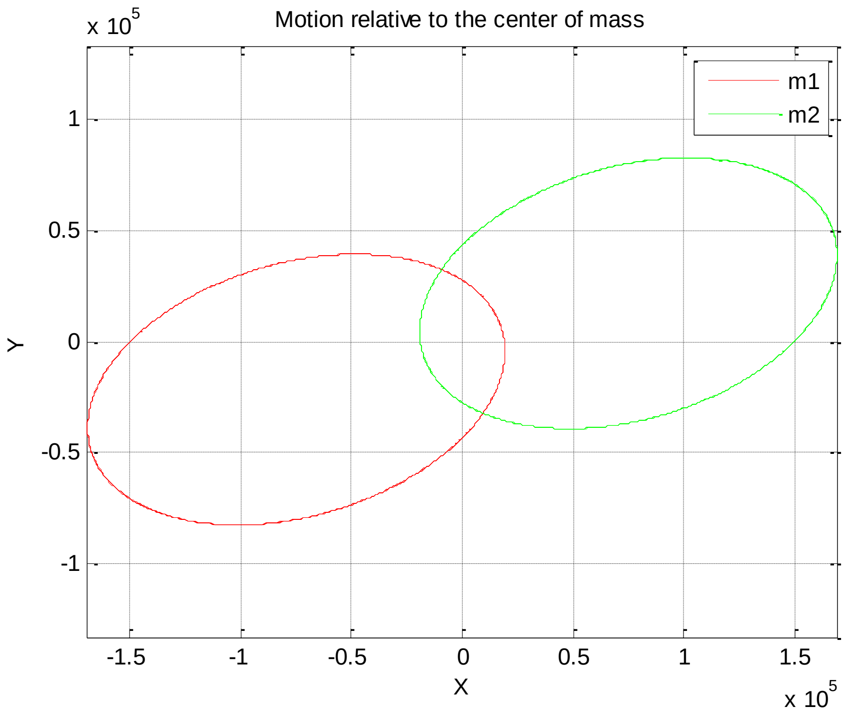
⇒ One is at rest at the origin of the inertial frame of reference.

⇒ The other one has a velocity directed upward to the right making a 45 degrees angle with the X axis.

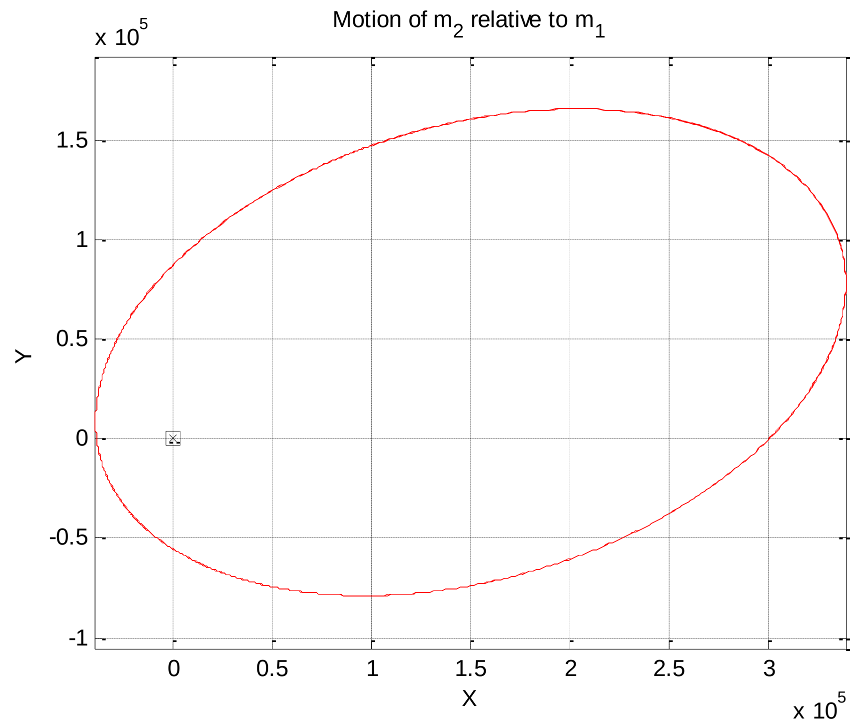




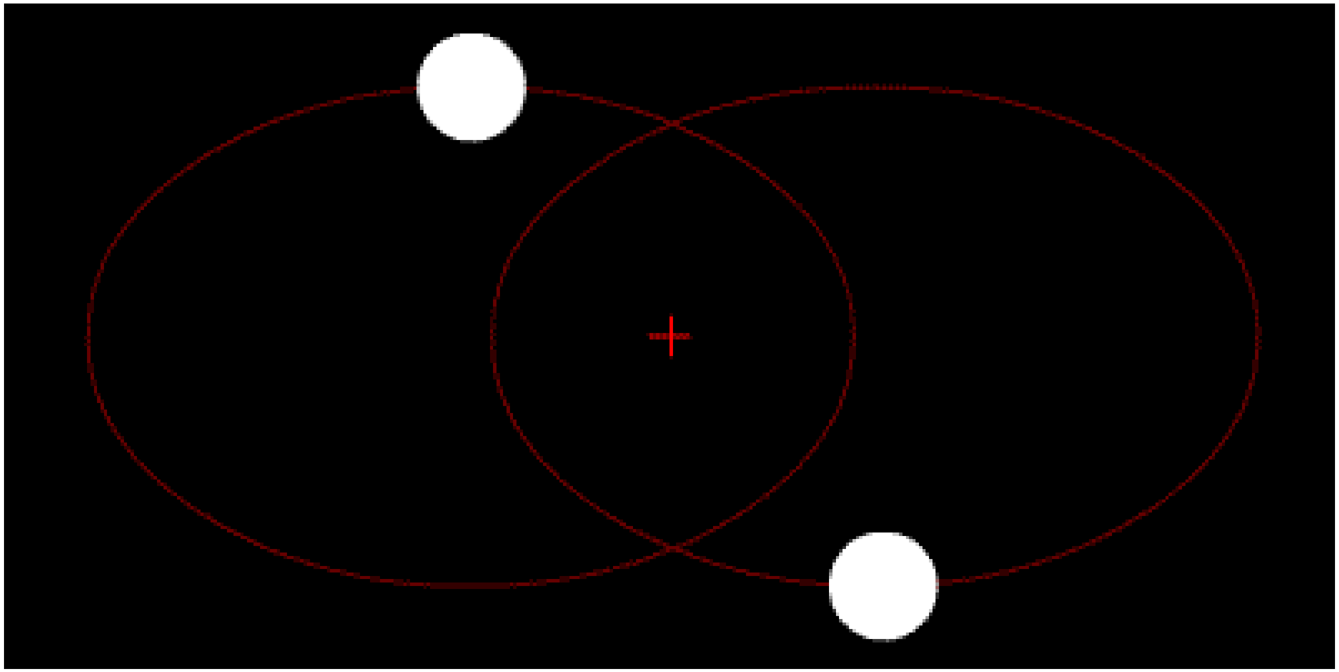
Seemingly complex motion in the inertial frame

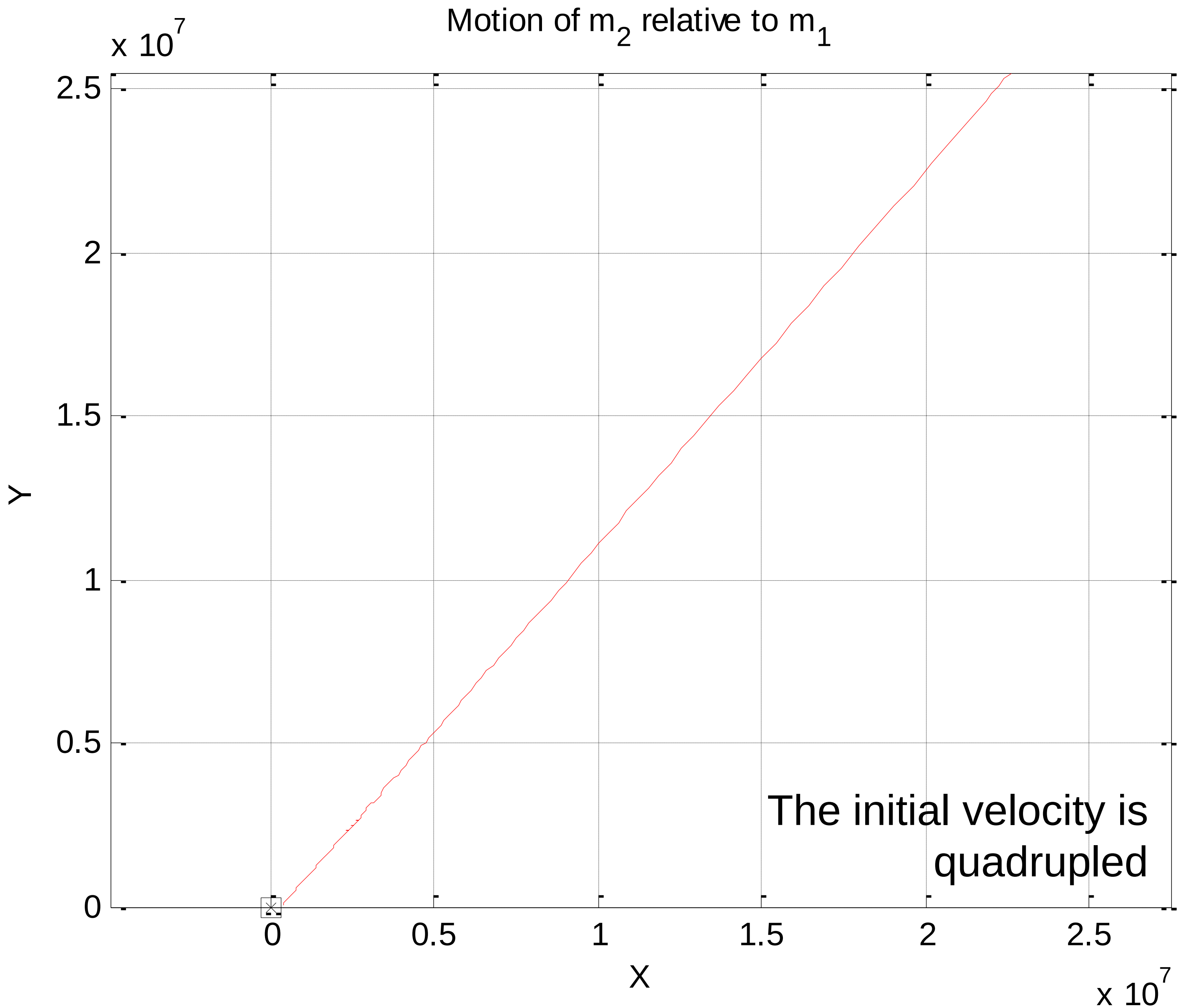


Much less complex motion when viewed from the c.o.m



Much less complex motion when viewed from m_1



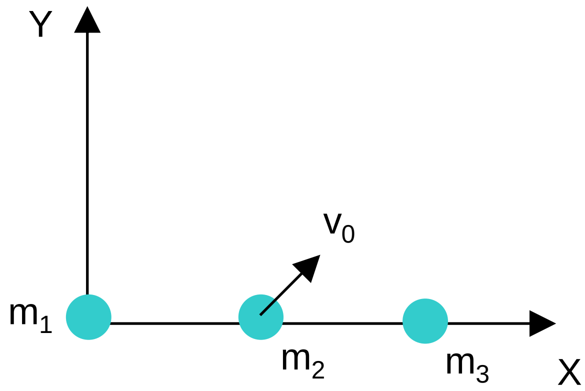

Motion of m_2 relative to m_1
x 10^7
2.5
2
1.5
1
0.5
0
Y
0
0.5
1
1.5
2
2.5
X
x 10^7
The initial velocity is quadrupled

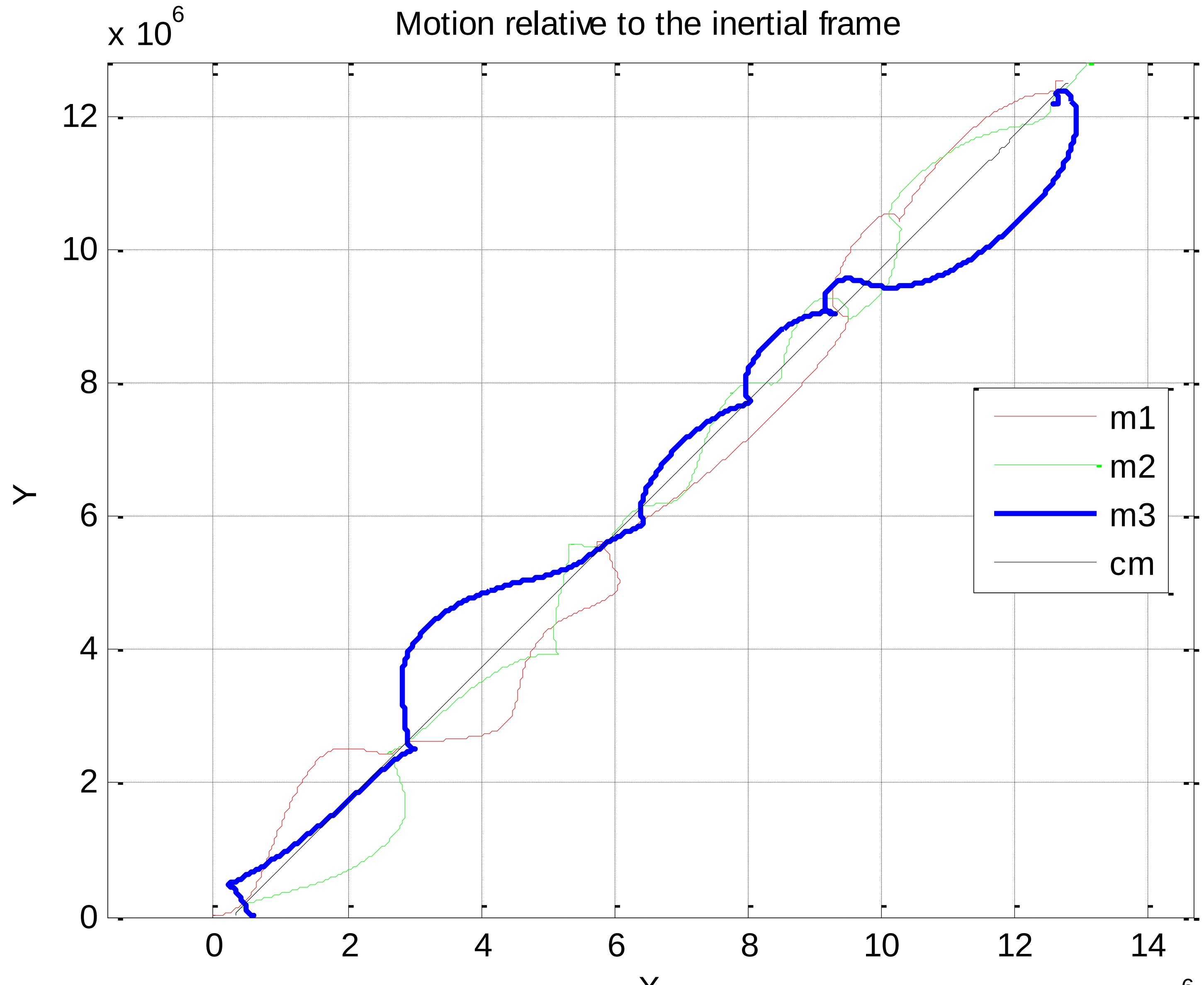
Three-Body Problem: Matlab Example

Three identical masses:

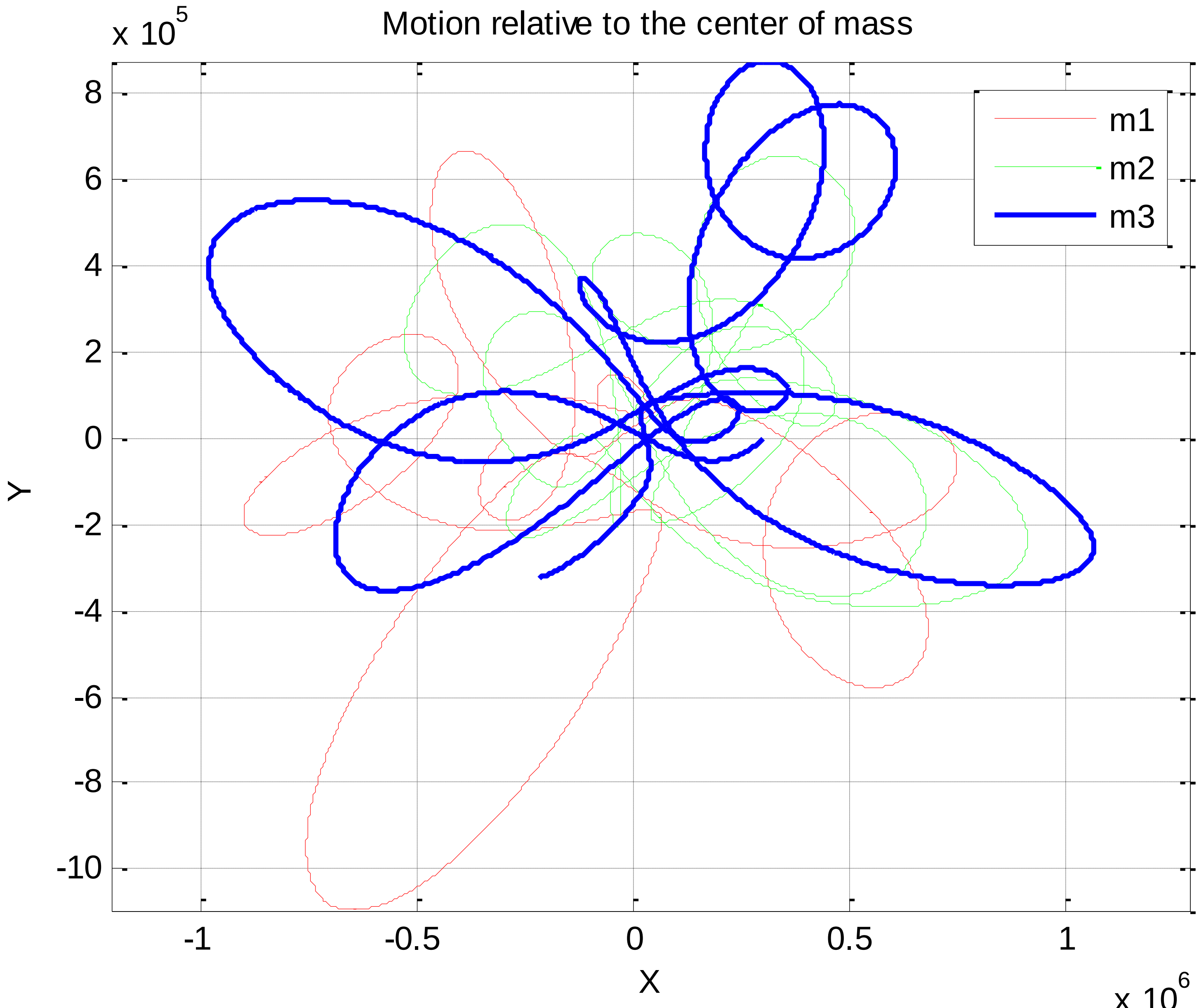
⇒ Two are at rest.

⇒ The third one has a velocity directed upward to the right making a 45 degrees angle with the X axis.

