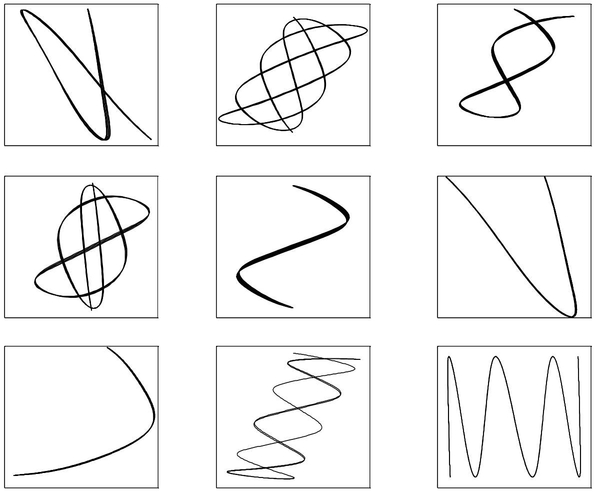
Nonlinear Vibrations of Aerospace Structures

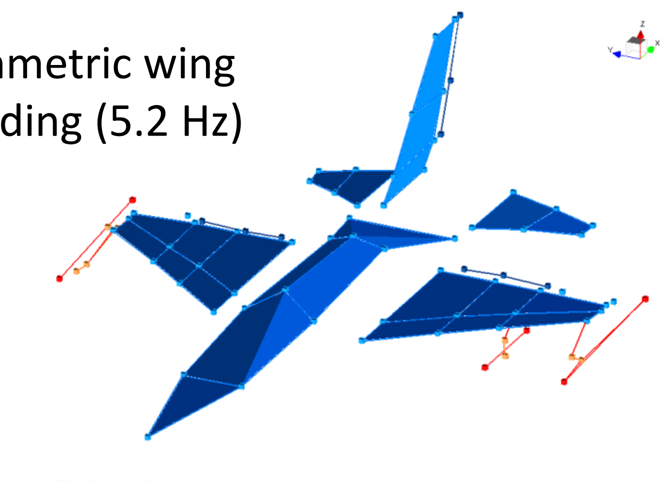
L03 | **Nonlinear modal analysis**

What Are Nonlinear Modes ?

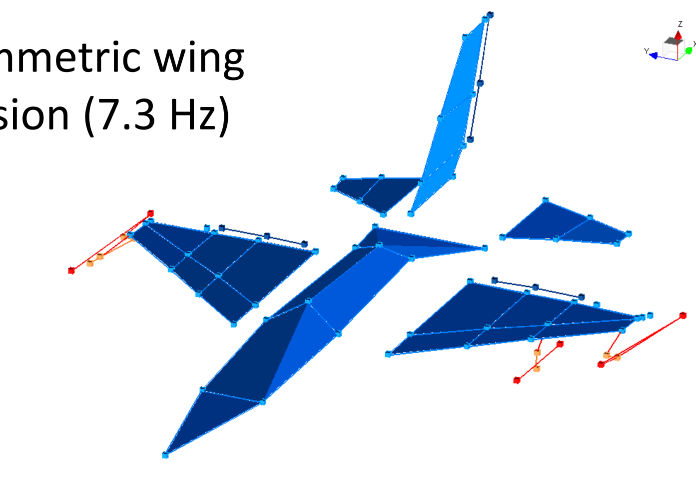


Modal Analysis Provides Key Information

Symmetric wing bending (5.2 Hz)



Symmetric wing torsion (7.3 Hz)



What Is a Linear Normal Mode ?