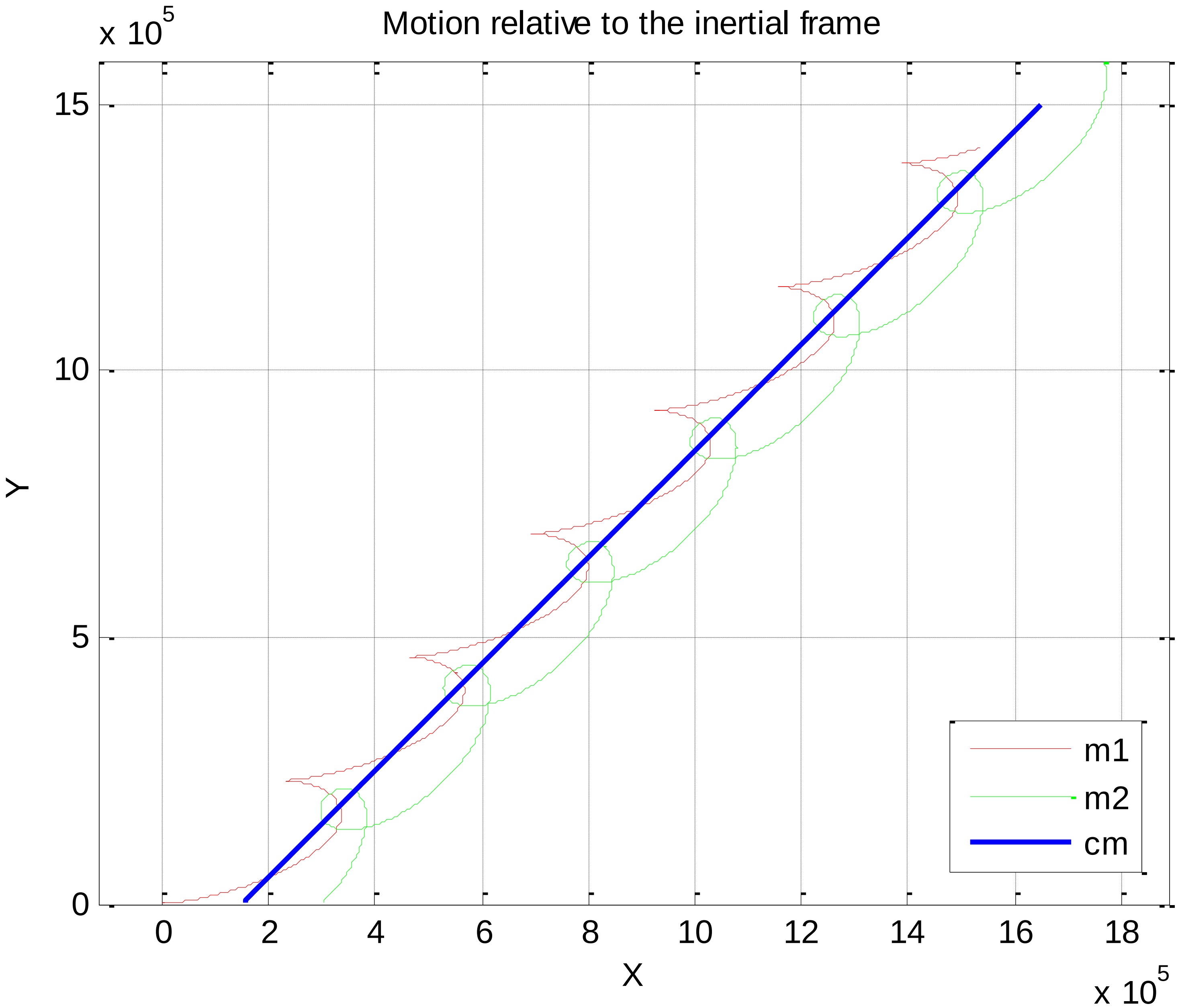




Motion relative to the inertial frame
x 10^6
12
10
8
6
4
2
0
Y
0
2
4
6
8
10
12
14
X
x 10^6
m1
m2
m3
cm

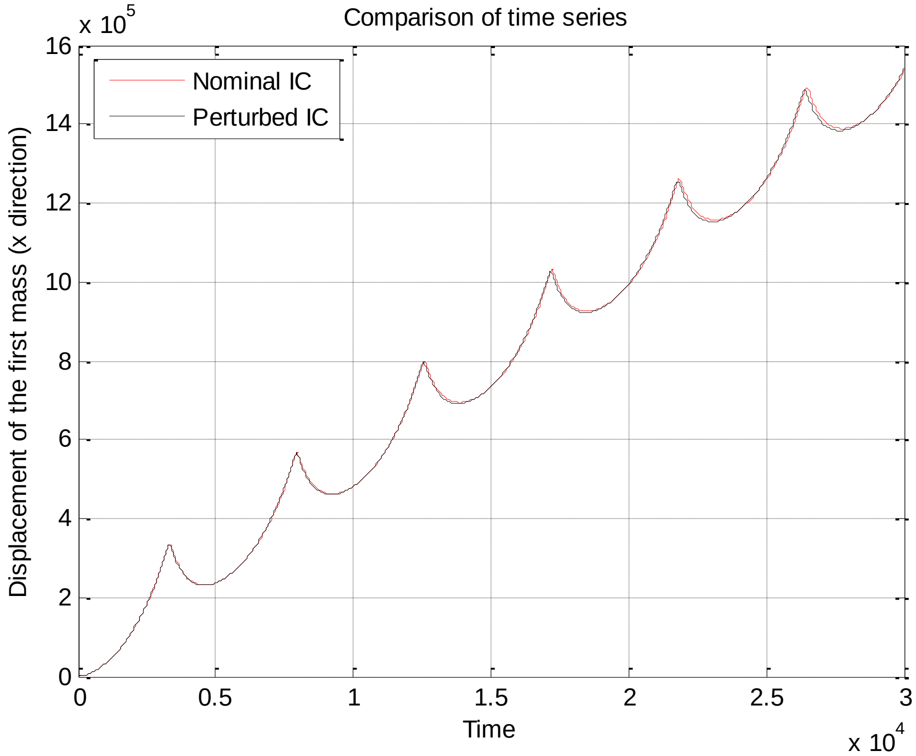


What Do You Conclude ?



Seemingly complex motion in the inertial frame

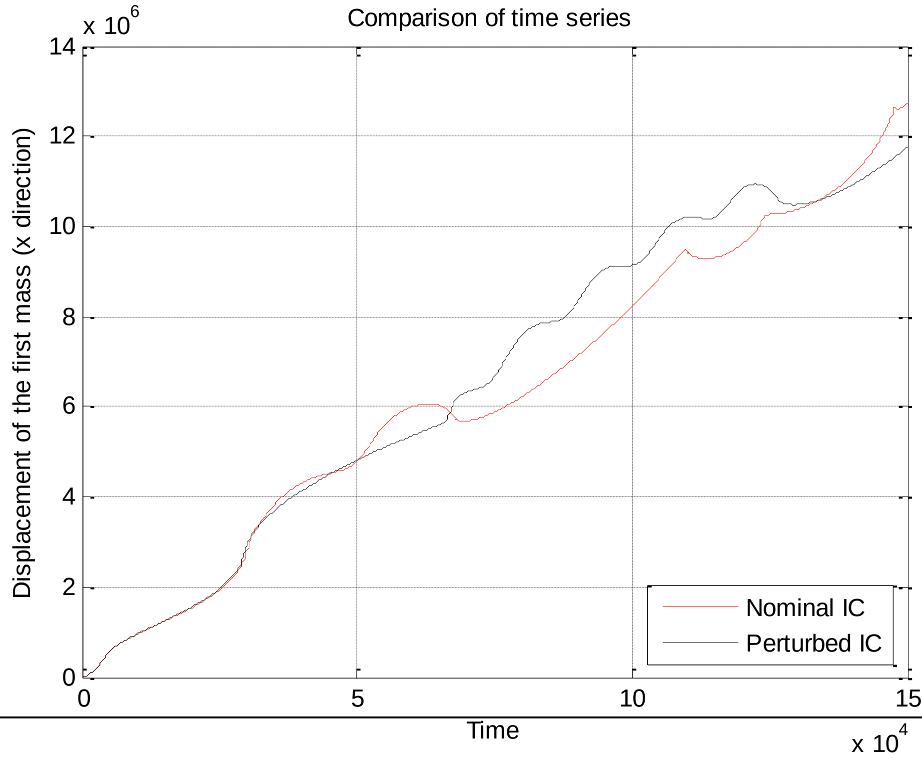
Why Is the 3-Body Problem So Difficult ?



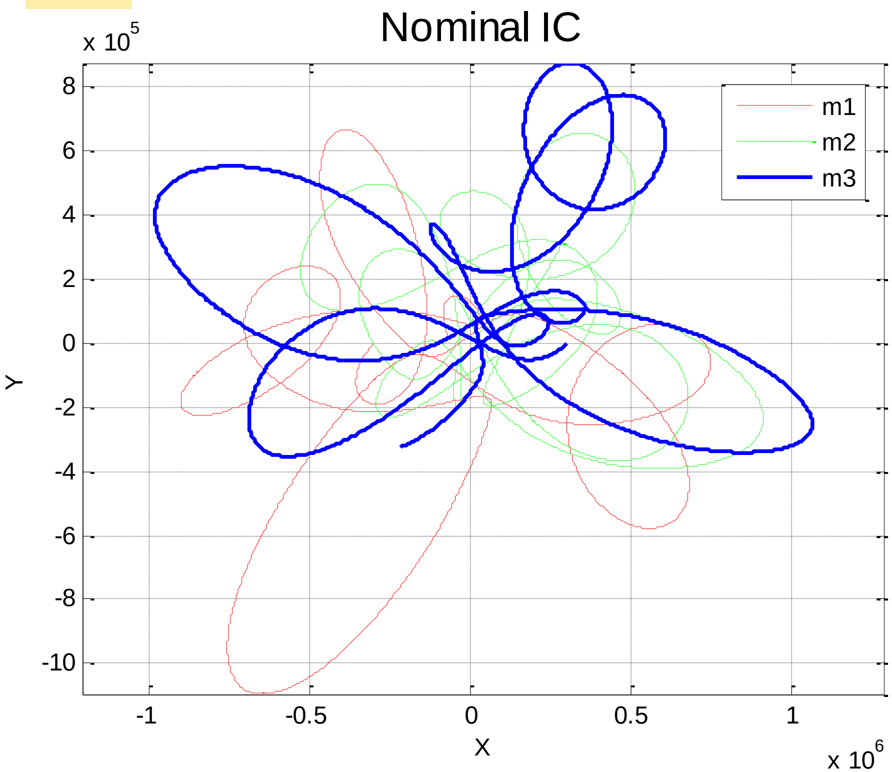
2-body problem
(initial conditions perturbed by 0.1%)

Chaotic by nature !

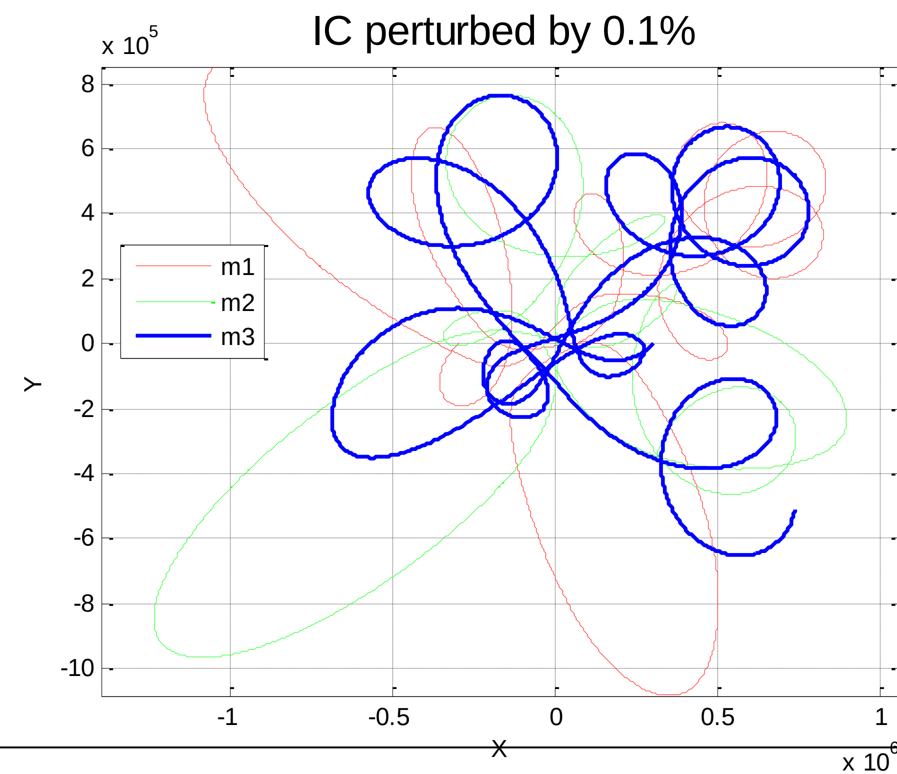
3-body problem
(initial conditions perturbed by 0.1%)



Why Is the 3-Body Problem So Difficult ?



Chaotic by nature !



Interest in the Two-Body Problem ?

Precise orbit propagation:

Elaborate models are necessary to compute the motion of satellites to the high level of accuracy required for many applications today (e.g., the GPS system). The 2-body problem is not helpful in that context.

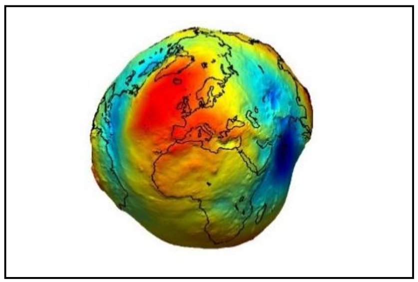
Interest in the Two-Body Problem ?

- *Qualitative understanding:*
  The main features of satellite and planet orbits can be described by a reasonably simple approximation, the two-body problem.

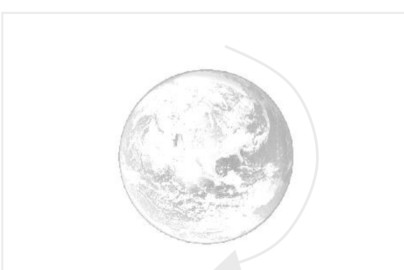
- *Mission design:*
  Some important quantities (ΔV and C_3) can be computed fairly accurately using the two-body assumption.

- *Interplanetary transfer:*
  In lecture 6, we will use a sequence of 2-body problems to approximate a complex interplanetary mission.

2. The Two-Body Problem



$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r}$$



2.2 Gravitational field:

2.2.1 Newton’s law of universal gravitation

2.2.2 The Earth

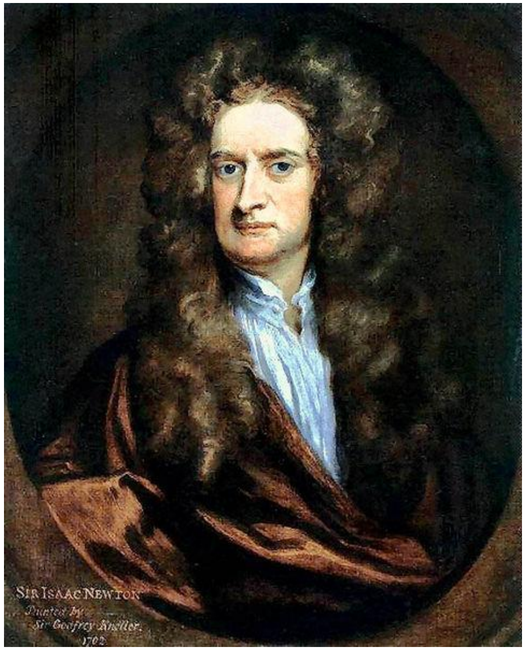
2.2.3 Gravity models and geoid

What is the highest point on Earth ?

Gravitational Force

The law of universal gravitation is an empirical law describing the gravitational attraction between bodies with mass.

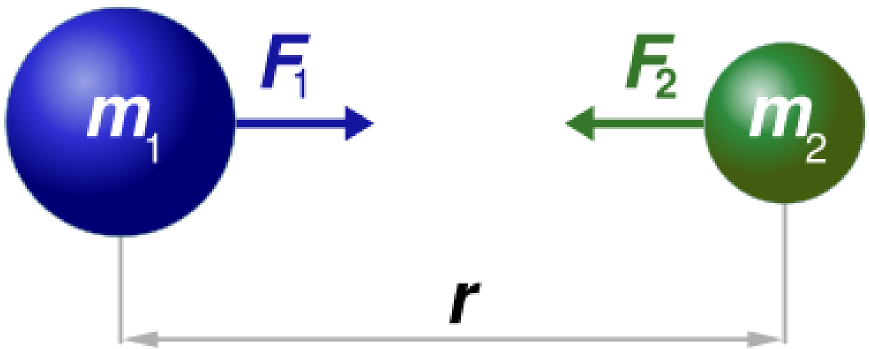
It was first formulated by Newton in *Philosophiae Naturalis Principia Mathematica* (1687). He was able to relate objects falling on the Earth to the motion of the planets.



Isaac Newton (1642-1727)

Gravitational Force

Every point mass attracts every other point mass by a force pointing along the line intersecting both points. The force is proportional to the product of the two masses and inversely proportional to the square of the distance between the point masses:



$$F_1 = F_2 = G\frac{m_1 \times m_2}{r^2}$$

In Vector Form

$$\mathbf{F}_{12} = -G \frac{m_1 m_2}{\left|\mathbf{r}_{12}\right|^2} \hat{\mathbf{r}}_{12}$$

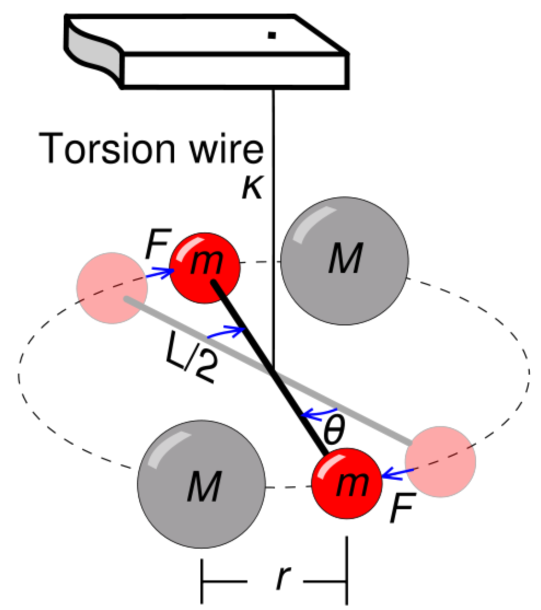
with

$$\hat{\mathbf{r}}_{12} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{\left|\mathbf{r}_2 - \mathbf{r}_{11}\right|}$$

$$\left|\mathbf{r}_{12}\right| = \left|\mathbf{r}_2 - \mathbf{r}_1\right|$$

Gravitational Constant

By measuring the mutual attraction of two bodies of known mass, the gravitational constant *G* can directly be determined from torsion balance experiments.



$$k\theta = L\frac{GmM}{r^2}$$

Due to the small size of the gravitational force, *G* is presently only known with limited accuracy and was first determined many years after Newton's discovery:

$$(6.67428 \pm 0.00067) \times 10^{-11}\ \mathrm{m^3.kg^{-1}.s^{-2}}$$

(http://www.physics.nist.gov/cgi-bin/cuu/Value?bg)

Gravitational Parameter of a Celestial Body

$$\mu = GM_{\oplus}$$

The gravitational parameter of the Earth has been determined with considerable precision from the analysis of laser distance measurements of artificial satellites:

$$398600.4418 \pm 0.0008 \ \text{km}^3.\text{s}^{-2}.$$

The uncertainty is 1 to 5e8, much smaller than the uncertainties in *G* and *M* separately (~1 to 1e4 each).

Satellite Laser Ranging

TIGO (Concepcion, Chile)

LAGEOS-1

Lasers measure ranges from ground stations to satellite borne retro-reflectors. Because the events of sending and receiving a pulse can be registered within a few picoseconds, the distance between the ground station and the satellite is determined within a few millimeters.

Acceleration of Gravity

$$g = \frac{GM}{r^2}$$

$$g_{earth,SL} = 9.807 \text{ m/s}^2$$

$$g_{earth,aircraft} = g_{earth,SL} - 0.3\,\%$$

$$g_{earth,ISS} = g_{earth,SL} - 10\ \%$$

We sense our own weight by feeling contact forces acting on us in opposition to the force of gravity: W=mg.

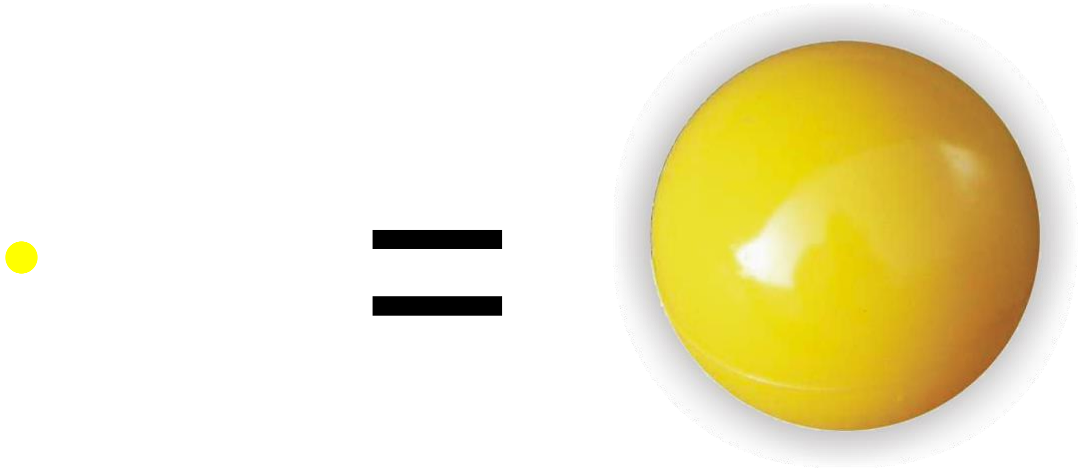
If planetary gravity is the only force acting on a body, then the body is said to be in free fall. There are, by definition, no contact forces, so there can be no sense of weight.

A person in free fall experiences weightlessness: gravity is still there, but he cannot feel it.

Bodies with Spatial Extent

Up to now, point masses were considered.

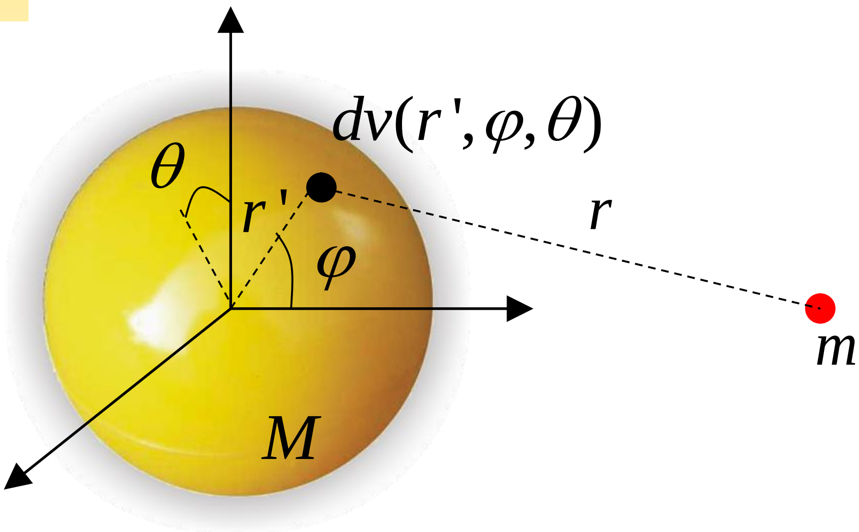
But an object with a spherically-symmetric distribution of mass exerts the same gravitational attraction on external bodies as if all the object's mass were concentrated at a point at its centre.

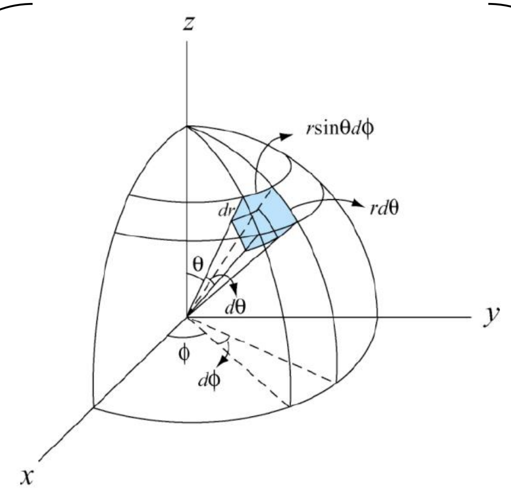


Point mass M

Sphere of mass M

Spherically Symmetric Mass Distribution





$$dV = r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$M = \iiint_v \rho dv$$

$$V = -Gm \iiint_v \frac{\rho dv}{r}$$

$$dv = r'^2 \sin\varphi \, \mathrm{d}\varphi \, \mathrm{d}\theta \, \mathrm{d}r'$$

$$r = \sqrt{R^2 + r'^2 - 2r'R\cos\varphi}$$

$$\frac{dr}{d\varphi} = \frac{r'R\sin\varphi}{r}$$

Spherically Symmetric Mass Distribution

$$M=\left(\int_0^{2\pi}\mathrm{d}\theta\right)\left(\int_0^{\pi}\sin\varphi\,\mathrm{d}\varphi\right)\left(\int_0^{R_0}\rho r'^2\,\mathrm{d}r'\right)=4\pi\left(\int_0^{R_0}\rho r'^2\,\mathrm{d}r'\right)$$

$$V=-2\pi Gm\left(\int_0^{R_0}\left(\int_0^{\pi}\frac{\sin\varphi\,\mathrm{d}\varphi}{r}\right)\rho r'^2\,\mathrm{d}r'\right)$$

$$=-2\pi Gm\left(\int_0^{R_0}\left(\frac{1}{r'R}\int_{R-r'}^{R+r'}\mathrm{d}r\right)\rho r'^2\,\mathrm{d}r'\right)$$

$$=-\frac{4\pi Gm}{R}\left(\int_0^{R_0}\rho r'^2\,\mathrm{d}r'\right)=-\frac{GMm}{R}$$

OK !

What is the Highest Point on Earth ?

Mount Chimborazo (6310 m), located in Ecuador, may be considered as the highest point on Earth. It is the spot on the surface farthest from the Earth’s center.

6384.4 km (Chimborazo) vs. 6382.3 km (Everest)

The Earth is not a Sphere…

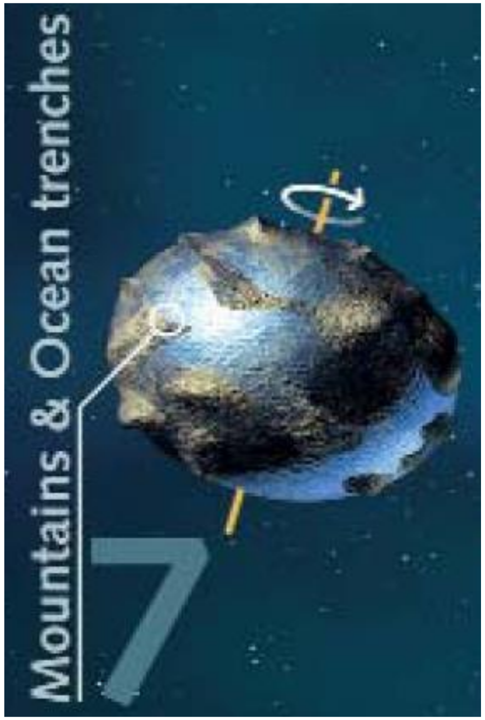
1st Order Effect: Equatorial Bulge

Because our planet rotates, the centrifugal force tends to pull material outwards around the Equator where the velocity of rotation is at its highest:

⇒ The Earth's radius is 21km greater at the Equator compared to the poles.

⇒ The force of gravity is weaker at the Equator (g=9.78 m/s^2) than it is at the poles (g=9.83 m/s^2).

2^{nd} Order Effect: Mountains and Oceans



Rather than being smooth, the surface of the Earth is relatively “lumpy”:

⇒ There is about a 20 km difference in height between the highest mountain and the deepest part of the ocean floor.

3^{rd} Order Effect: Internal Mass Distribution

The different materials that make up the layers of the Earth's crust and mantle are far from homogeneously distributed:

⇒ For instance, the crust beneath the oceans is a lot thinner and denser than the continental crust.

The Idealized Geometrical Figure of the Earth

Because of its relative simplicity, a flattened ellipsoid, called the **reference ellipsoid**, is typically used as the idealized Earth:

⇒ Ellipsoid of revolution.

⇒ The size is represented by the radius at the equator, a.

⇒ The shape of the ellipsoid is given by the flattening, f, which indicates how much the ellipsoid departs from spherical. $f=(a-b)/a$, where b is the polar radius.

Most Common Reference Ellipsoid

WGS84 (World Geodetic System 1984, revised in 2004):

⇒ Origin at the center of mass of Earth.

⇒ a=6378.137 km, b=6356.752 km, f=0.335 %.

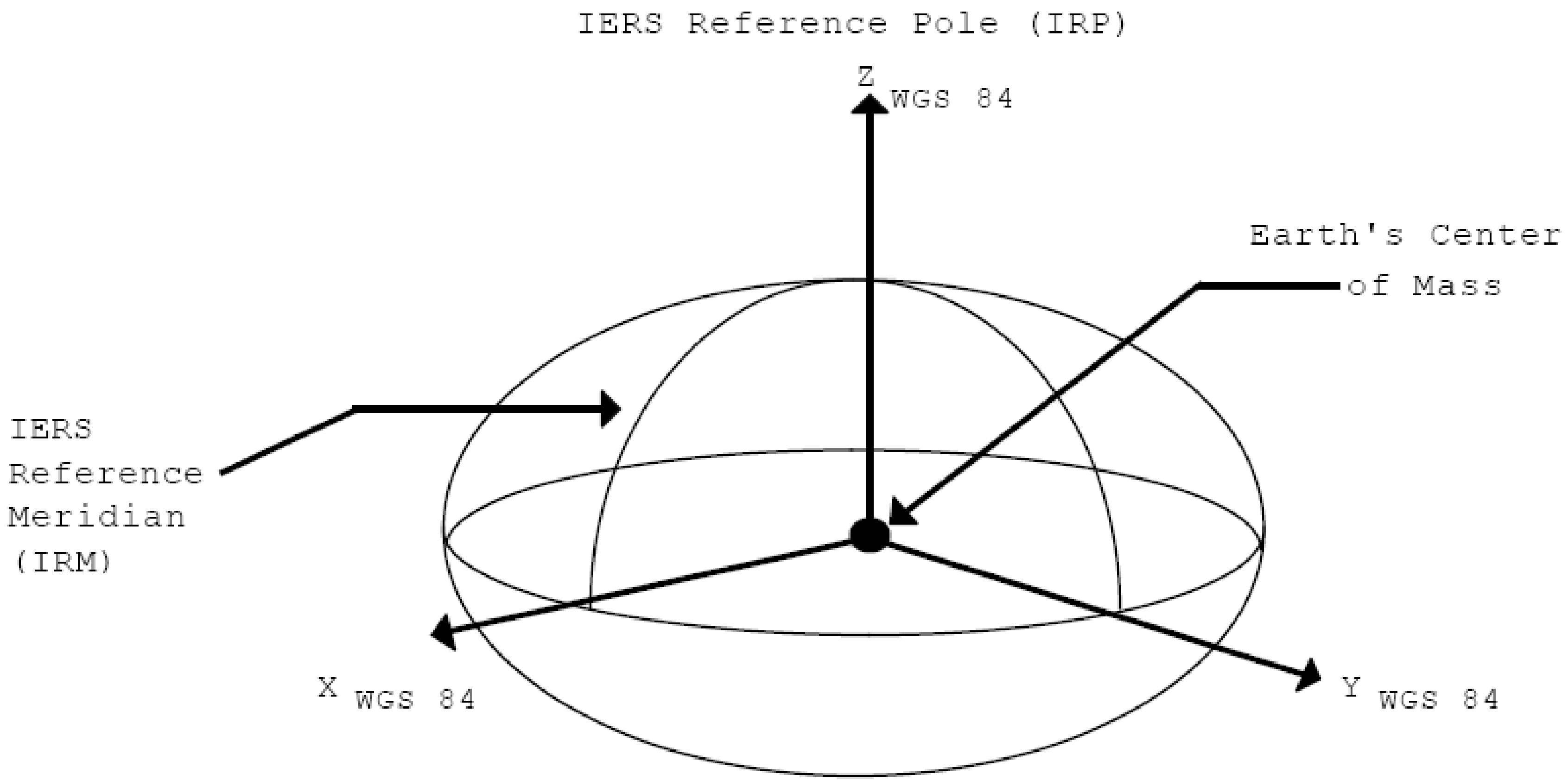
⇒ Reference system used by the GPS.

⇒ Official document on the course web site (interesting to read !).

Parameter	Notation	Value
Semi-major Axis	a	6378137.0 meters
Reciprocal of Flattening	1/f	298.257223563
Angular Velocity of the Earth	ω	$7292115.0 \times 10^{-11}$ rad/s
Earth's Gravitational Constant (Mass of Earth's Atmosphere Included)	GM	$3986004.418 \times 10^{8} m^3/s^2$

WGS84 four defining parameters

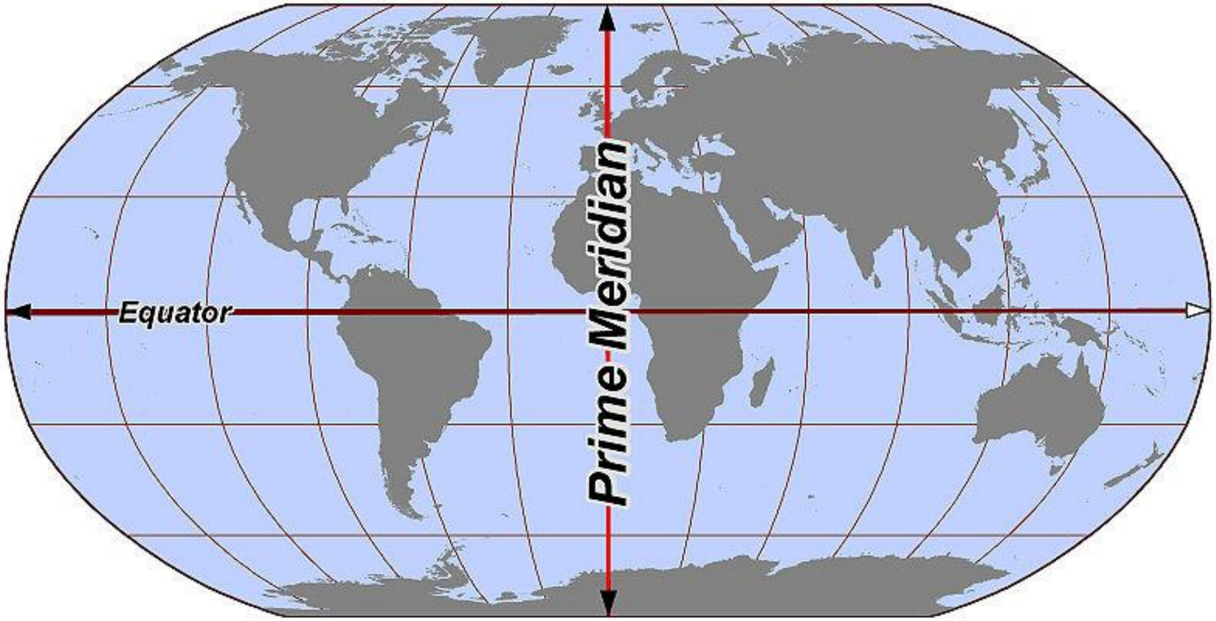
WGS84 Coordinate System



Longitude

Point coordinates such as latitude, longitude and elevation are defined from the reference ellipsoid.

The meridian of zero longitude is the IERS Reference Meridian, which lies 5.31'' east of the Greenwich Meridian.

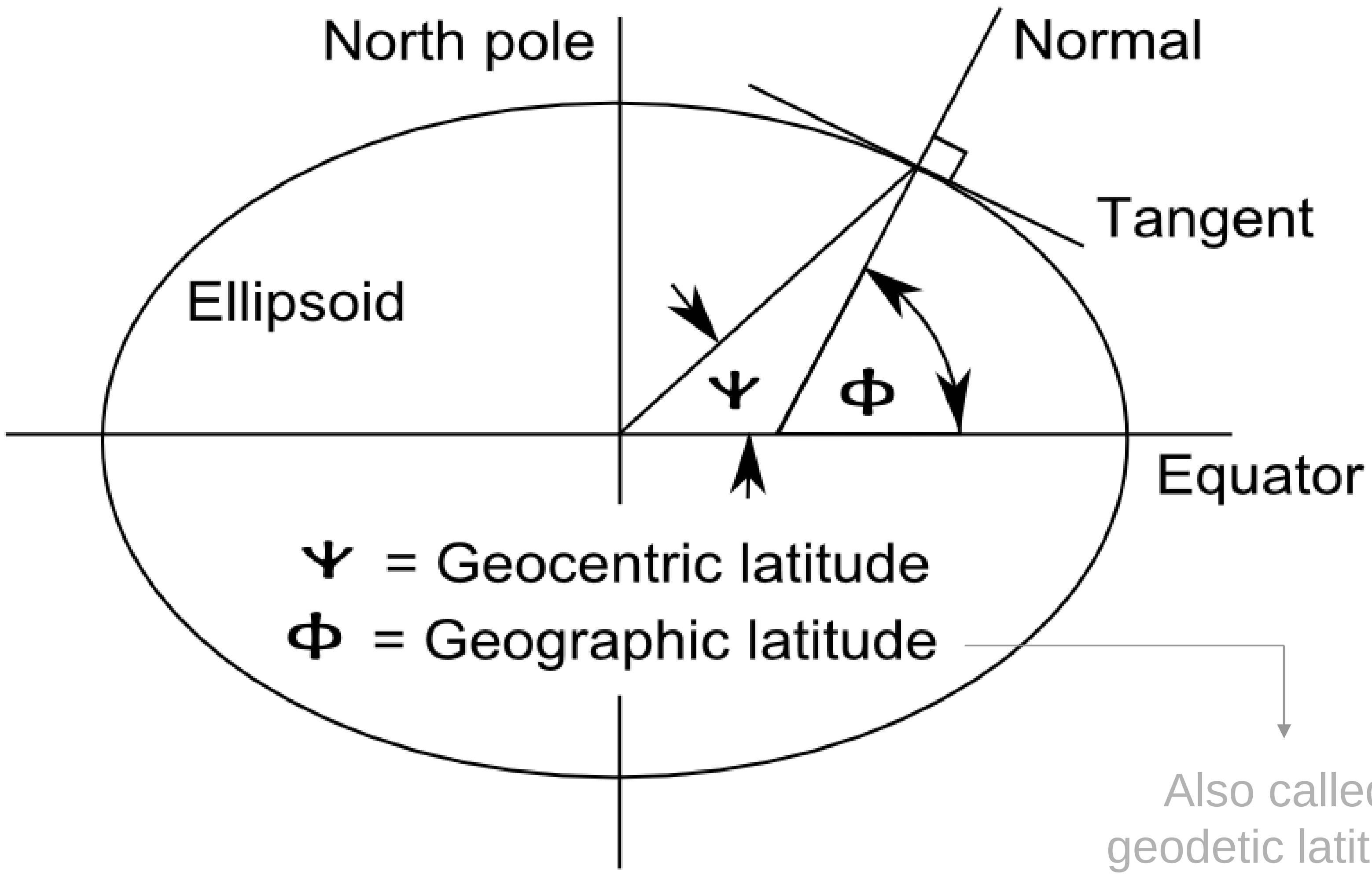


GPS Receiver at the Greenwich Meridian

5.31/3600=0.0015

OK !

Latitude



Also called
geodetic latitude

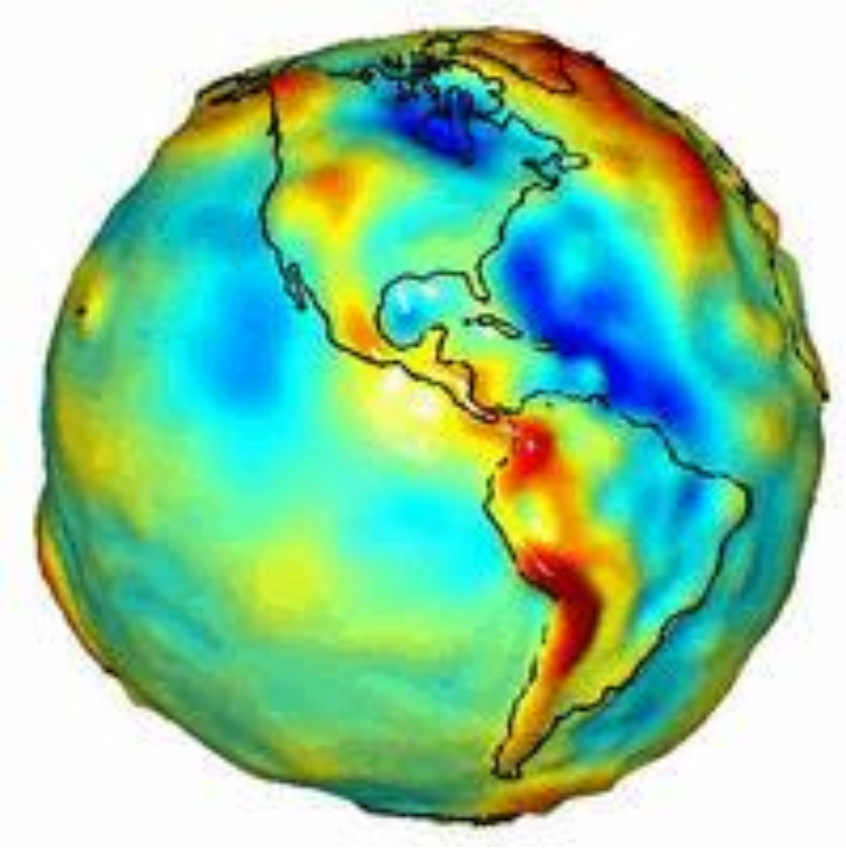
The True Figure of the Earth

The **geoid** is that equipotential surface which would coincide exactly with the mean ocean surface of the Earth, if the oceans were in equilibrium, at rest, and extended through the continents:

$\Rightarrow$ It is by definition a surface to which the force of gravity is everywhere perpendicular.

$\Rightarrow$ It is an irregular surface but considerably smoother than Earth's physical surface. While the latter has excursions of almost 20 km, the total variation in the geoid is less than 200 m.

The True Figure of the Earth



Gravitational Modeling

Spherical harmonics are used to model the Earth gravitational model:

$$V = \frac{GM}{r}\left[1 + \sum_{n=2}^{n_{max}} \sum_{m=0}^{n} \left(\frac{a}{r}\right)^{n} \overline{P}_{nm}(\sin\phi')\left(\overline{C}_{nm}\cos m\lambda + \overline{S}_{nm}\sin m\lambda\right)\right]$$

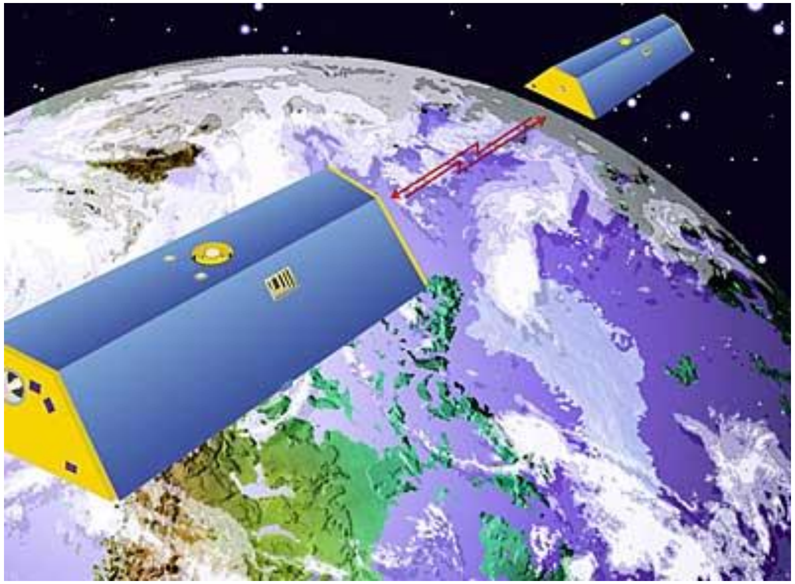
Gravitational potential function

⇒ The current set is **EGM2008** (Earth Gravity Model 2008). The model comprises 4.6 million terms in the spherical expansion (order and degree 2159).

⇒ Geoid with a resolution approaching 10 km (5'x5').

⇒ More details in Chapter 4 (Non-Keplerian motion).