2.2. Modes normaux de vibration

Pour résoudre les équations des petites oscillations libres (2.1.12)

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = 0$$

cherchons une solution particulière dans laquelle toutes les coordonnées généralisées suivent, à un facteur près, la même loi temporelle

$$\mathbf{q} = \mathbf{x}\ \phi(t) \tag{2.2.1}$$

où $\mathbf{x}$ est un vecteur de constantes constituant *la forme propre du mouvement*, propre dans ce sens que le rapport de deux coordonnées est indépendant du temps et est toujours égal au rapport des éléments correspondants de $\mathbf{x}$. L'essai d'une solution de ce type fournit

$$\ddot{\phi}(t)\mathbf{M}\mathbf{x} + \phi(t)\mathbf{K}\mathbf{x} = 0 \tag{2.2.2}$$

Cours de théorie des vibrations

How Do We Calculate Linear Normal Modes ?

$$\ddot{q}_1 + (2q_1 - q_2) = 0$$
$$\ddot{q}_2 + (2q_2 - q_1) = 0$$

How Do We Calculate Linear Normal Modes ?

$$\ddot{q}_1 + (2q_1 - q_2) = 0$$
$$\ddot{q}_2 + (2q_2 - q_1) = 0$$

$q_{1,2} = A, B cos\omega t$

$$-\omega^2 A + 2A - B = 0$$
$$-\omega^2 B + 2B - A = 0$$