EGM2008 Made Use of Grace Satellites



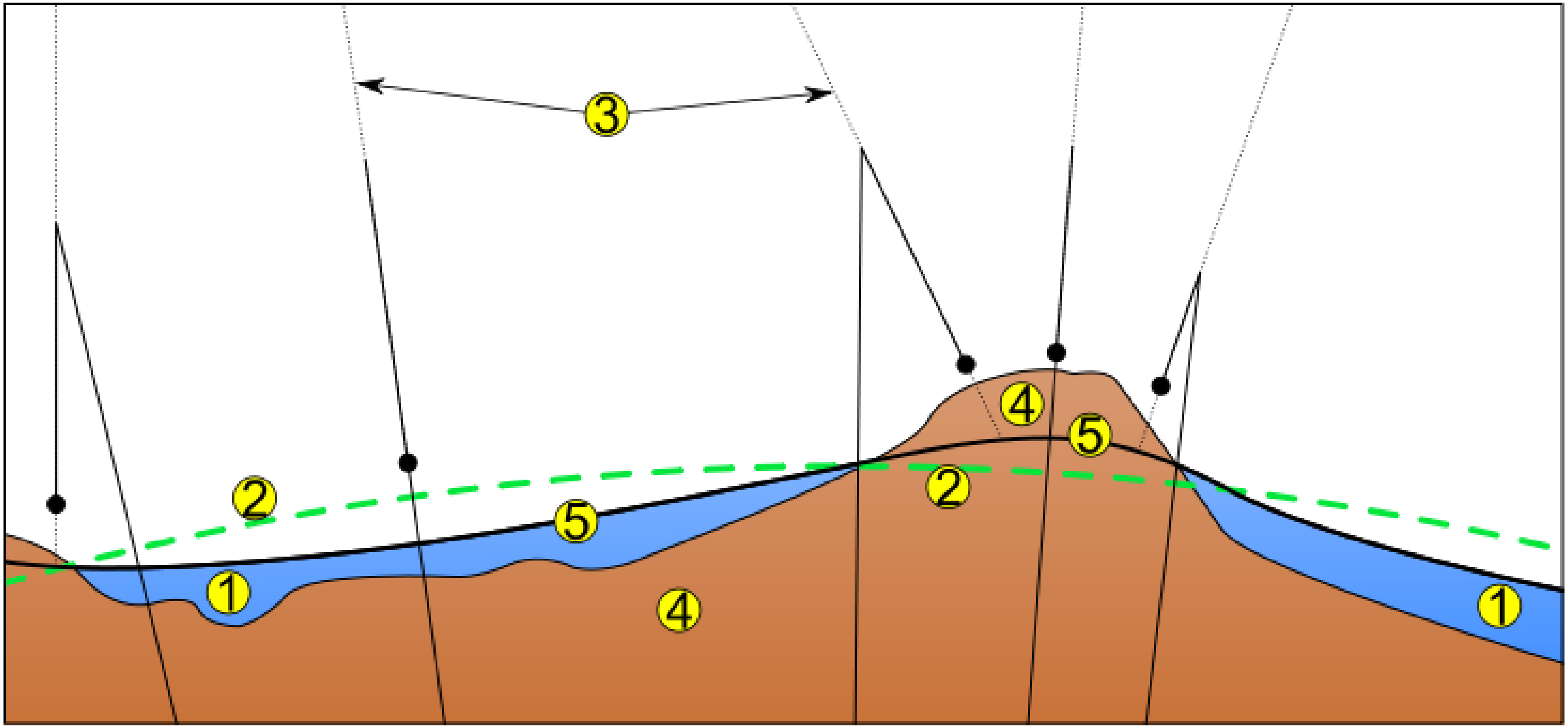
GRACE employs microwave ranging system to measure changes in the distance between two identical satellites as they circle Earth. The ranging system detects changes as small as 10 microns over a distance of 220 km.

Geoid Definition

EGM2008 contains no explicit information about which level surface, out of the infinitely many that may be generated from the potential coefficients, is "the" geoid.

EGM2008 model is therefore used to compute geoid undulations with respect to WGS84 ellipsoid. The result is referred to as **WGS84-EGM08 geoid**.

Geoid calculator for EGM96:
http://earth-info.nga.mil/GandG/wgs84/gravitymod/egm96/intpt.html



1: ocean

2: reference ellipsoid

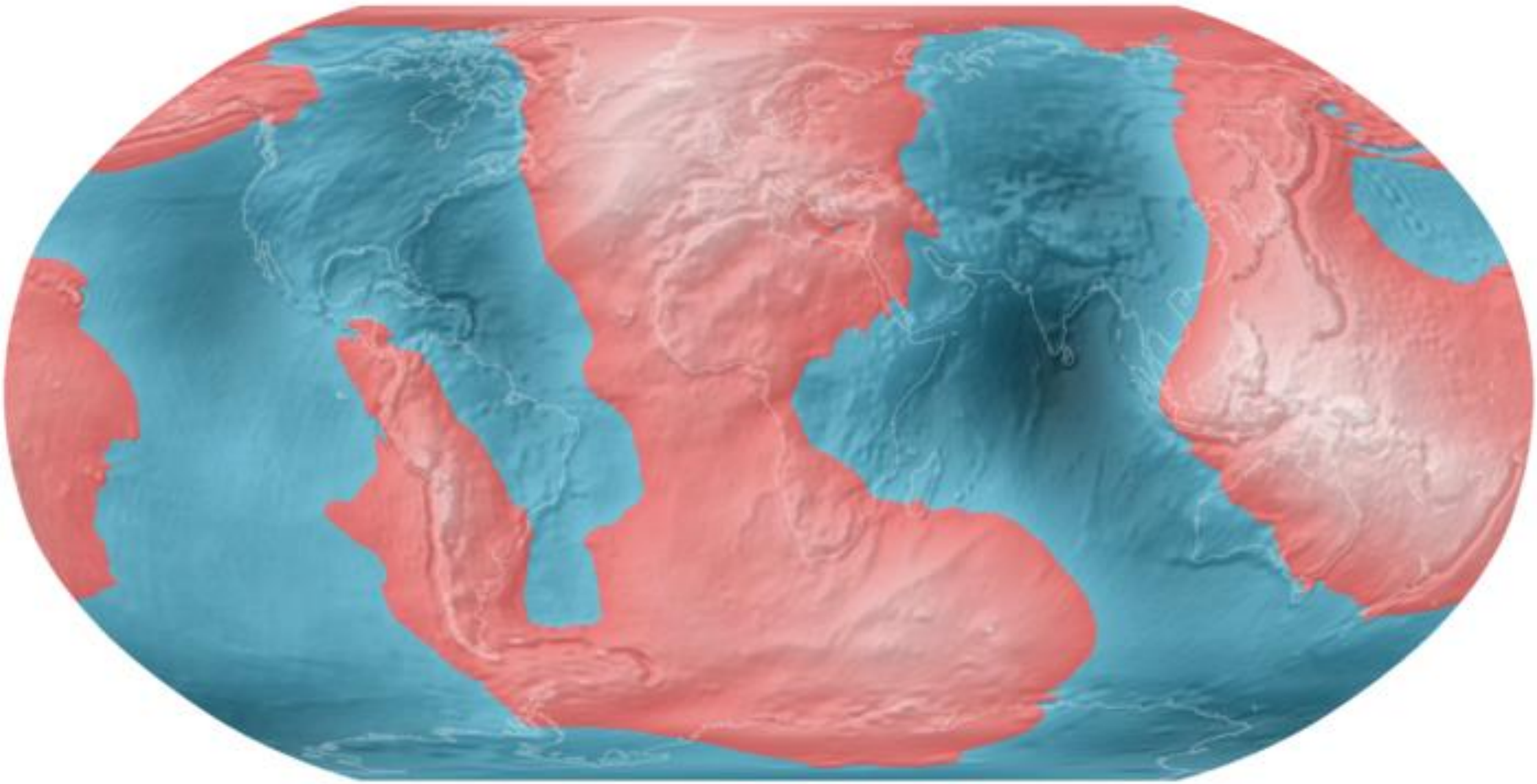
3: local plumb

4: continent

5: geoid

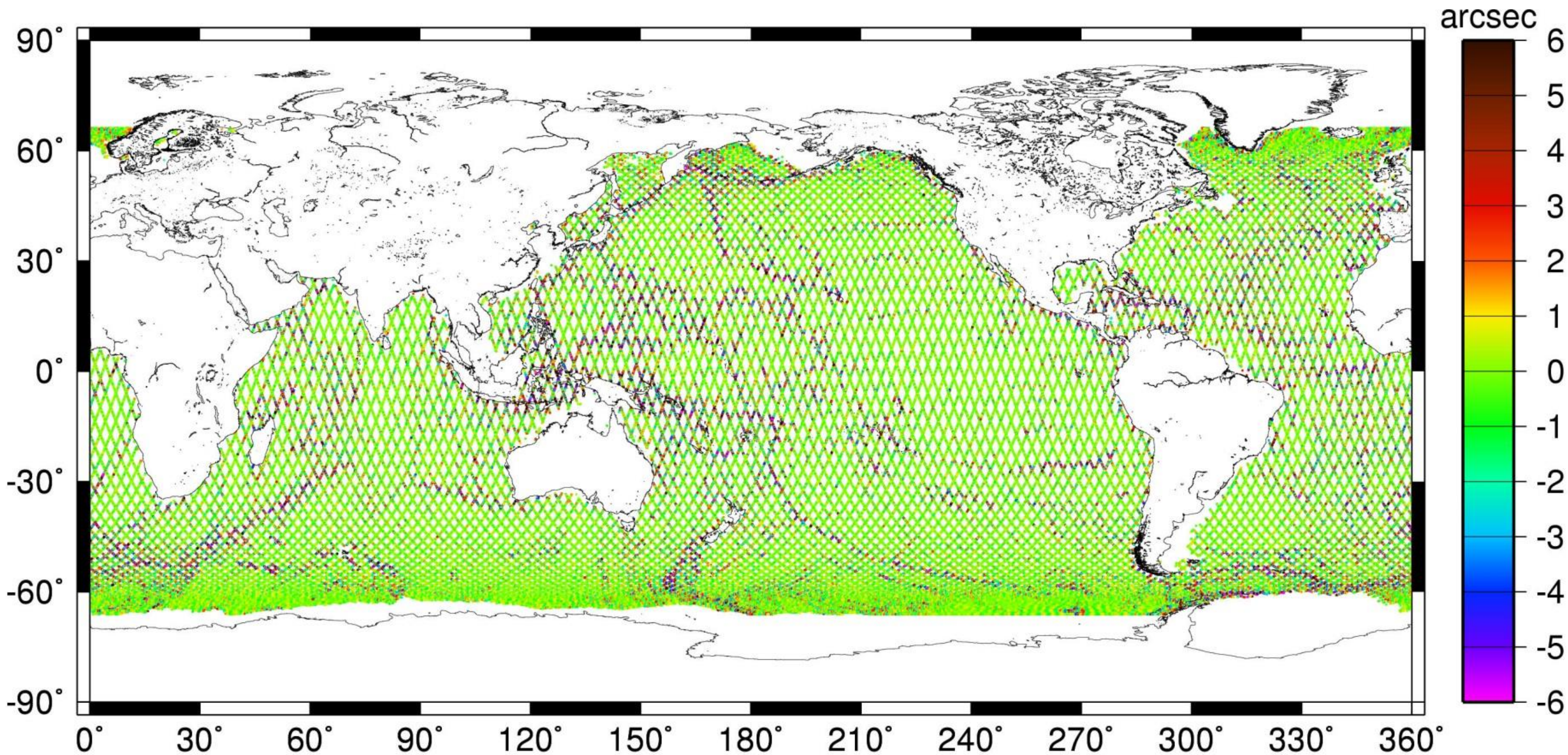
Deviation of the Geoid from the idealized figure of the Earth

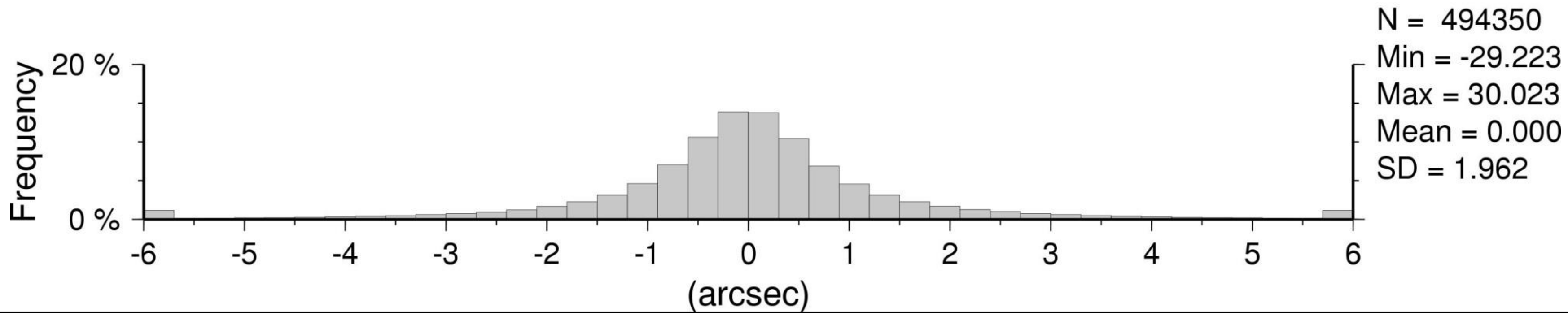
(difference between the EGM96 geoid and the WGS84 reference ellipsoid)



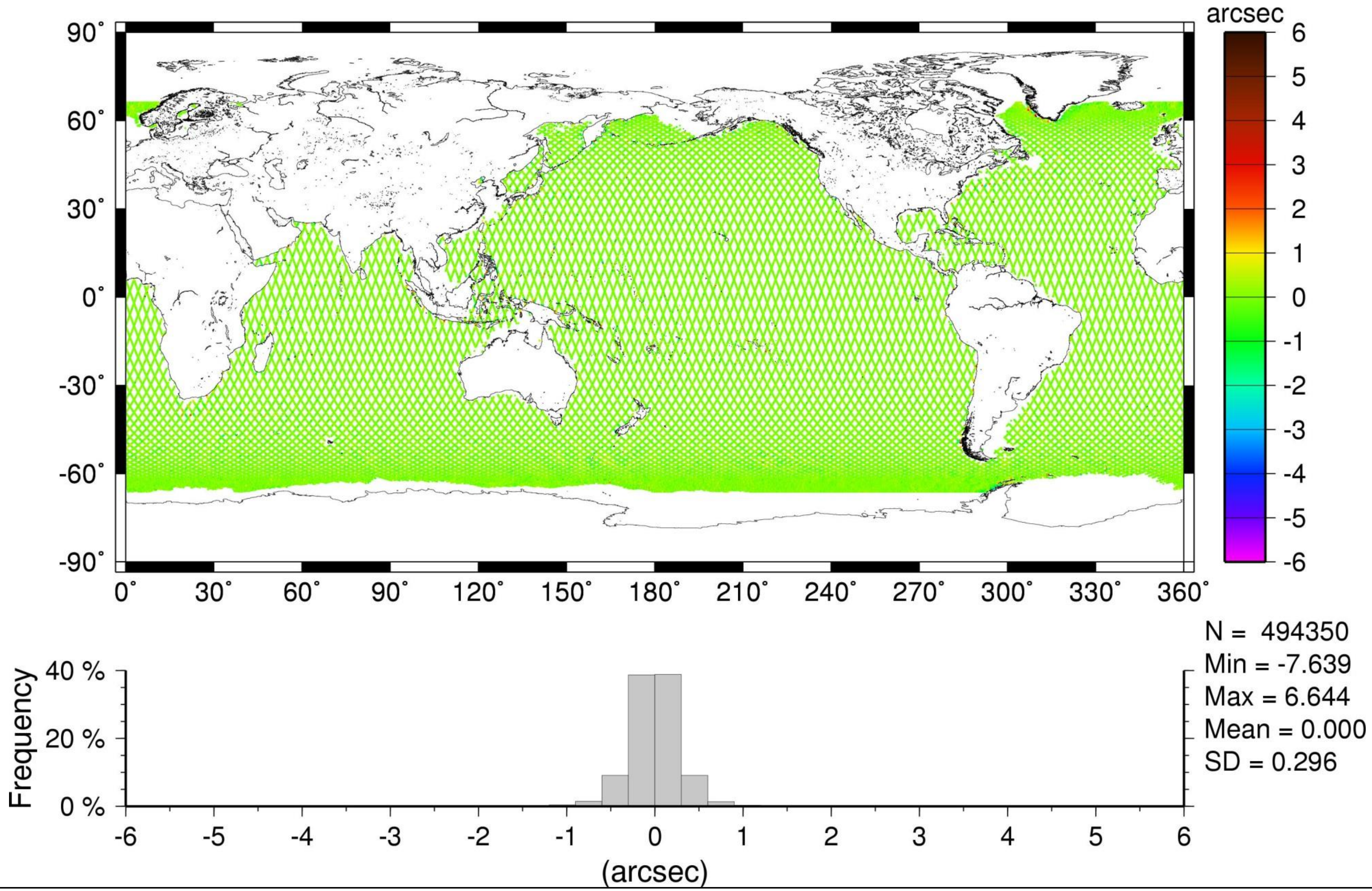
Red areas are above the idealized ellipsoid; blue areas are below.

Residual Sea Surface Slopes (EGM-96)

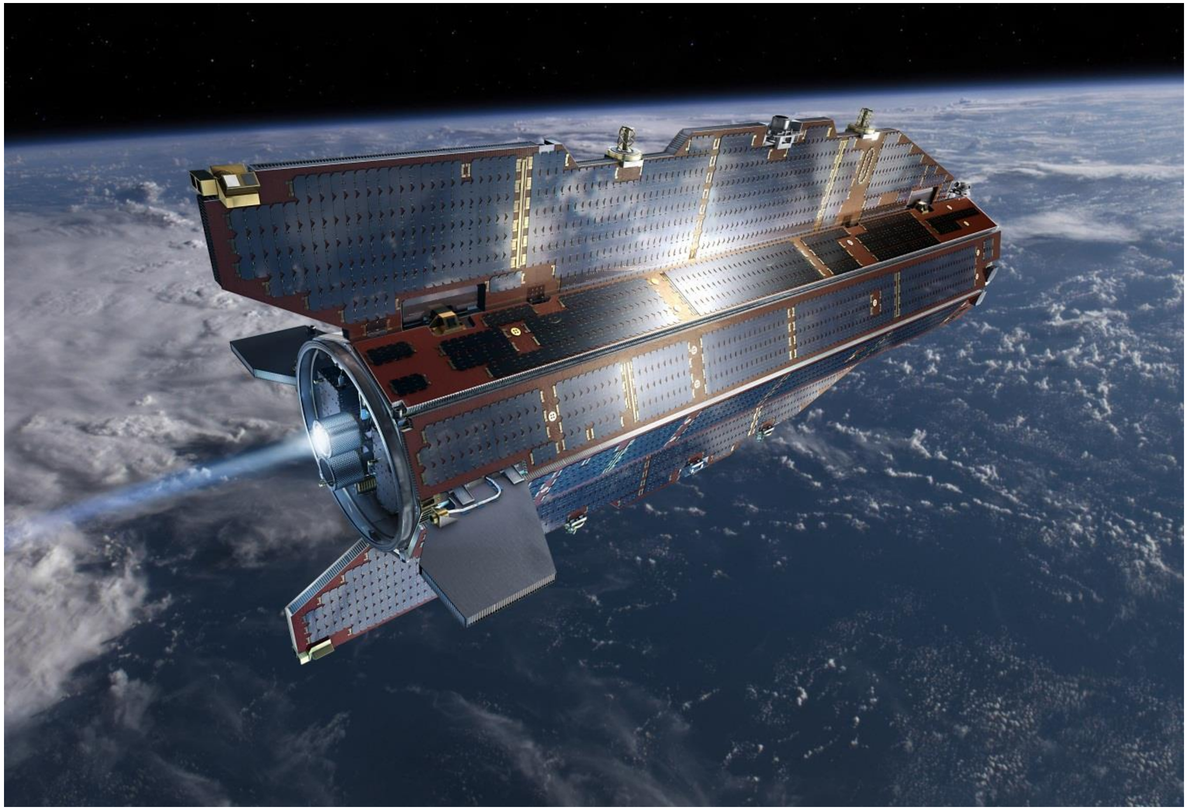




Residual Sea Surface Slopes (EGM-2008)



Future Improvements: GOCE, 2009



Parameter	Requirement		
	Accuracy	Resolution (km)	Spherical Harmonic Degree
Geoid (m)	0.01–0.02	100	200
Gravity Anomalies (mGal)	1.0	100	200

(EGM96: ~0.5 m)

(1mGal = 10^{-5} m/s^2)

Why So Many Efforts ???

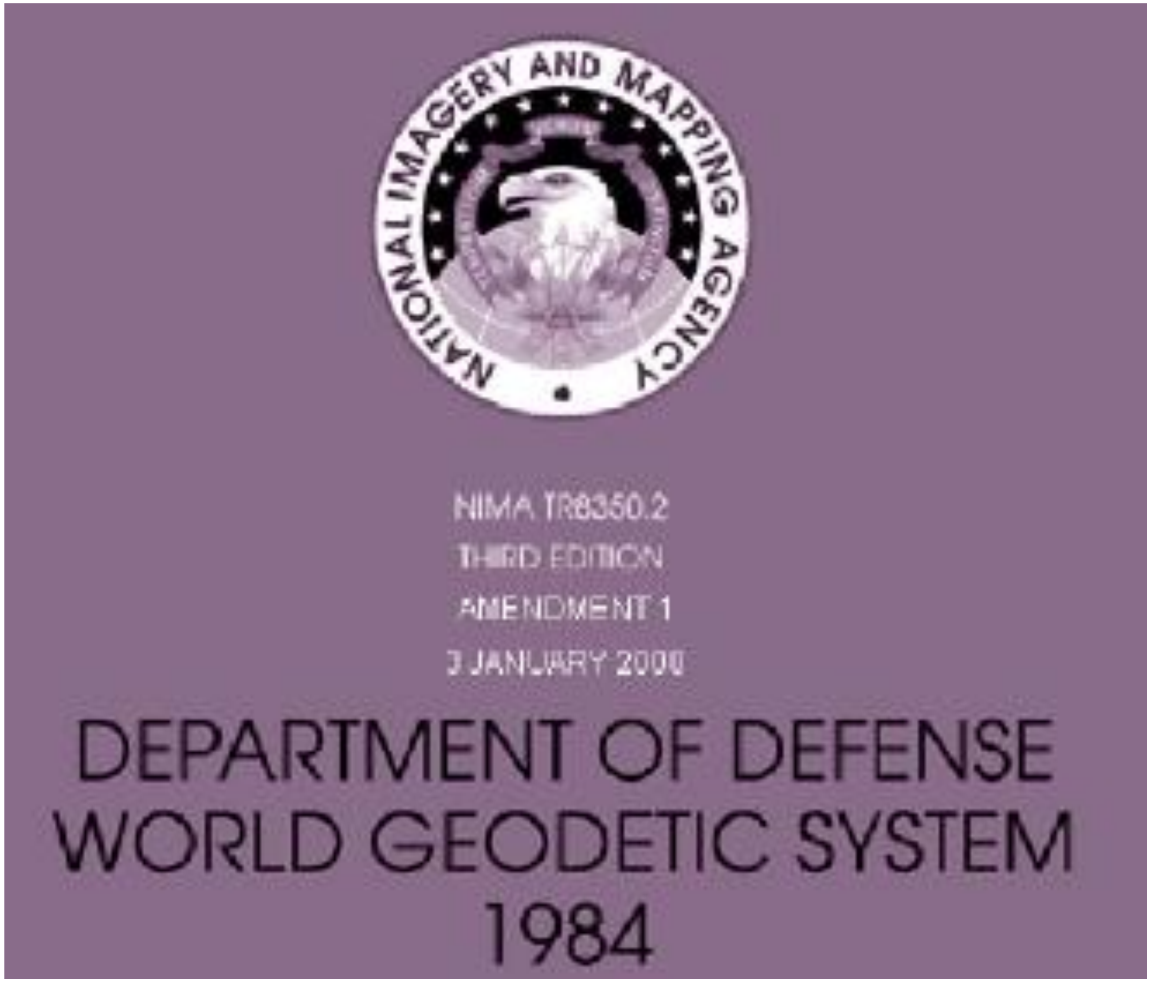
1. GPS and an advanced map of the geoid can replace time-consuming leveling procedures.