$$B = A(2 - \omega^2)$$
$$-\omega^2 B + 2B - A = 0$$

$$-\omega^2 A(2 - \omega^2)$$
$$+ 2A(2 - \omega^2) - A = 0$$

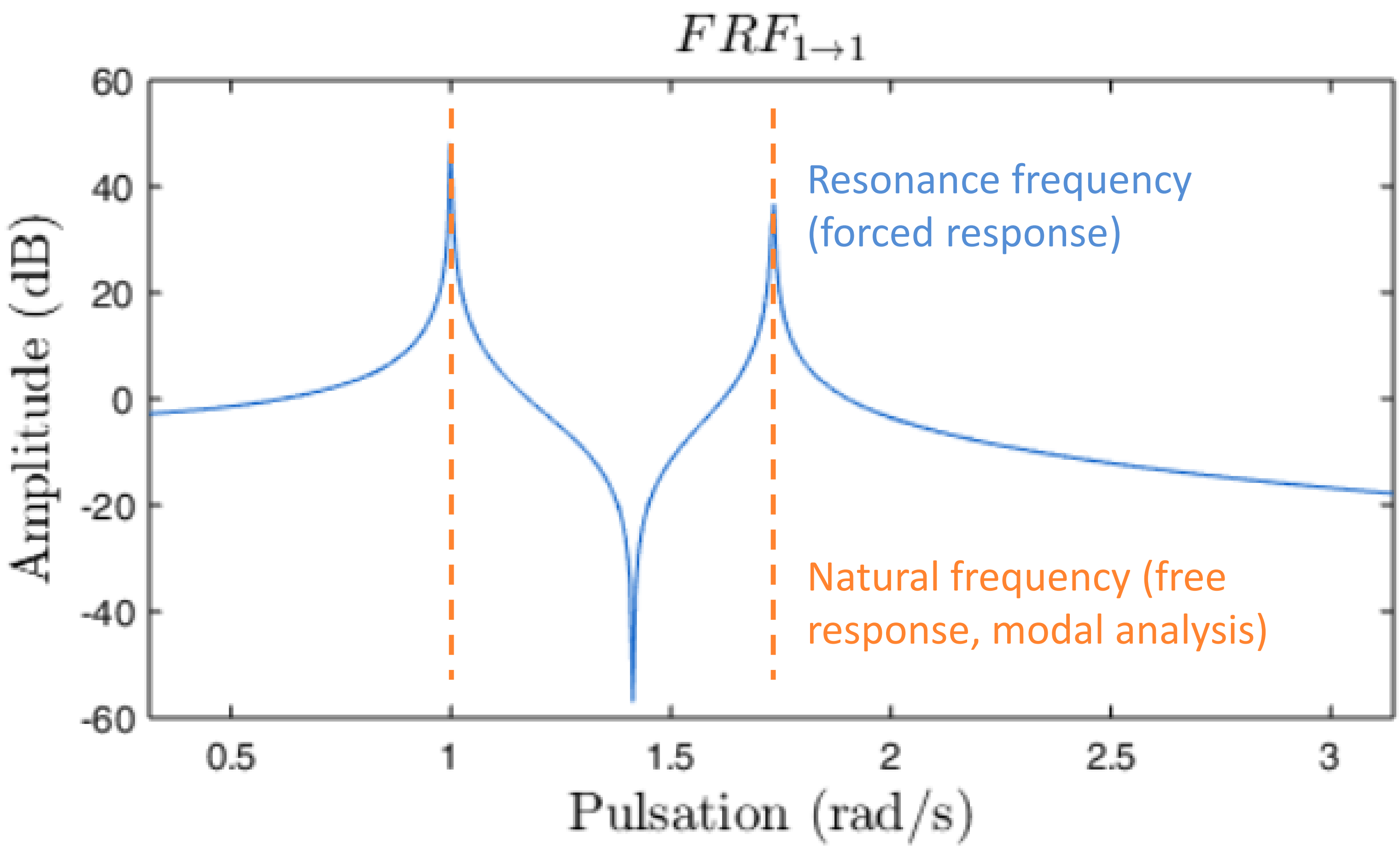
$$\omega^4 - 4\omega^2 + 3 = 0$$

$\omega_1 = 1$ rad/s with $A = B$,

$\omega_2 = \sqrt{3}$ rad/s with $A = -B$,

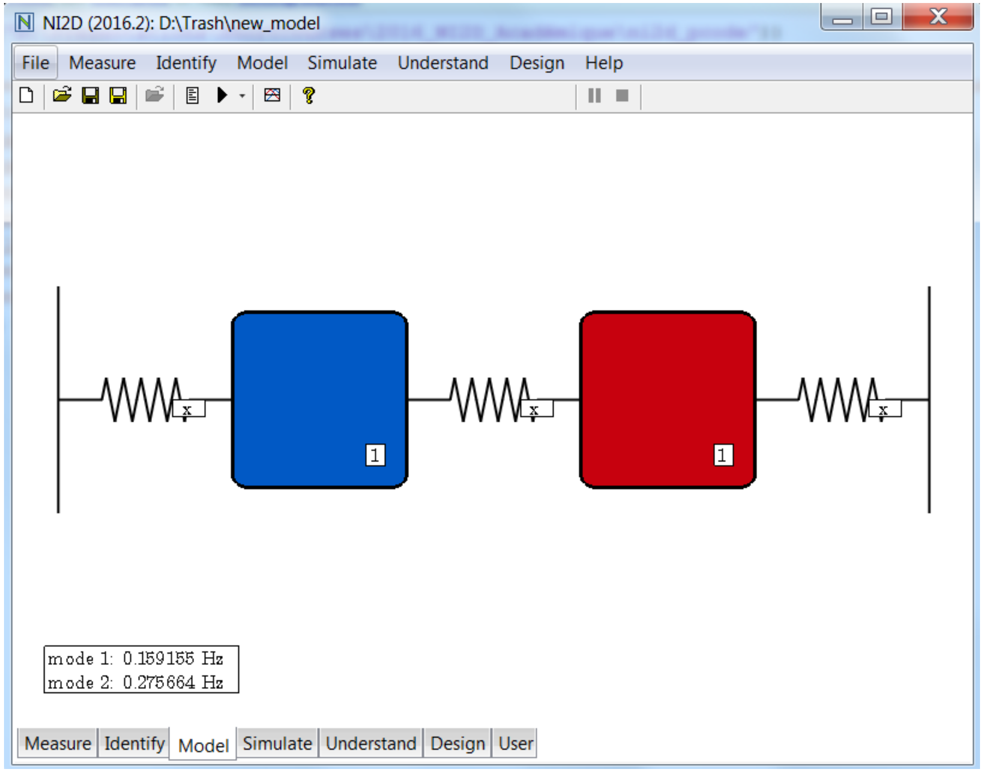
Linear modes are invariant

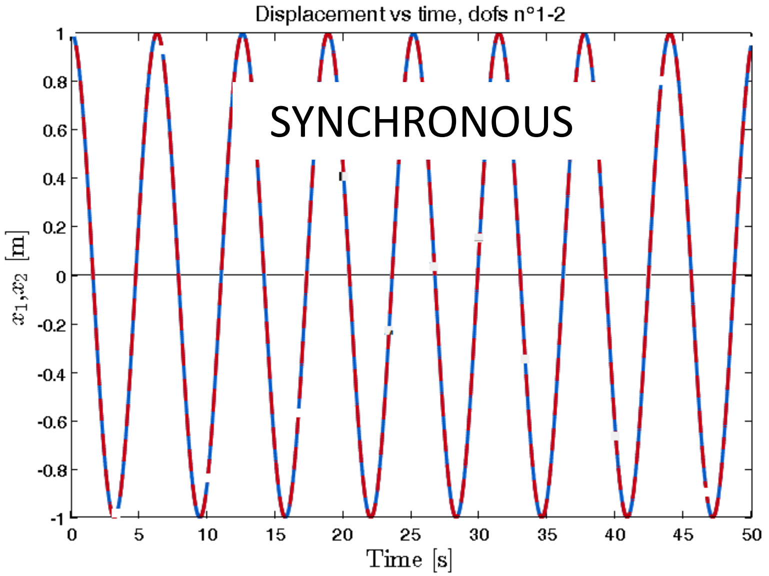
Link Between Natural and Resonance Frequencies

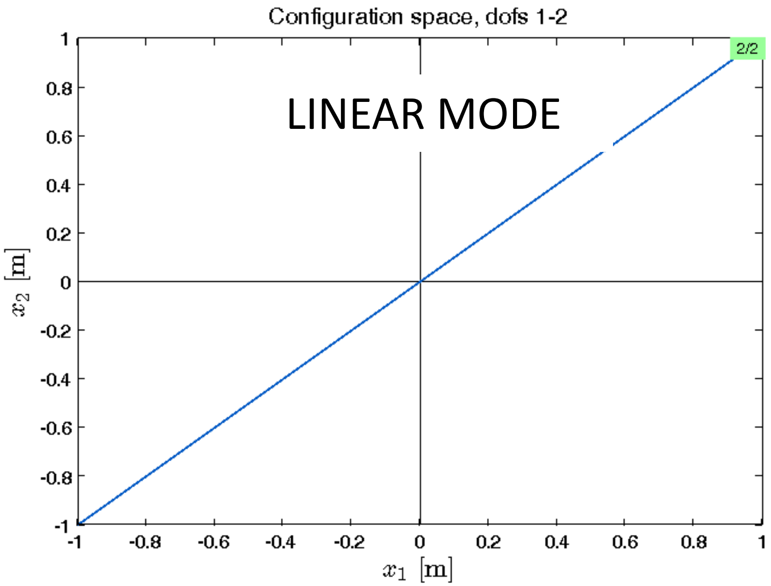


NI2D – 2DOF\Modes_Linear

Properties of Linear Modes ?