2. Physics of the Earth's interior (gravity is directly linked to the distribution of mass).

3. Understanding of ocean circulation, which plays a key role in energy exchanges around the globe.

4. Computation of the motion of satellites to the level of accuracy required today.

Further Reading on the Course Web Site



Digression: General Relativity

Einstein's theory is the current description of gravity in modern physics.

This course will not cover the theory of general relativity, but Newton's law is still an excellent approximation of the effects of gravity if:

$$\frac{\Phi}{c^2} = \frac{GM}{rc^2} <<< 1, \text{ and } \left(\frac{v}{c}\right)^2 <<< 1$$

General Relativity: Earth-Sun Example

$$\frac{\Phi}{c^2} = \frac{GM_{sun}}{r_{orbit}c^2} \sim 10^{-8}, \text{ and } \left(\frac{v}{c}\right)^2 = \left(\frac{2\pi r_{orbit}}{1 \text{ year}.c}\right)^2 \sim 10^{-8}$$

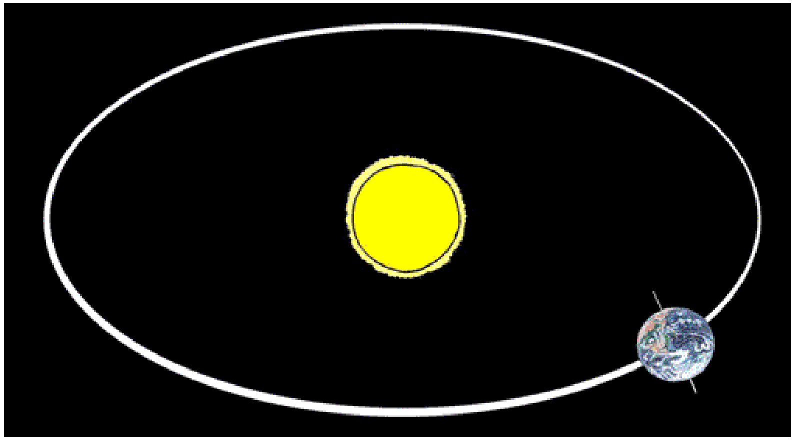
OK !

$G=6.67428 \times 10^{-11}$ m^3.kg^{-1}.s^{-2}

$r_{orbit}=1.5 \times 10^{11}$ m (1 AU)

$M_{sun}=1.9891 \times 10^{30}$ kg

$c=3e8$ m.s^{-1}



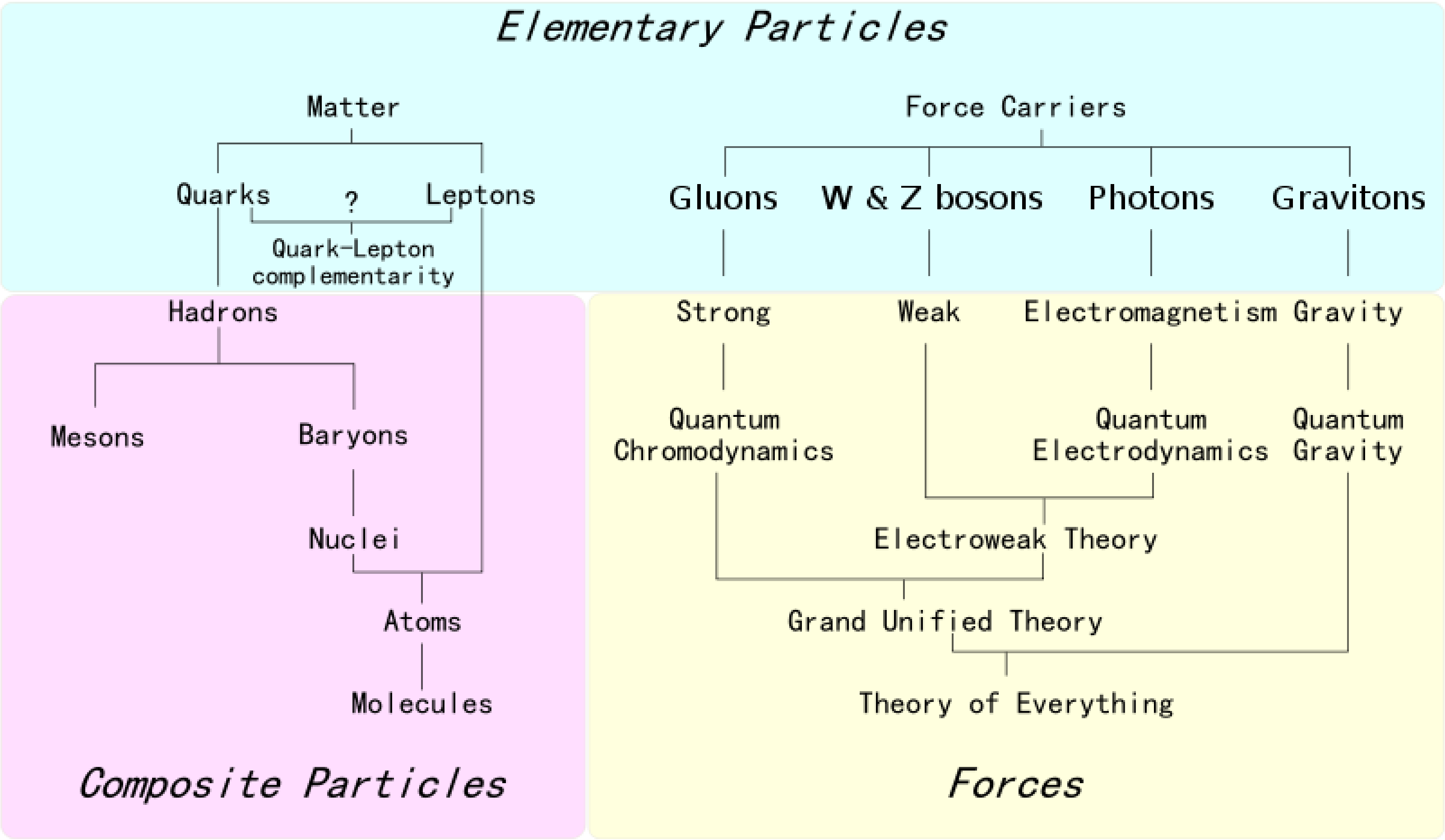
The Quest of a Unifying Theory

What is the relationship between the gravitational force and other known fundamental forces ?

That one body may act upon another at a distance through a vacuum without the mediation of anything else, by and through which their action and force may be conveyed from one another, is to me so great an absurdity that, I believe, no man who has in philosophic matters a competent faculty of thinking could ever fall into it. (Newton, 1692)

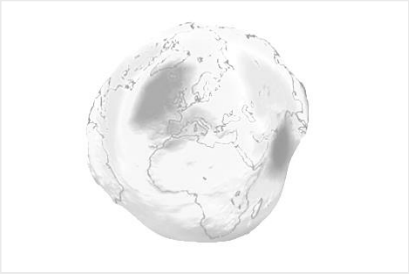
The question is not yet fully resolved today !

The Quest of a Unifying Theory

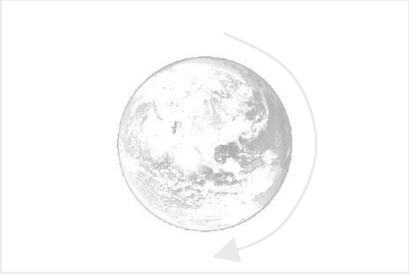


[End of digression]

2. The Two-Body Problem



$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r}$$



2.3 Relative motion:

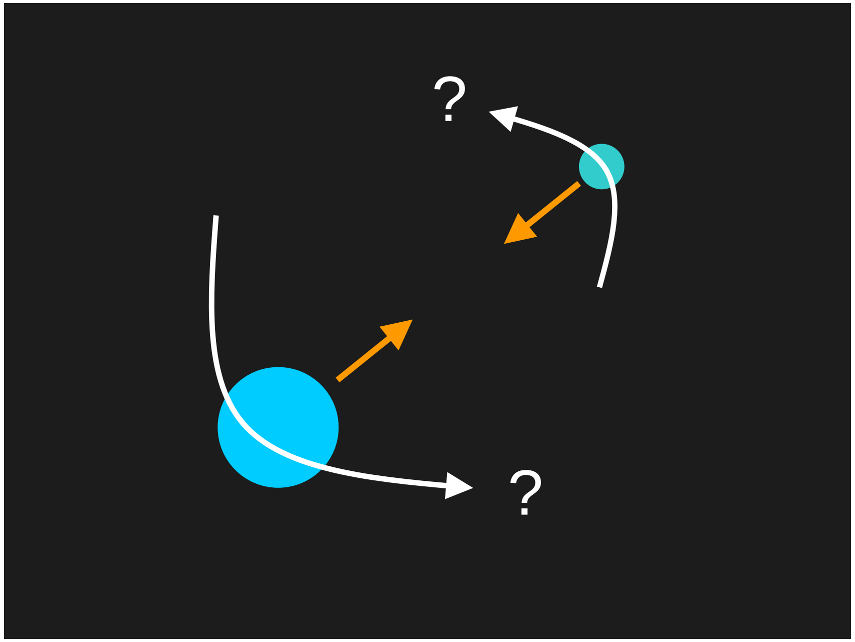
2.3.1 Equations of motion

2.3.2 Closed-form solution

Definition of the 2-Body Problem

Motion of two bodies due solely to their own mutual gravitational attraction. Also known as **Kepler problem**.

Assumption: two point masses (or equivalently spherically symmetric objects).



Motion of the Center of Mass

$$m_1\ddot{\mathbf{R}}_1 = \frac{Gm_1m_2}{r^2}\hat{\mathbf{u}}_r$$

$$+$$

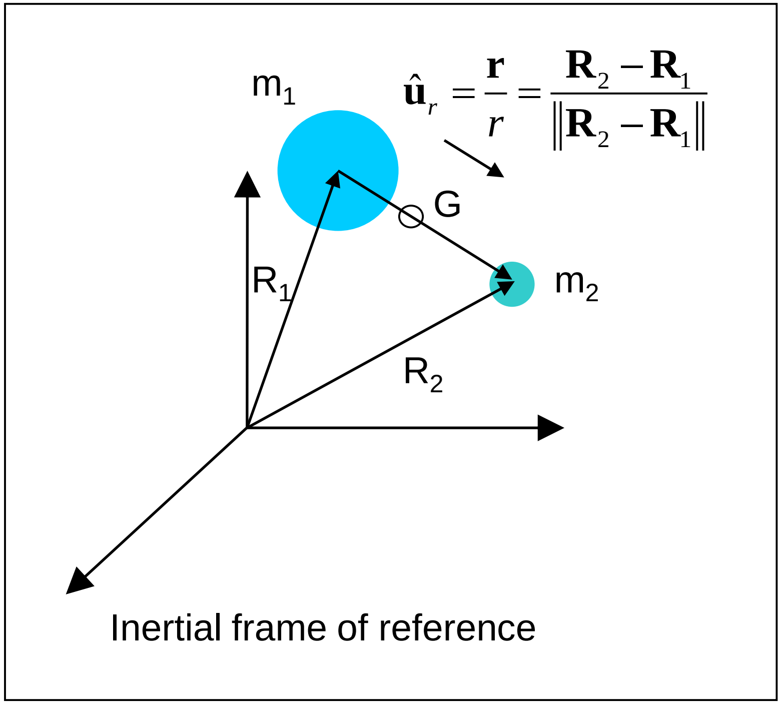
$$m_2\ddot{\mathbf{R}}_2 = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{u}}_r$$

$$m_1\ddot{\mathbf{R}}_1 + m_2\ddot{\mathbf{R}}_2 = 0$$

$$+$$

$$\mathbf{R}_G = \frac{m_1\mathbf{R}_1 + m_2\mathbf{R}_2}{m_1 + m_2}$$

$$\mathbf{R}_G = \mathbf{R}_{G0} + \mathbf{v}_G t$$



The c.o.m. of a 2-body system may serve as the origin of an inertial frame.

Equations of Relative Motion

$$-m_1 m_2 \ddot{\mathbf{R}}_1 = \frac{-G m_1 m_2^2}{r^2}\hat{\mathbf{u}}_r$$

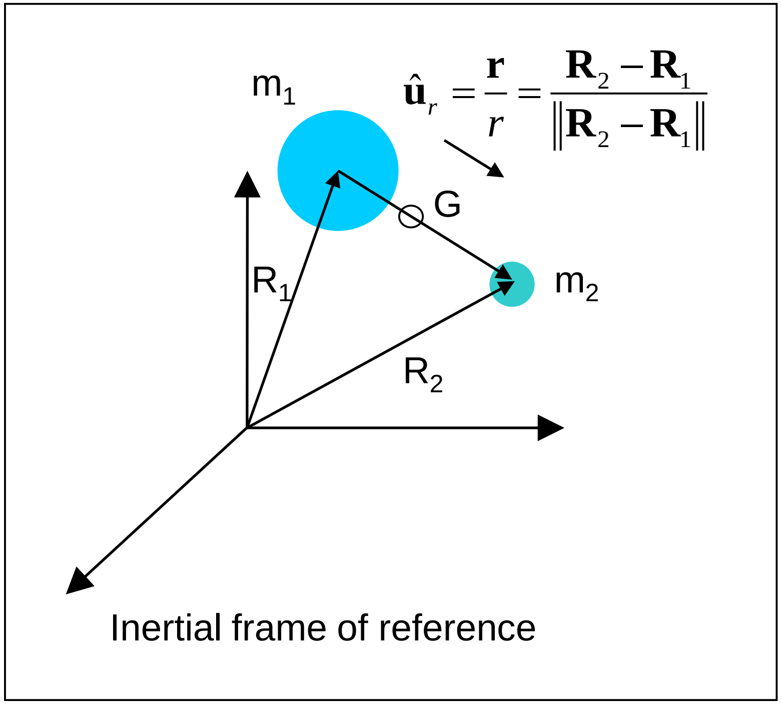
$$+$$

$$m_1 m_2 \ddot{\mathbf{R}}_2 = -\frac{G m_1^2 m_2}{r^2}\hat{\mathbf{u}}_r$$

$$\ddot{\mathbf{R}}_2 - \ddot{\mathbf{R}}_1 = -\frac{G\left(m_1 + m_2\right)}{r^2}\hat{\mathbf{u}}_r$$

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r}$$

μ is the gravitational parameter



The motion of m_2 as seen from m_1 is the same as the motion of m_1 as seen from m_2.

Equations of Relative Motion

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r}$$

This is a nonlinear dynamical system.

How to solve it ?

⇒ Find constants of the motion !

⇒ How many ?

Energy Conservation

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r}$$

$$\ddot{\mathbf{r}}.\dot{\mathbf{r}} = \frac{1}{2}\frac{d}{dt}(\dot{\mathbf{r}}.\dot{\mathbf{r}}) = \frac{1}{2}\frac{d}{dt}\left(\dot{r}^{\mathbf{2}}\right) = \frac{1}{2}\frac{d}{dt}\left(v^{\mathbf{2}}\right)$$

$$\mu\frac{\mathbf{r}.\dot{\mathbf{r}}}{r^3} = \mu\frac{r.\dot{r}}{r^3} = \mu\frac{\dot{r}}{r^2} = -\frac{d}{dt}\left(\frac{\mu}{r}\right)$$

⇩

$$\frac{v^2}{2} - \frac{\mu}{r} = E$$

Constant Angular Momentum

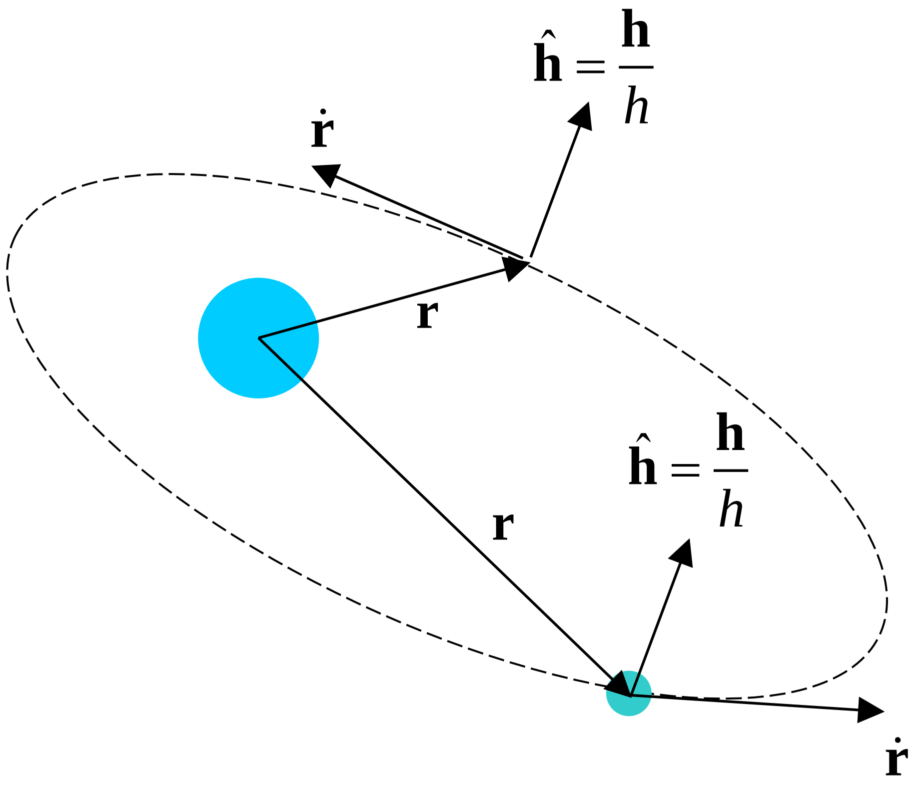
$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} \quad \xrightarrow{\mathbf{r}\times} \quad \mathbf{r}\times\ddot{\mathbf{r}} = \mathbf{r}\times\left(-\frac{\mu}{r^3}\mathbf{r}\right) = 0$$

$$\mathbf{h} = \mathbf{r}\times\dot{\mathbf{r}} \quad \xrightarrow{d/dt} \quad \frac{d\mathbf{h}}{dt} = \dot{\mathbf{r}}\times\dot{\mathbf{r}} + \mathbf{r}\times\ddot{\mathbf{r}} = \mathbf{r}\times\ddot{\mathbf{r}}$$

Specific angular momentum

$$\frac{d\mathbf{h}}{dt} = 0 \rightarrow \mathbf{r}\times\dot{\mathbf{r}} = \text{constant} = \mathbf{h}$$

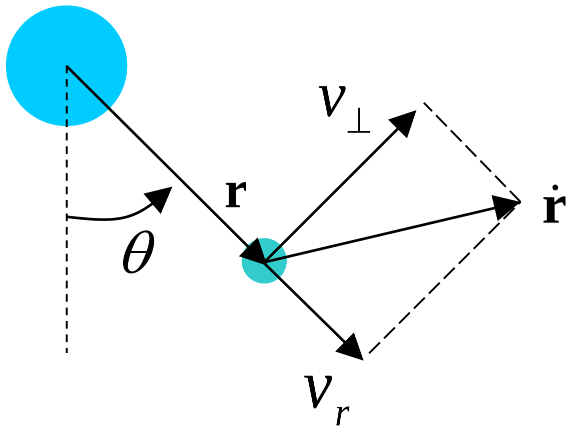
The Motion Lies in a Fixed Plane



The fixed plane is the **orbit plane** and is normal to the angular momentum vector.

$$\mathbf{r} \times \dot{\mathbf{r}} = \text{constant} = \mathbf{h}$$

Azimuth Component of the Velocity



$$\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}} = r\hat{\mathbf{u}}_r \times (v_r \hat{\mathbf{u}}_r + v_\perp \hat{\mathbf{u}}_\perp) = r v_\perp \hat{\mathbf{h}}$$

$$h = r v_\perp = r^2 \dot{\theta}$$

The angular momentum depends only on the azimuth component of the relative velocity

First Integral of Motion

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} \quad \overset{\times\mathbf{h}}{\Rightarrow} \quad \ddot{\mathbf{r}} \times \mathbf{h} = -\frac{\mu}{r^3}\mathbf{r} \times \mathbf{h} = -\frac{\mu}{r^3}\mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}})$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a}.\mathbf{c}) - \mathbf{c}(\mathbf{a}.\mathbf{b}) \quad \Rightarrow \quad \ddot{\mathbf{r}} \times \mathbf{h} = \frac{\mu}{r^3}\left[\dot{\mathbf{r}}(\mathbf{r}.\mathbf{r}) - \mathbf{r}(\mathbf{r}.\dot{\mathbf{r}})\right]$$

$$\mathbf{r}.\dot{\mathbf{r}} = r\dot{r} \quad \Rightarrow \quad = \mu\left(\frac{\dot{\mathbf{r}}}{r} - \frac{\mathbf{r}\dot{r}}{r^2}\right) = \mu\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right)$$

$$\int \quad \Rightarrow \quad \dot{\mathbf{r}} \times \mathbf{h} - \mu\frac{\mathbf{r}}{r} = \text{constant} = \mu\mathbf{e}$$

e lies in the orbit plane (***e***.**h**)=0: the line defined by **e** is the apse line. Its norm, *e*, is the eccentricity.