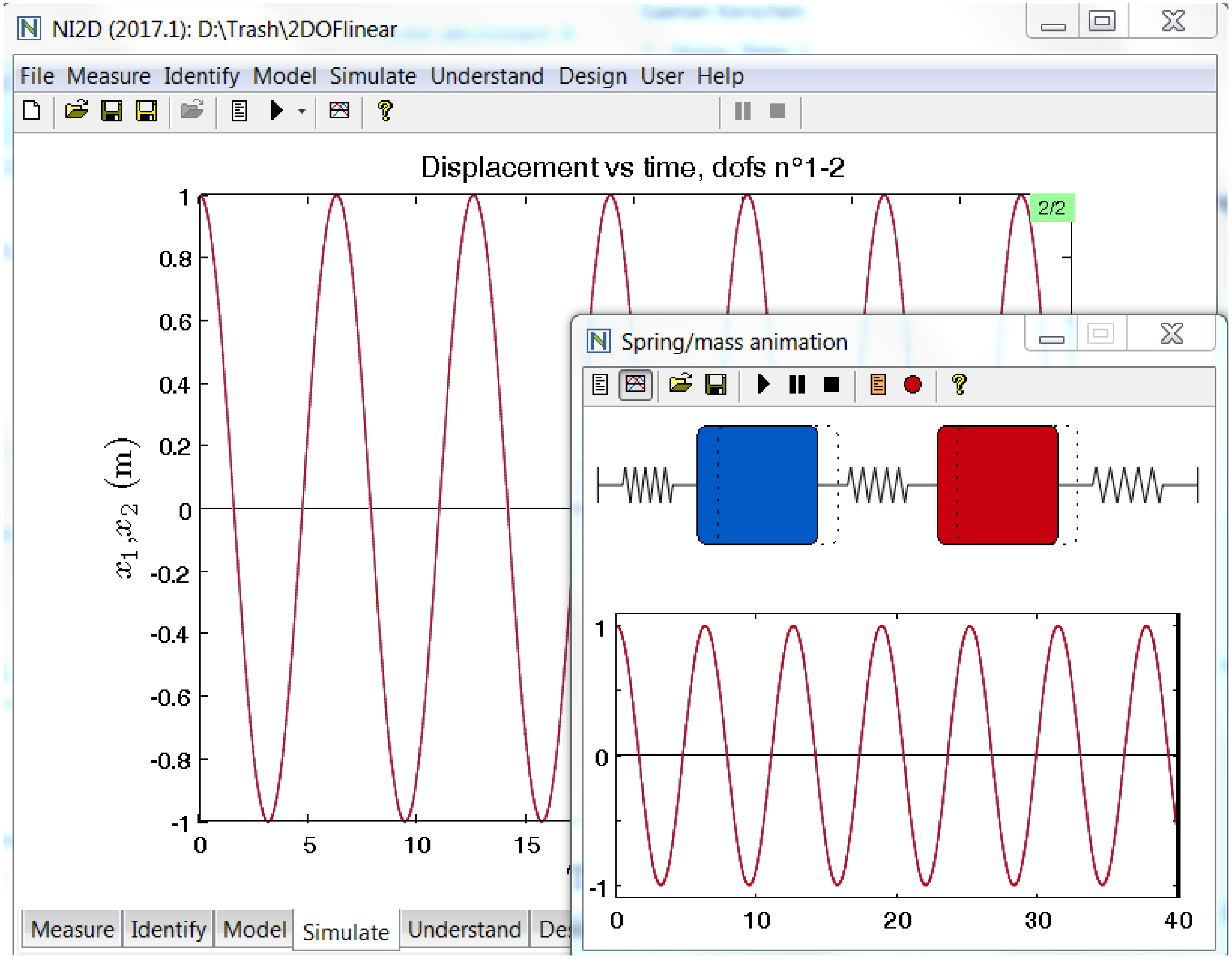






NI2D – 2DOF\Modes_Linear

Properties of Linear Modes ?



Phase Quadrature

a. ***Critère de la quadrature de phase***

Lorsque l'on réalise un essai de vibration harmonique sur un système amorti, les amplitudes des forces appliquées et les amplitudes des réponses observées aux différents points vérifient la relation complexe (3.1.18)

$$\left(\mathbf{K} - \omega^2 \mathbf{M} + i\omega \mathbf{C}\right)\mathbf{z} = \mathbf{f} \qquad (3.1.21)$$

Isoler un mode propre par une excitation appropriée revient à réaliser

$$\mathbf{z} = \mathbf{x}_{(k)} \qquad \text{et} \qquad \omega = \omega_{0k}$$

L'équation (3.1.21) devient alors

$$\left(\mathbf{K} - \omega_{0k}^2 \mathbf{M} + i\omega_{0k} \mathbf{C}\right)\mathbf{x}_{(k)} = \mathbf{f}_{(k)} \qquad (3.1.22)$$

où $\mathbf{f}_{(k)}$ est le mode de sollicitation qui permet de réaliser l'excitation appropriée. ω_{0k}^2 et $\mathbf{x}_{(k)}$ étant solutions propres du système conservatif associé, on a

$$\left(\mathbf{K} - \omega_{0k}^2 \mathbf{M}\right)\mathbf{x}_{(k)} = \mathbf{0}$$