Note: demonstrate the Identity $\mathbf{r}.\dot{\mathbf{r}} = r\dot{r}$

$$\frac{d}{dt}(\mathbf{r}.\mathbf{r}) = \mathbf{r}.\frac{d\mathbf{r}}{dt} + \frac{d\mathbf{r}}{dt}.\mathbf{r} = 2\mathbf{r}.\frac{d\mathbf{r}}{dt} = 2\mathbf{r}.\dot{\mathbf{r}}$$

$$\mathbf{r}.\mathbf{r} = r^2 \Rightarrow \frac{d}{dt}(\mathbf{r}.\mathbf{r}) = 2r\frac{dr}{dt} = 2r\dot{r}$$

$$\Downarrow$$

$$\mathbf{r}.\dot{\mathbf{r}} = 2r\dot{r}$$

Orbit Equation

$$\frac{\dot{\mathbf{r}}\times\mathbf{h}}{\mu}=\frac{\mathbf{r}}{r}+\mathbf{e} \quad \xrightarrow{\ \mathbf{r}.\ } \quad \frac{\mathbf{r}.\left(\dot{\mathbf{r}}\times\mathbf{h}\right)}{\mu}=\frac{\mathbf{r}.\mathbf{r}}{r}+\mathbf{r}.\mathbf{e}$$

$\mathbf{a}.\left(\mathbf{b}\times\mathbf{c}\right)=\left(\mathbf{a}\times\mathbf{b}\right).\mathbf{c}$

$$\frac{\mathbf{r}.\left(\dot{\mathbf{r}}\times\mathbf{h}\right)}{\mu}=\frac{\left(\mathbf{r}\times\dot{\mathbf{r}}\right).\mathbf{h}}{\mu}=\frac{\mathbf{h}.\mathbf{h}}{\mu}=\frac{h^2}{\mu}=r+\mathbf{r}.\mathbf{e}$$

$$r=\frac{h^2}{\mu}\frac{1}{1+e\cos\theta}$$

Closed form of the nonlinear equations of motion

Conic Section in Polar Coordinates

Constant: angular momentum

Semi-latus rectum

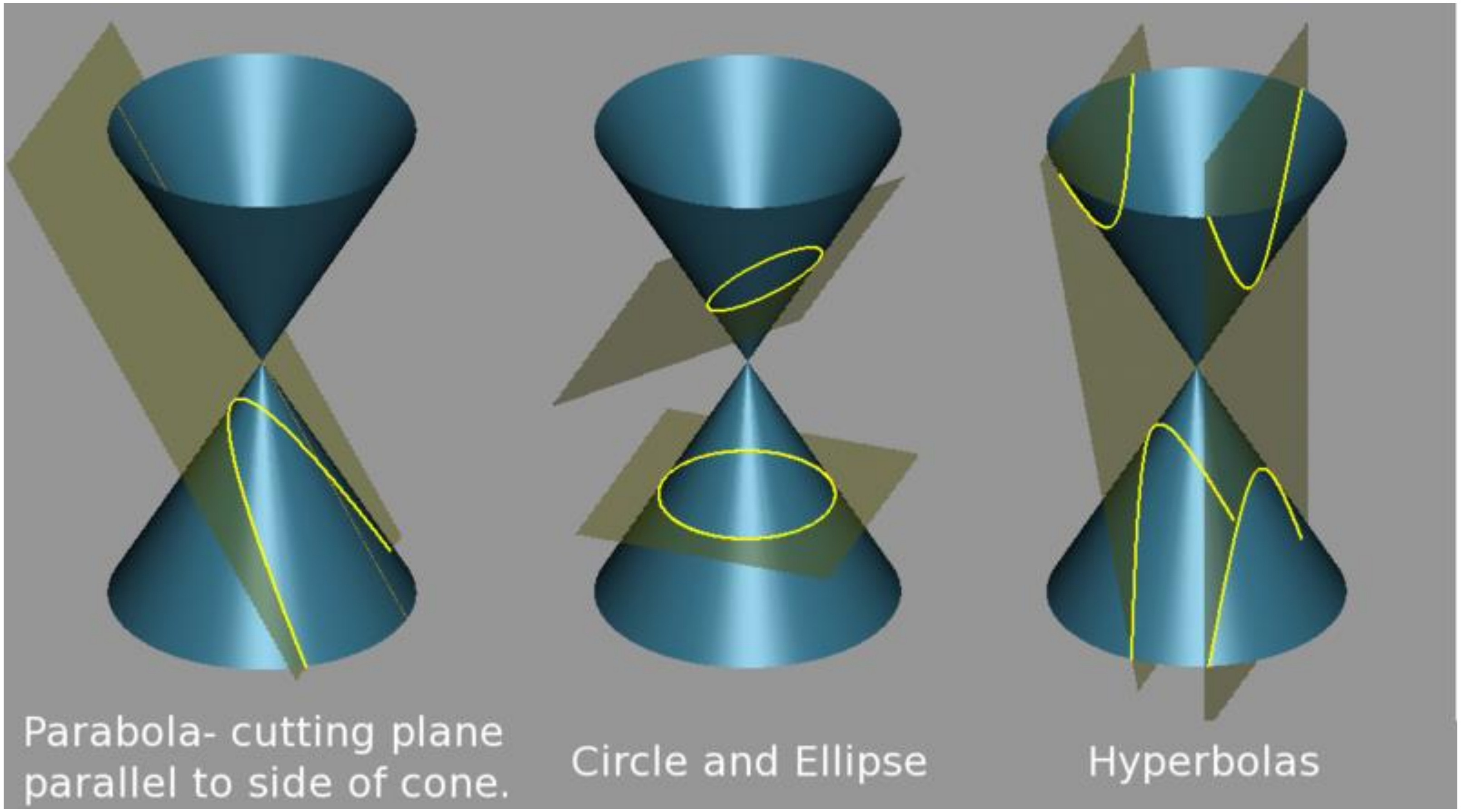
$$r = \frac{h^2}{\mu} \frac{1}{1 + e\cos\theta} = \frac{p}{1 + e\cos\theta}$$

Constant: gravitational parameter

Independent variable: true anomaly (=0 at the periapsis)

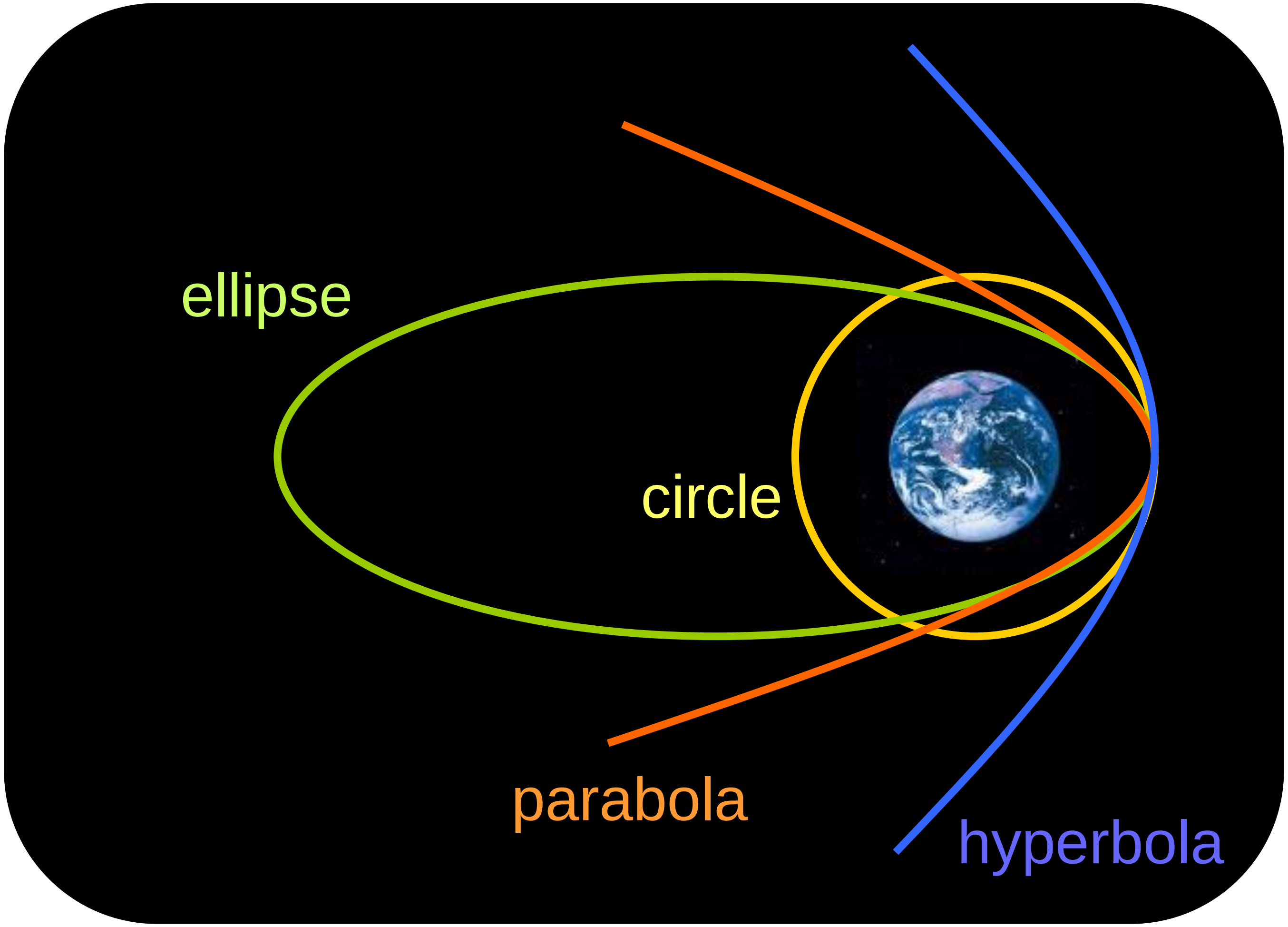
Constant: eccentricity

Conic Section



e=1 e=0 0<e<1 e>1

Possible Motions in the 2-Body System



In Summary

+ We can calculate *r* for all values of the true anomaly.

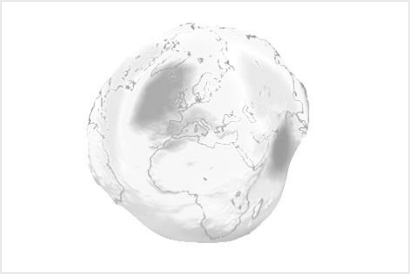
+ The orbit equation is a mathematical statement of Kepler's first law.

- The solution of the "simple" problem of two bodies cannot be expressed in a closed form, explicit function of time.

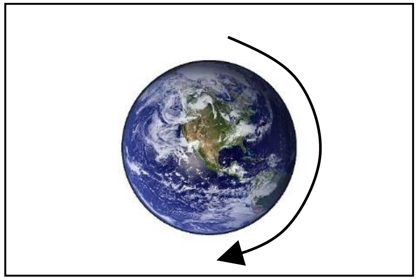
Do we have 6 independent constants ?

The two vector constants **h** and **e** provide only 5 independent constants: **h.e**=0

2. The Two-Body Problem



$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r}$$



2.4 Resulting orbits:

2.4.1 Circular orbits

2.4.2 Elliptic orbits

2.4.3 Parabolic orbits

2.4.4 Hyperbolic orbits

Circular Orbits (e=0)

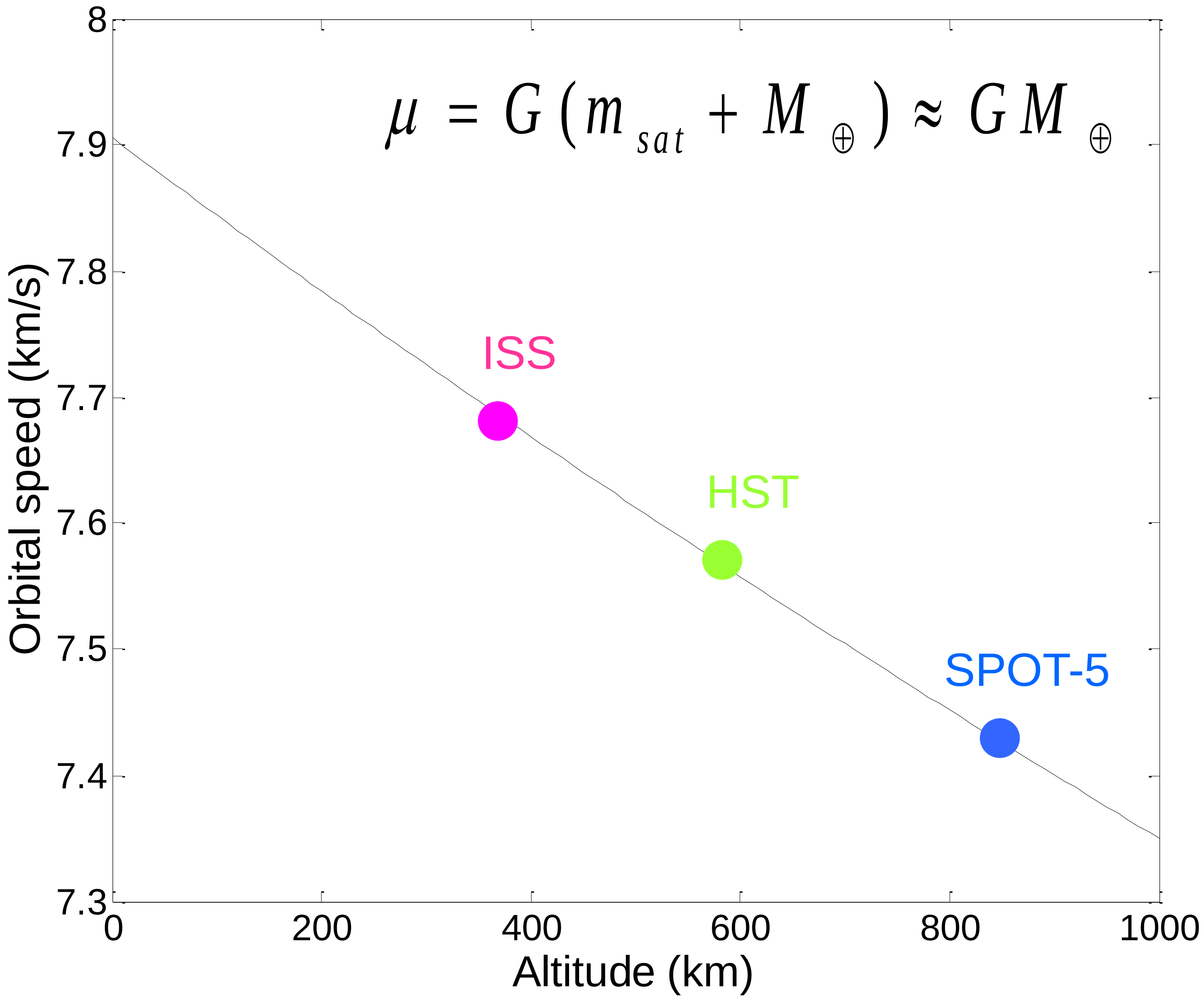
$$r = \frac{h^2}{\mu} = \text{Constant} \qquad h = r v_{\perp} = r v_{circular}$$

$$v_{circ} = \sqrt{\frac{\mu}{r}}$$

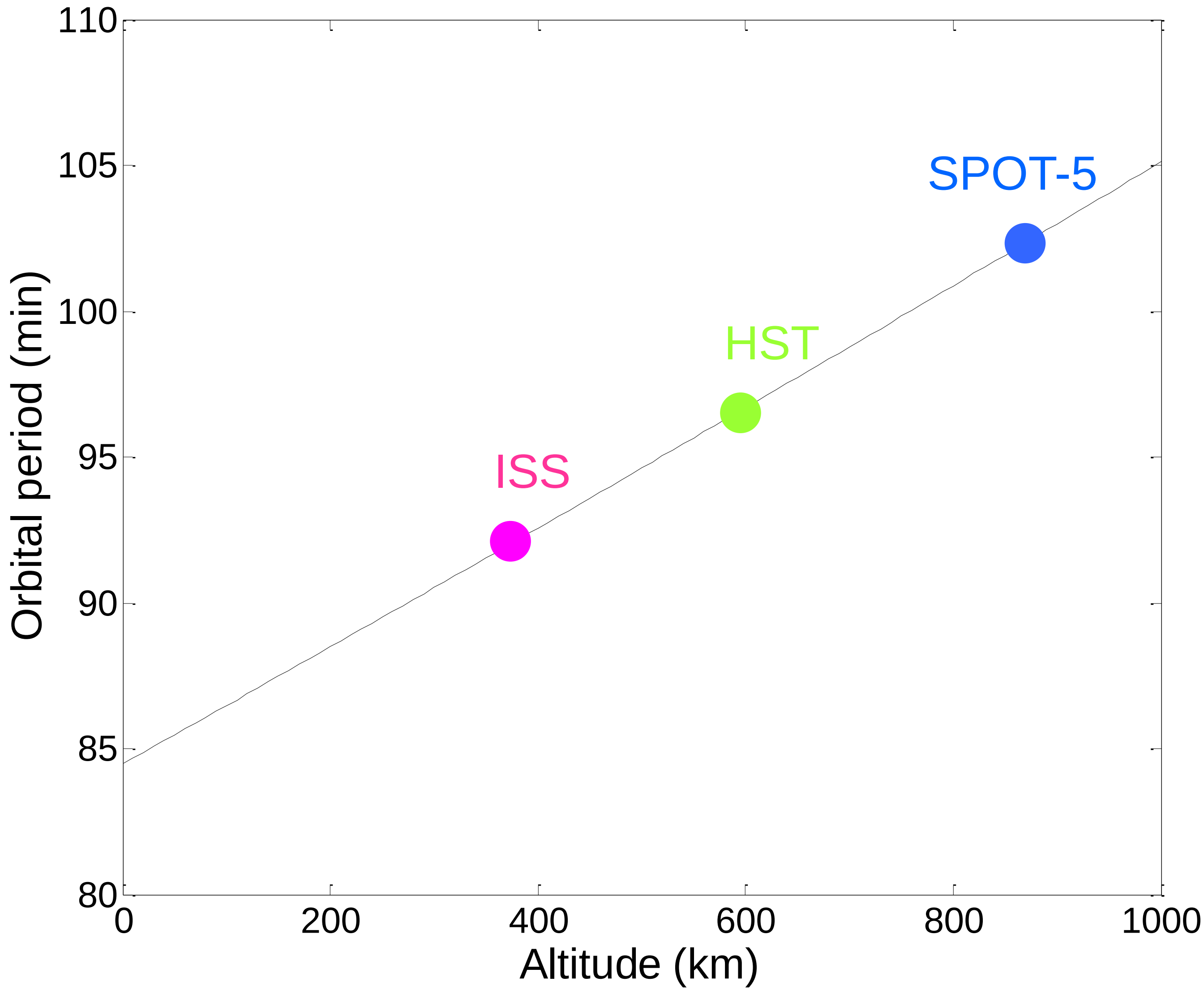
$$T_{circ} = 2\pi r \Big/ \sqrt{\frac{\mu}{r}} = \frac{2\pi}{\sqrt{\mu}} r^{3/2}$$

$$\varepsilon_{circ} = -\frac{\mu}{2r} < 0$$

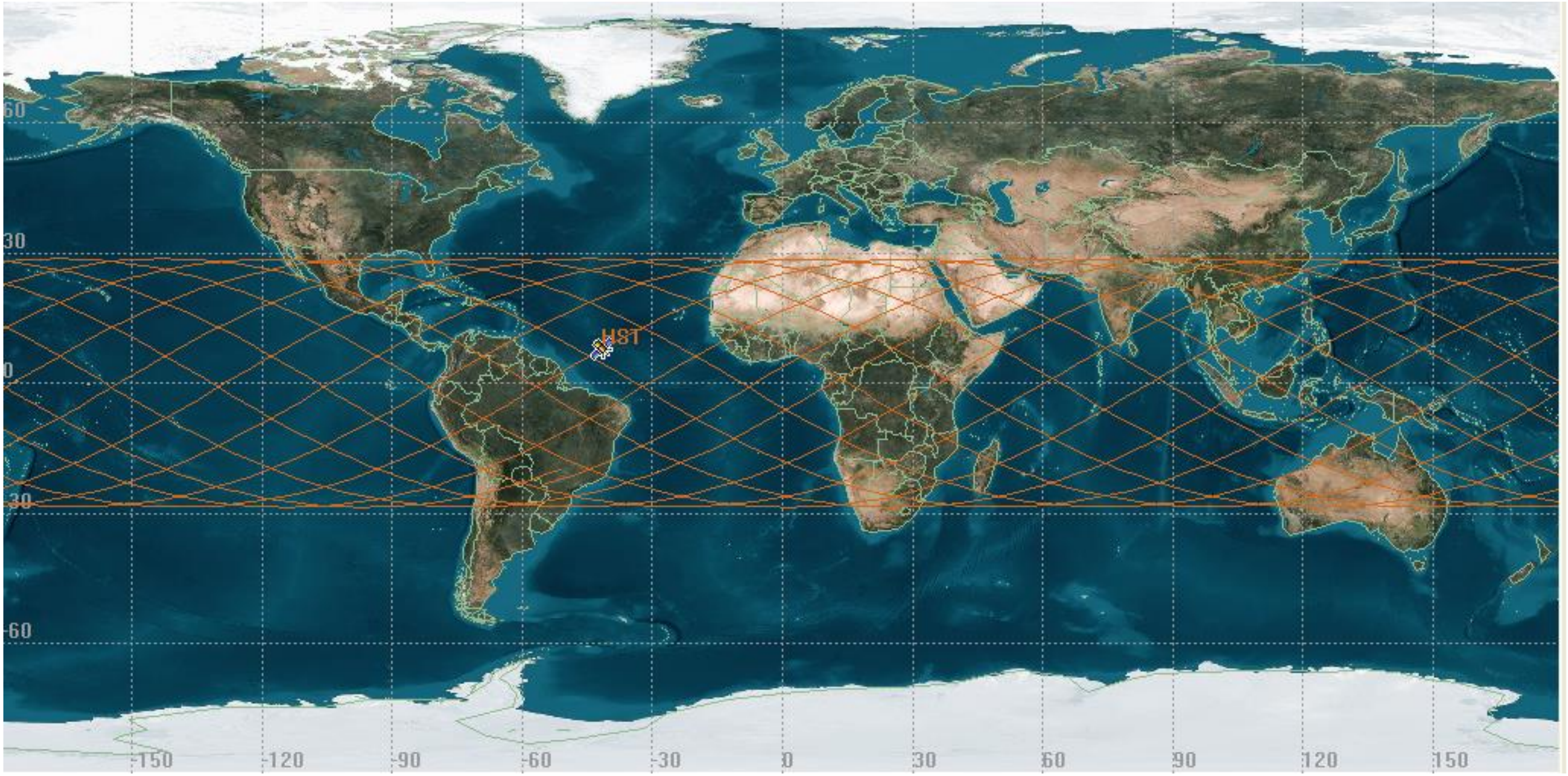
Orbital Speed



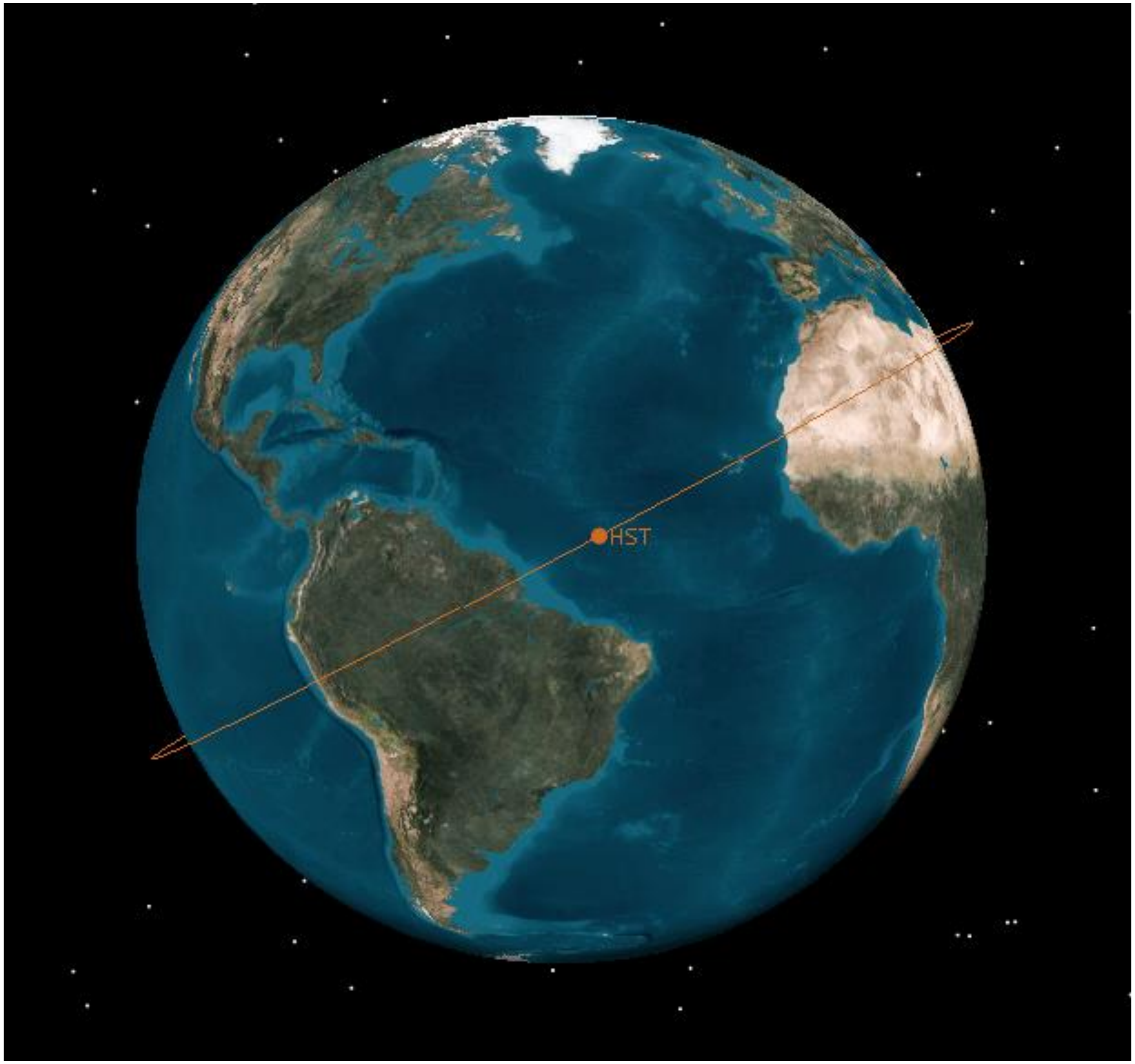
Orbital Period



Hubble Space Telescope



Hubble Space Telescope



Two Particular Cases

1. 7.9 km/s is the **first cosmic velocity**; i.e., the minimum velocity (theoretical velocity, $r = 6378$ km) to orbit the Earth.

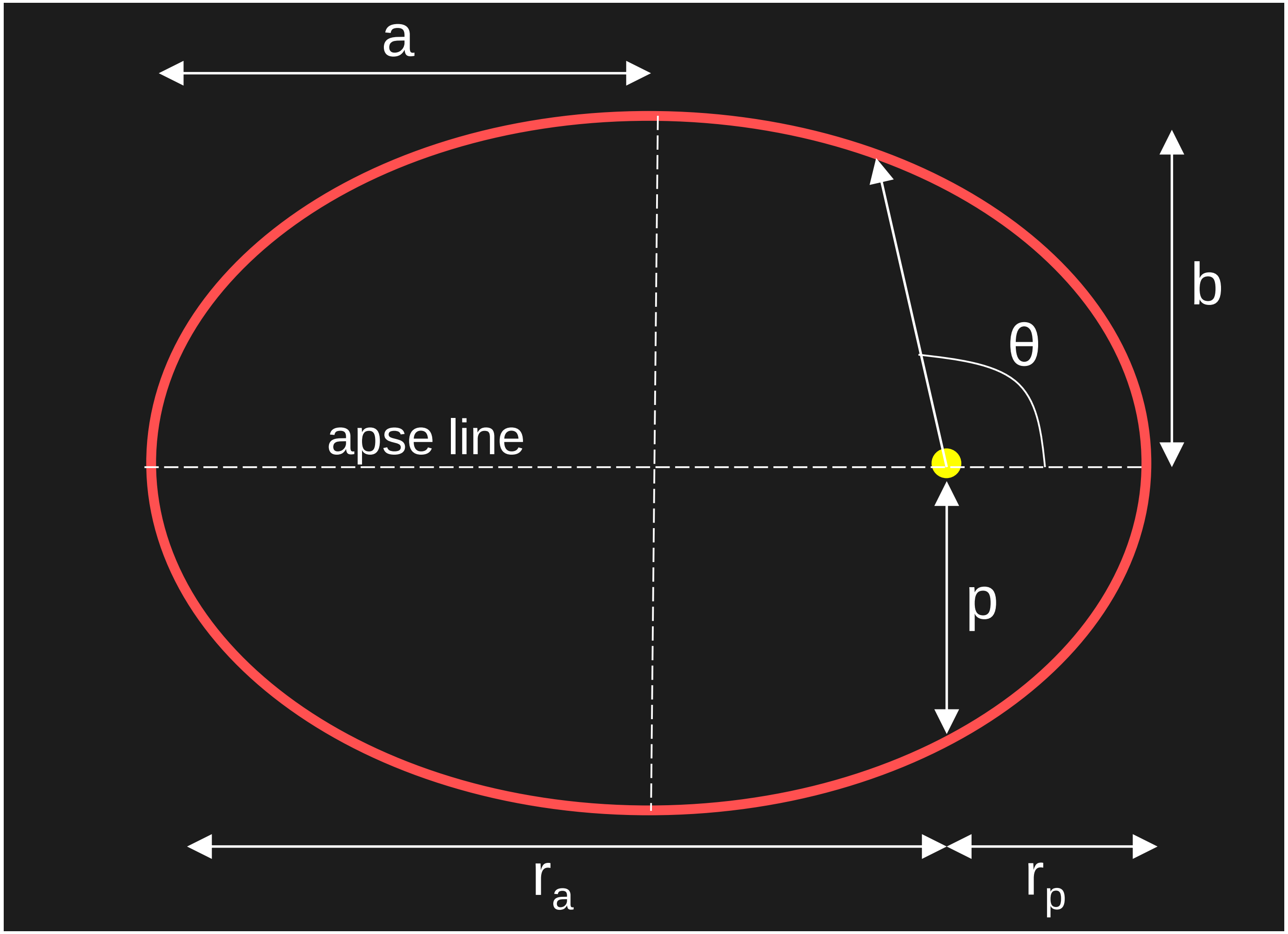
2. 35786 km is the altitude of the **geostationary orbit**. It is the orbit at which the satellite angular velocity is equal to that of the Earth, $\omega = \omega_E = 7.292\ 10^{-5}$ rad/s, in inertial space (*).

$$r_{GEO} = \left(\frac{T_{circ}\sqrt{\mu}}{2\pi} \right)^{2/3}$$

---

* A sidereal day, 23h56m4s, is the time it takes the Earth to complete one rotation relative to inertial space. A synodic day, 24h, is the time it takes the sun to apparently rotate once around the earth. They would be identical if the earth stood still in space.

Geometry of the Elliptic Orbit



Elliptic Orbits (0<e<1)

$$r = \frac{h^2}{\mu} \frac{1}{1 + e\cos\theta}$$

The relative position vector remains bounded.

θ=0, minimum separation, **periapse**

$$r_p = \frac{h^2}{\mu(1+e)}$$

θ=π, greatest separation, **apoapse**

$$r_a = \frac{h^2}{\mu(1-e)}$$

θ=π/2, **semi-latus rectum** *p*

$$e = \frac{r_a - r_p}{r_a + r_p}$$

Energy of an Elliptical Orbit

$$\frac{v^2}{2} - \frac{\mu}{r} = E \qquad \frac{v_p^2}{2} - \frac{\mu}{r_p} = E_{perigee}$$

$h = v_p r_p$ See part 1

$$\frac{h^2}{2r_p^2} - \frac{\mu}{r_p} = E_{perigee}$$

$$r_p = \frac{h^2}{\mu(1+e)}$$

$$-\frac{1}{2}\frac{\mu^2}{h^2}(1 - e^2) = E_{perigee}$$

Link between energy and the other constants **h** and **e**!

$h = \sqrt{\mu a(1-e^2)}$ See next slide

$$-\frac{\mu}{2a} = E_{perigee}$$

Note: Angular Momentum

$$r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}$$

Orbit equation

$$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

Polar equation of an ellipse
(*a*, semimajor axis)

$$h = \sqrt{\mu a(1-e^2)}$$

Velocity in an Elliptical Orbit

$$\frac{v^2}{2} - \frac{\mu}{r} = E \qquad\qquad -\frac{\mu}{2a} = E_{perigee}$$

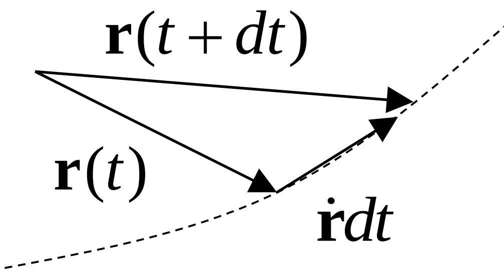
⇩

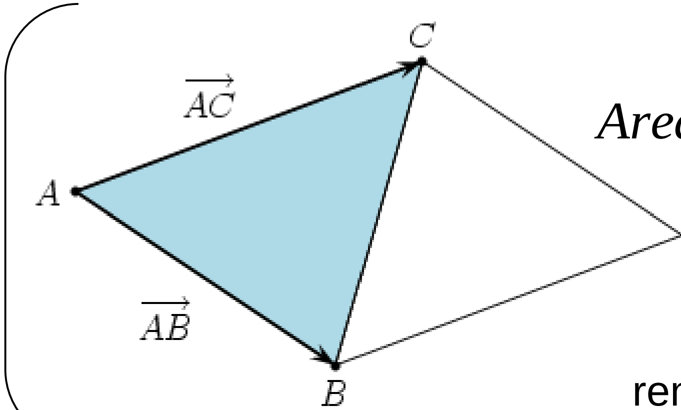
$$\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$$

⇩

$$v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$

Kepler's Second Law





$$Area = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$

reminder

$$dA = \frac{1}{2}\left|\mathbf{r} \times \dot{\mathbf{r}}dt\right| = \frac{1}{2}\left|\mathbf{h}\right|dt = \frac{1}{2}hdt$$

$$\frac{dA}{dt} = \frac{h}{2} = \frac{1}{2}r^2\frac{d\theta}{dt} = \text{constant}$$

The line from the sun to a planet sweeps out equal areas inside the ellipse in equal lengths of time.

Kepler's Third Law

$$T = \frac{\text{enclosed area}}{dA/dt} = \frac{2\pi ab}{h}$$

$$h = \sqrt{\mu a(1-e^2)} \qquad b = a\sqrt{1-e^2}$$

$$T_{ellip} = 2\pi\sqrt{\frac{a^3}{\mu}}$$

The elliptic orbit period depends only on the semimajor axis and is independent of the eccentricity.

$$\frac{T_1^2}{T_2^2} = \frac{a_1^3}{a_2^3}$$

The squares of the orbital periods of the planets are proportional to the cubes of their mean distances from the sun.

Satellite in Elliptic Orbit

$$r_p = 354 + 6378 = 6732\,\text{km} \qquad r_a = 1447 + 6378 = 7825\,\text{km}$$

$$\Downarrow$$

$$e = \frac{r_a - r_p}{r_a + r_p} = 0.075, \quad a = \frac{r_a + r_p}{2} = 7278.5\,\text{km}$$

$$T = 2\pi\sqrt{\frac{a^3}{\mu}} = 6179.79\,\text{s} = 103\,\text{min}$$

$$v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)} \rightarrow \begin{cases} v_p = 7.98\,\text{km/s} \\ v_a = 6.86\,\text{km/s} \end{cases}$$

GTO and GEO

For an orbit with a perigee at 320 km and an apogee at 35786 km, what is the velocity increment required to reach the geostationary orbit ?

$$v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$

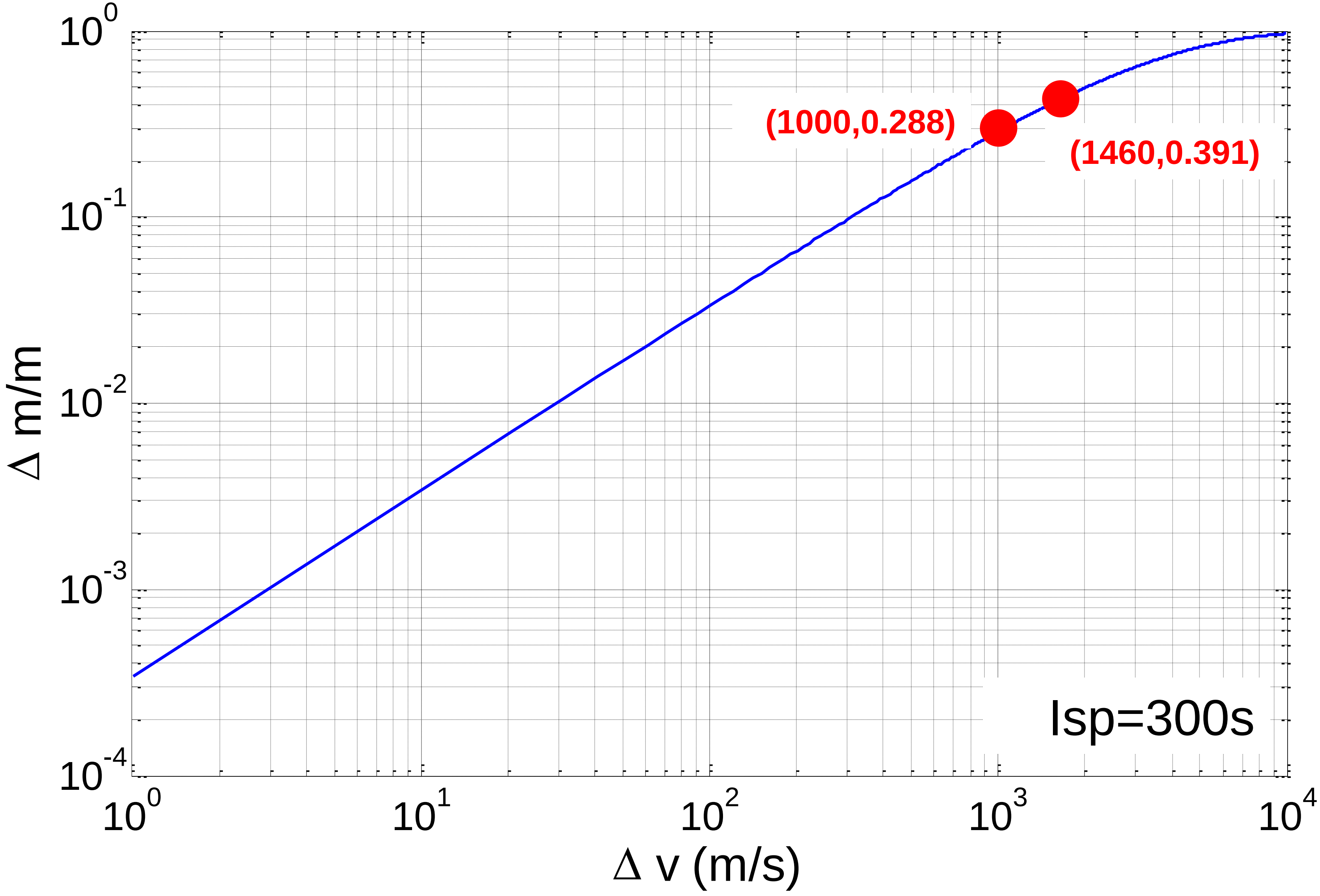
GTO and GEO

For an orbit with a perigee at 320 km and an apogee at 35786 km, what is the velocity increment required to reach the geostationary orbit ?

GTO	GEO
$a = \dfrac{r_a + r_p}{2} = 24430\,\text{km}$	$v_{circ} = \sqrt{\dfrac{398000}{35786 + 6378}}$
$v_p = 10.13\,\text{km/s}$	$= 3.07\,\text{km/s}$
$v_a = 1.61\,\text{km/s}$	

Answer: 1.46 km/s (apogee motor)

GTO and GEO



(1000,0.288)
(1460,0.391)
Isp=300s
Δ m/m
Δ v (m/s)
10^0
10^-1
10^-2
10^-3
10^-4
10^1
10^2
10^3
10^4

Parabolic Orbits (e=1)

$$r = \frac{h^2}{\mu} \frac{1}{1+\cos\theta} \qquad \theta \to \pi,\ r \to \infty$$

$$\varepsilon_{parab} = -\frac{1}{2}\frac{\mu^2}{h^2}\left(1-e^2\right) = 0$$

The satellite has just enough energy to escape from the attracting body.

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r}$$

$$v_{parab} = \sqrt{\frac{2\mu}{r}}$$

The satellite will coast to infinity, arriving there with zero velocity relative to the central body.

Escape Velocity, V_{esc}

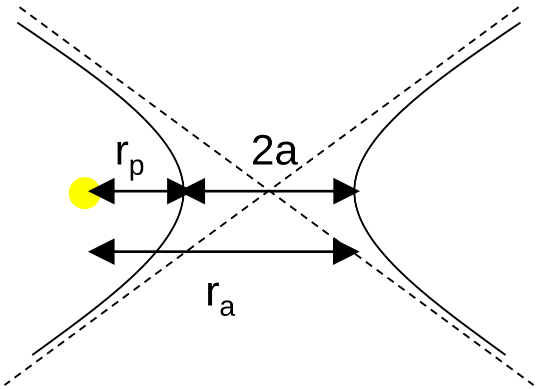
11.2 km/s is the **second cosmic velocity**; i.e., the minimum velocity (theoretical velocity, $r = 6378\text{km}$) to orbit the Earth.

$$v_{circ} = \sqrt{\frac{\mu}{r}} \qquad v_{parab} = \sqrt{\frac{2\mu}{r}}$$

$$11.2\,\text{km/s} = \sqrt{2} \times 7.9\,\text{km/s}$$

Hyperbolic Orbits (e>1)

$$r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}$$



$$r_p = \frac{h^2}{\mu(1+e)}$$

$$r_a = \frac{h^2}{\mu(1-e)} < 0$$

$$2a = -\left(r_a + r_p\right)$$

$$a = \frac{h^2}{\mu}\frac{1}{e^2-1}$$

$$\varepsilon = -\frac{1}{2}\frac{\mu^2}{h^2}\left(1-e^2\right)$$

$$\varepsilon_{hyper} = \frac{\mu}{2a} > 0$$

C_3 Velocity

$$\varepsilon = \frac{\mu}{2a} = \frac{v^2}{2} - \frac{\mu}{r}$$

$$v_\infty = \sqrt{\frac{\mu}{a}}$$

Hyperbolic excess speed

$$\frac{v_\infty^2}{2} = \frac{v^2}{2} - \frac{\mu}{r}$$

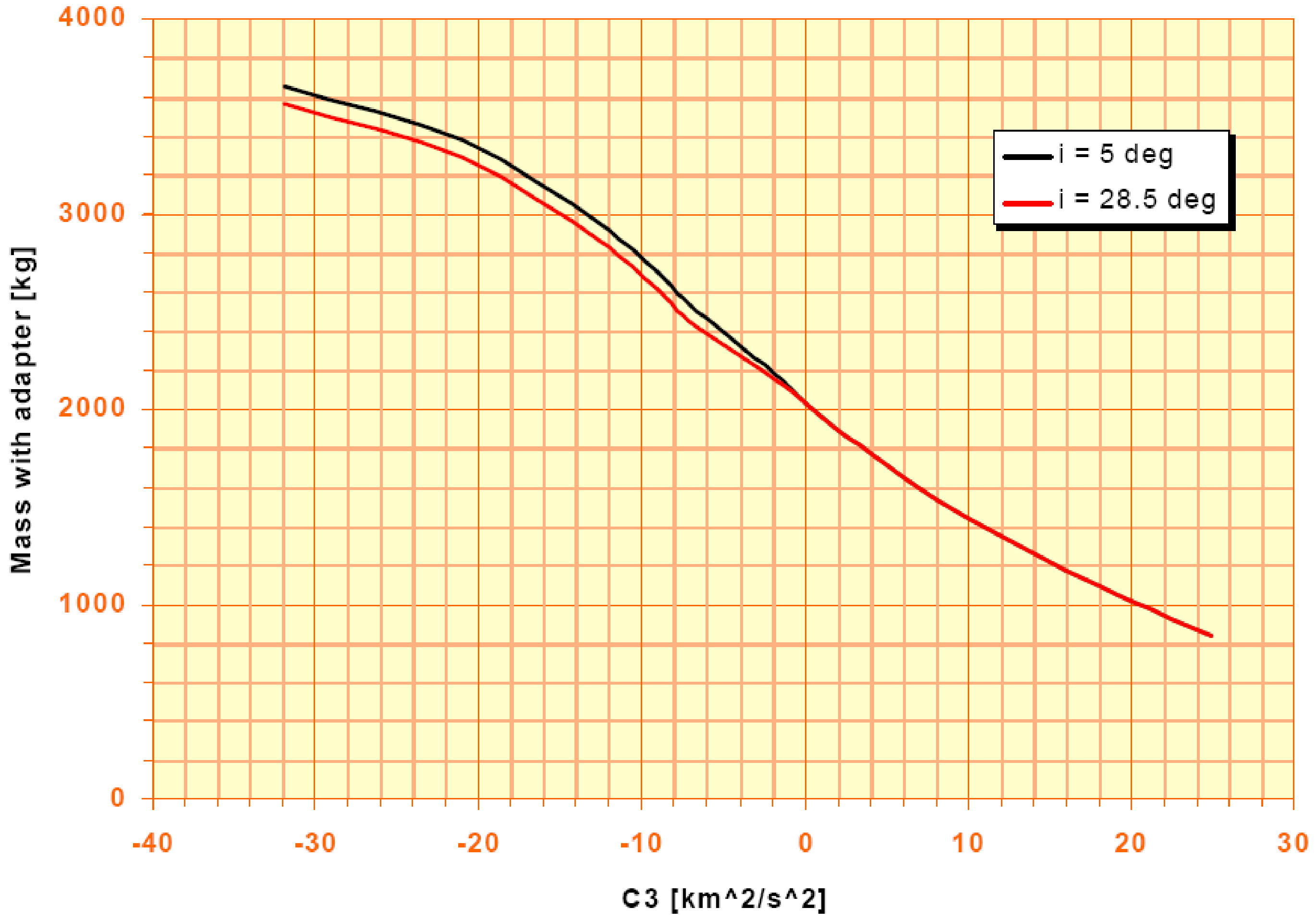
$$v^2 = v_\infty^2 + v_{esc}^2 = C_3 + v_{esc}^2$$

C_3 is a measure of the energy for an interplanetary mission:

16.6 km^2/s^2 (Cassini-Huygens)

8.9 km^2/s^2 (Solar Orbiter, phase A)

Soyuz ST v2-1b (Kourou Launch)



Delta II, Delta III and Atlas IIIA

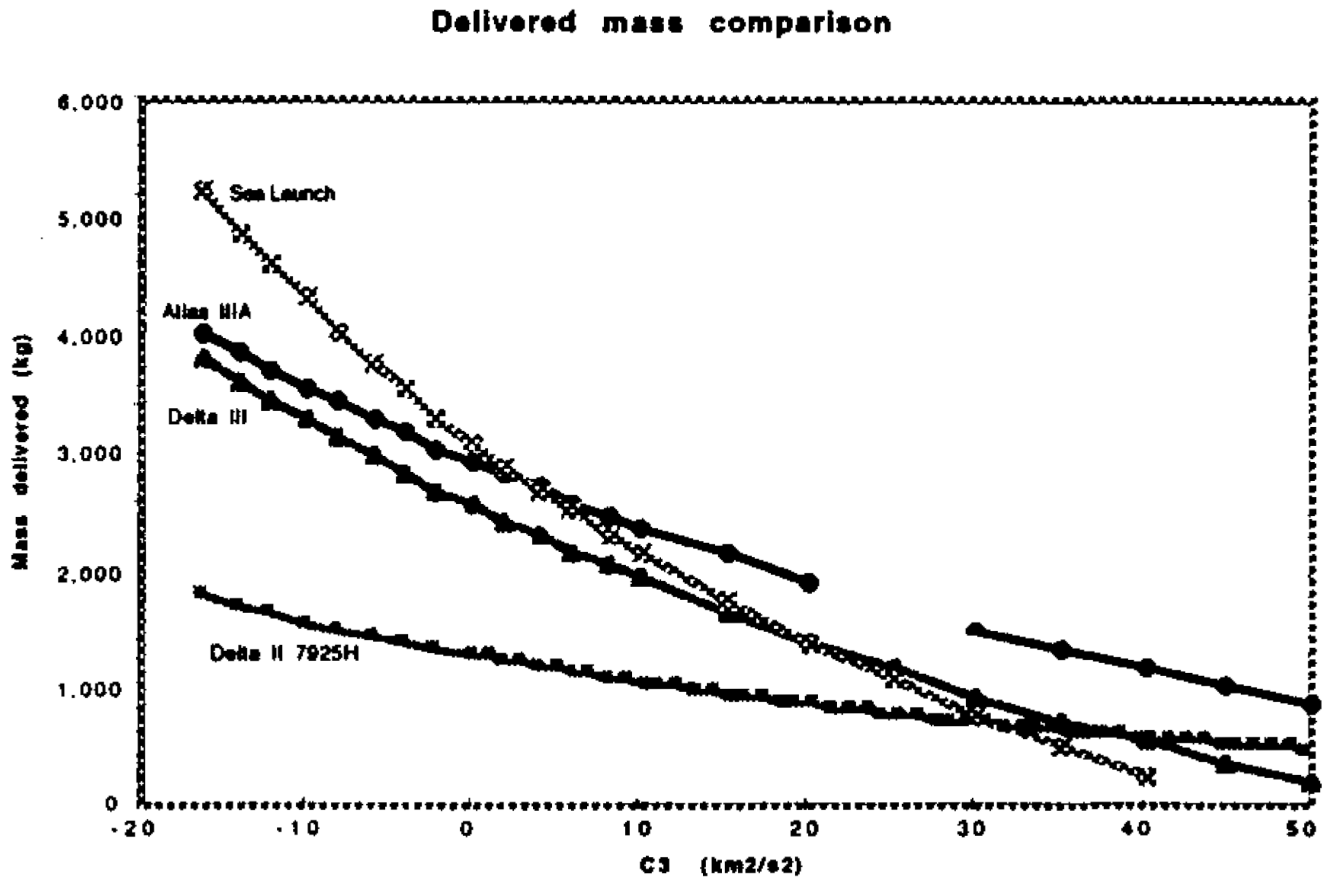


Figure 1. Delivered mass as a function of C3 for the Delta II 7927, Delta III, Altlas IIIA, and Sea Launch.

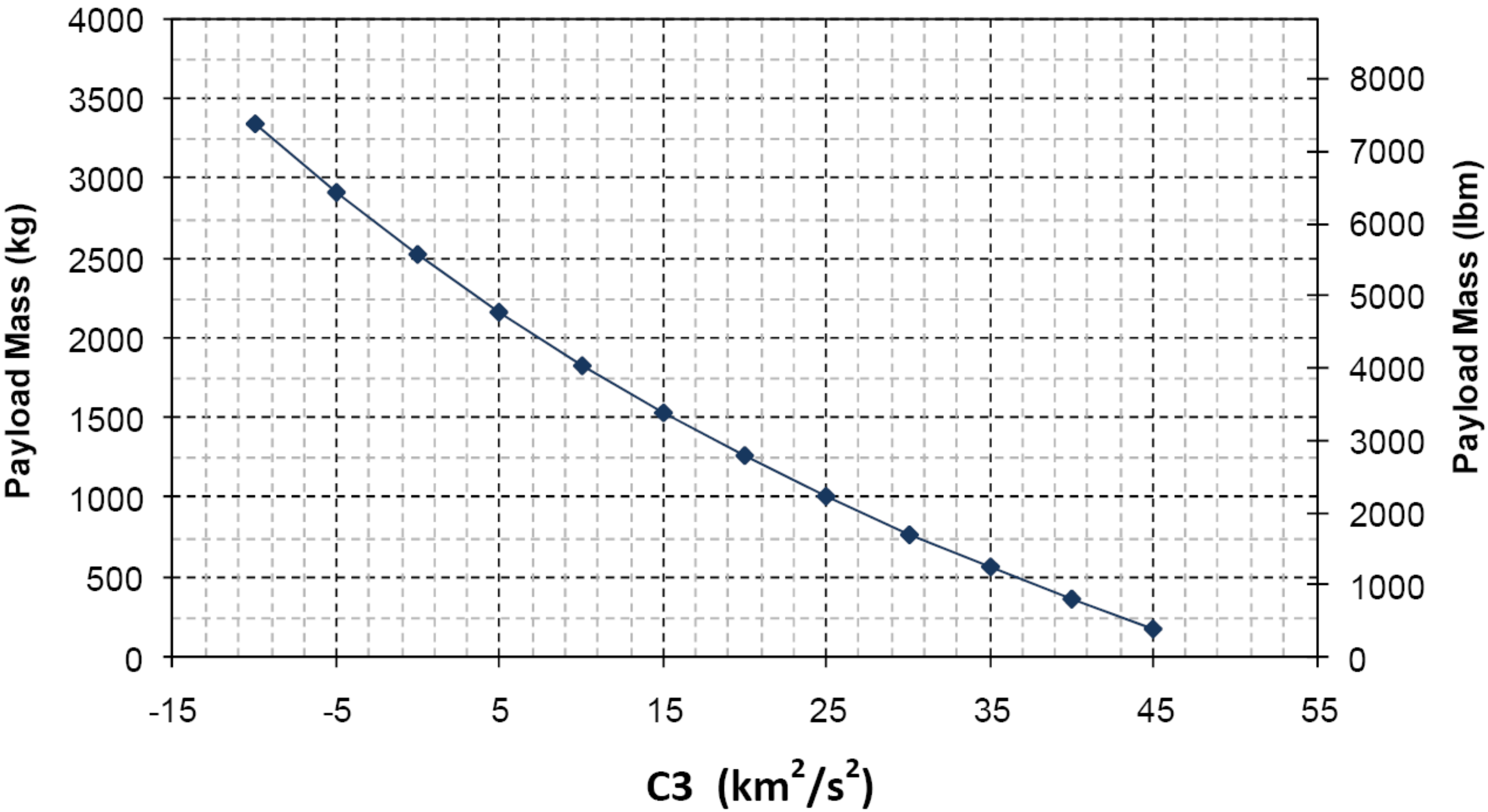
Proton

Table 2.9.1-1: Earth Escape Proton M Breeze M Missions

C3 Parameter (km^2/s^2)	Payload Systems Mass (kg)
-5	6270
-2	5890
0	5650
5	5090
10	4580
15	4110
20	3685
25	3295
30	2920
35	2575
40	2260
45	1990
50	1750
55	1525
60	1305
65	1120

C3 Parameter = $V^2 - 2\mu/R$.
Performance based on the use of 15255 mm PLF (standard).
At fairing jettison, FMHF shall be no more than 1135 W/m^2.
PSM includes LV adapter system mass.
PSM is calculated assuming a 2.33-sigma LV propellant margin.

Falcon 9



4000
3500
3000
2500
2000
1500
1000
500
0
Payload Mass (kg)
8000
7000
6000
5000
4000
3000
2000
1000
0
Payload Mass (lbm)
-15
-5
5
15
25
35
45
55
C3 (km²/s²)

The Two-Body Problem

2.1 JUSTIFICATION OF THE 2-BODY MODEL

2.2 GRAVITATIONAL FIELD

2.2.1 Newton’s law of universal gravitation

2.2.2 The Earth

2.2.3 Gravity models and geoid

2.3 RELATIVE MOTION

2.3.1 Equations of motion

2.3.2 Closed-form solution

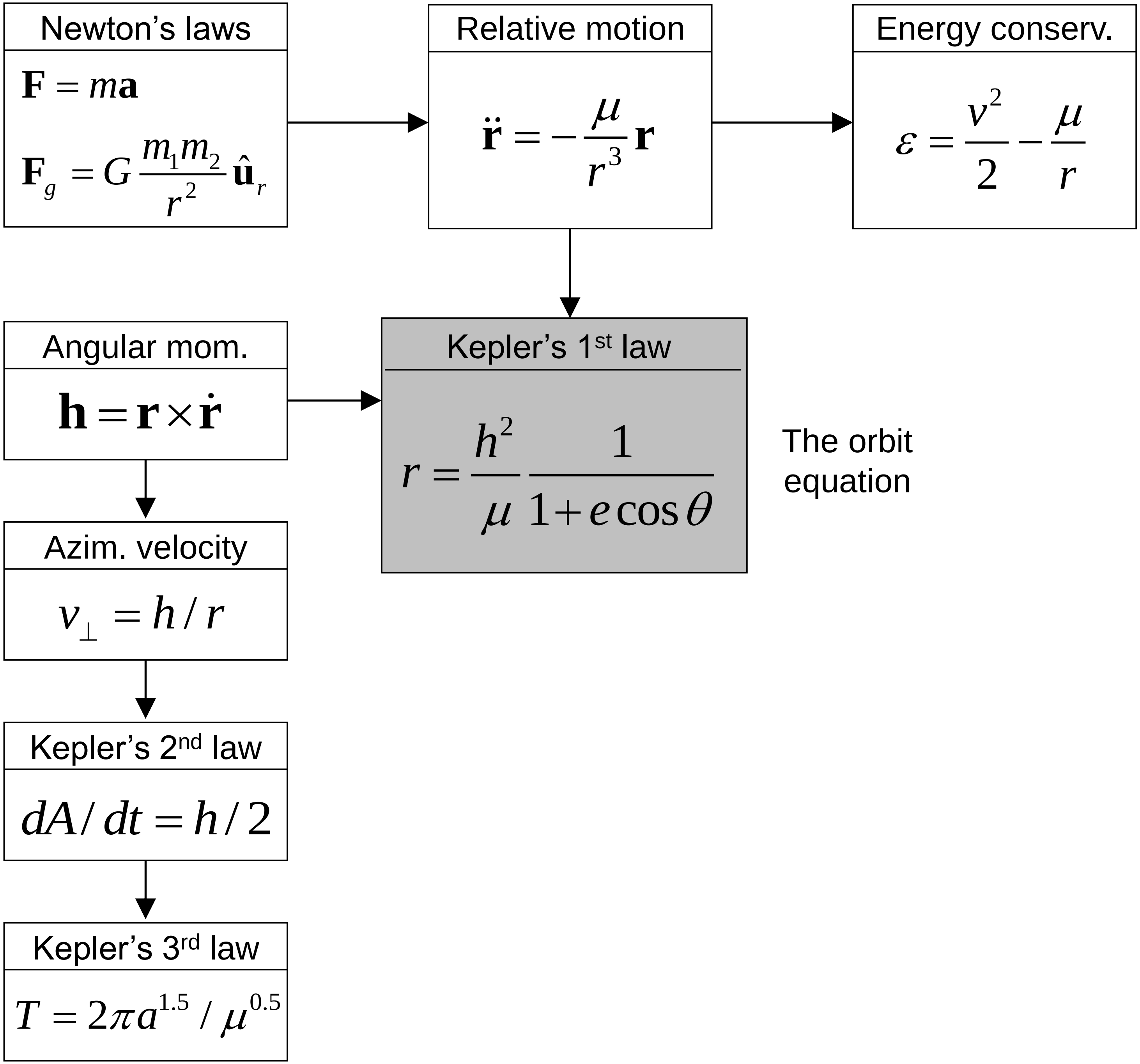
2.4 RESULTING ORBITS

2.4.1 Circular orbits

2.4.2 Elliptic orbits

2.4.3 Parabolic orbits

2.4.4. Hyperbolic orbits


Newton's laws
$\mathbf{F} = m\mathbf{a}$
$\mathbf{F}_g = G\frac{m_1 m_2}{r^2}\hat{\mathbf{u}}_r$
Relative motion
$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r}$
Energy conserv.
$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r}$
Angular mom.
$\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}}$
Kepler's 1st law
$r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}$
The orbit equation
Azim. velocity
$v_\perp = h/r$
Kepler's 2nd law
$dA/dt = h/2$
Kepler's 3rd law
$T = 2\pi a^{1.5} / \mu^{0.5}$

Concluding Remarks

Closed-form solution from which we deduced Kepler's laws.

Analytic formulas for orbital energy, velocity and period.

Two-body propagator available in STK. Often used in early studies to perform trending analysis.

But …

We have lost track of the time variable !

Did you Know ?

Compactness of the solar system: measured by the ratio of the distance *a* of a planet from the Sun to the radius R of the Sun.

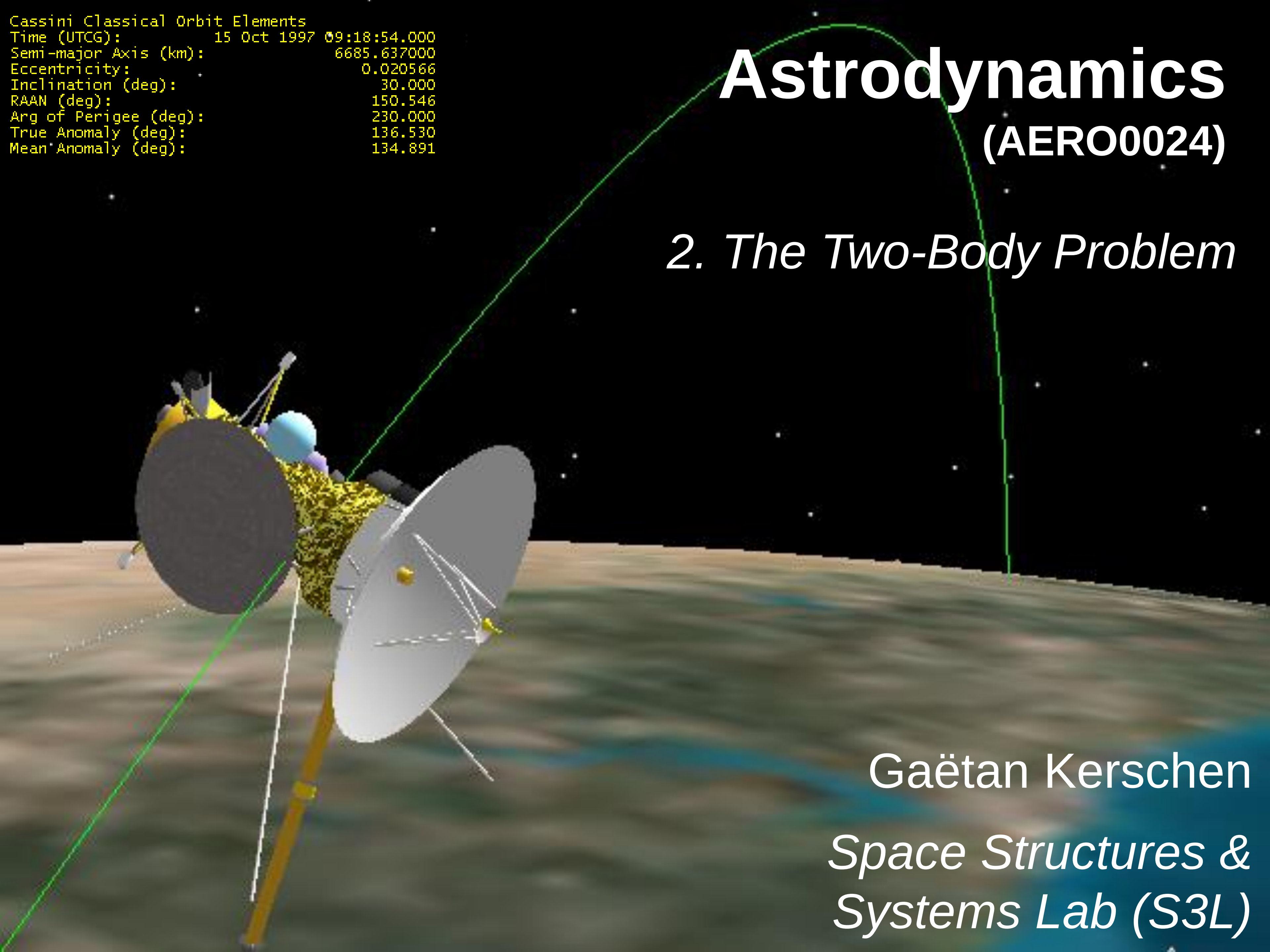
$$\frac{a}{R} \approx 200$$

Compactness of the hydrogen atom: measured by the ratio of the distance *a* of an electron from the nucleus to the radius R of the nucleus.

$$\frac{a}{R} \approx 5e4$$

Astrodynamics
(AERO0024)

2. The Two-Body Problem



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