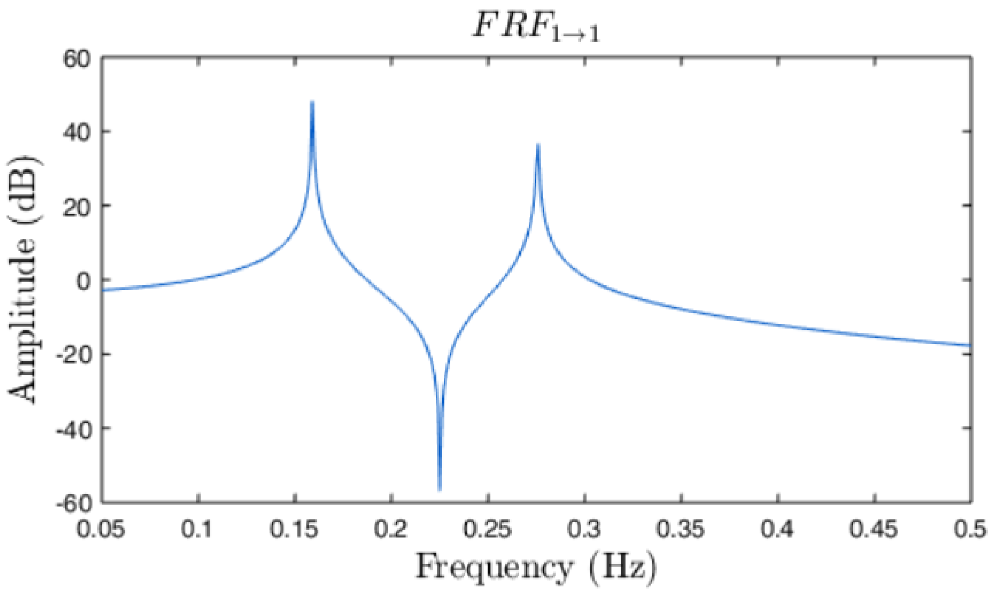
et l'expression de la sollicitation qui permet d'exciter le mode $\mathbf{x}_{(k)}$ à sa fréquence de résonance découle de l'équation (3.1.22)

$$\mathbf{f}_{(k)} = i\omega_{0k} \mathbf{C} \mathbf{x}_{(k)} \qquad (3.1.23)$$

Elle montre que la sollicitation est alors en phase avec les forces de dissipation et présente donc un déphasage de 90° par rapport à la réponse.

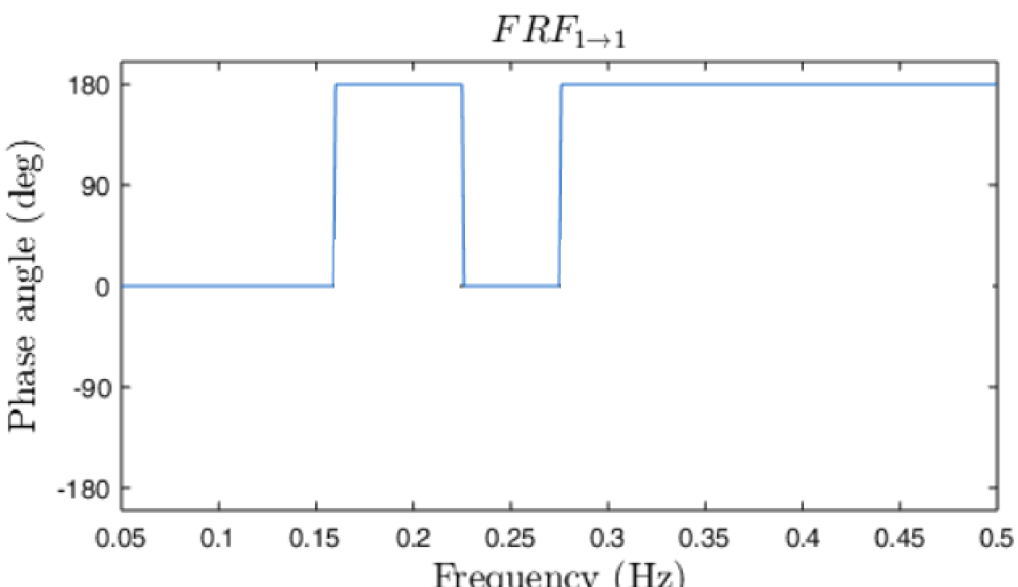
Cours de théorie des vibrations

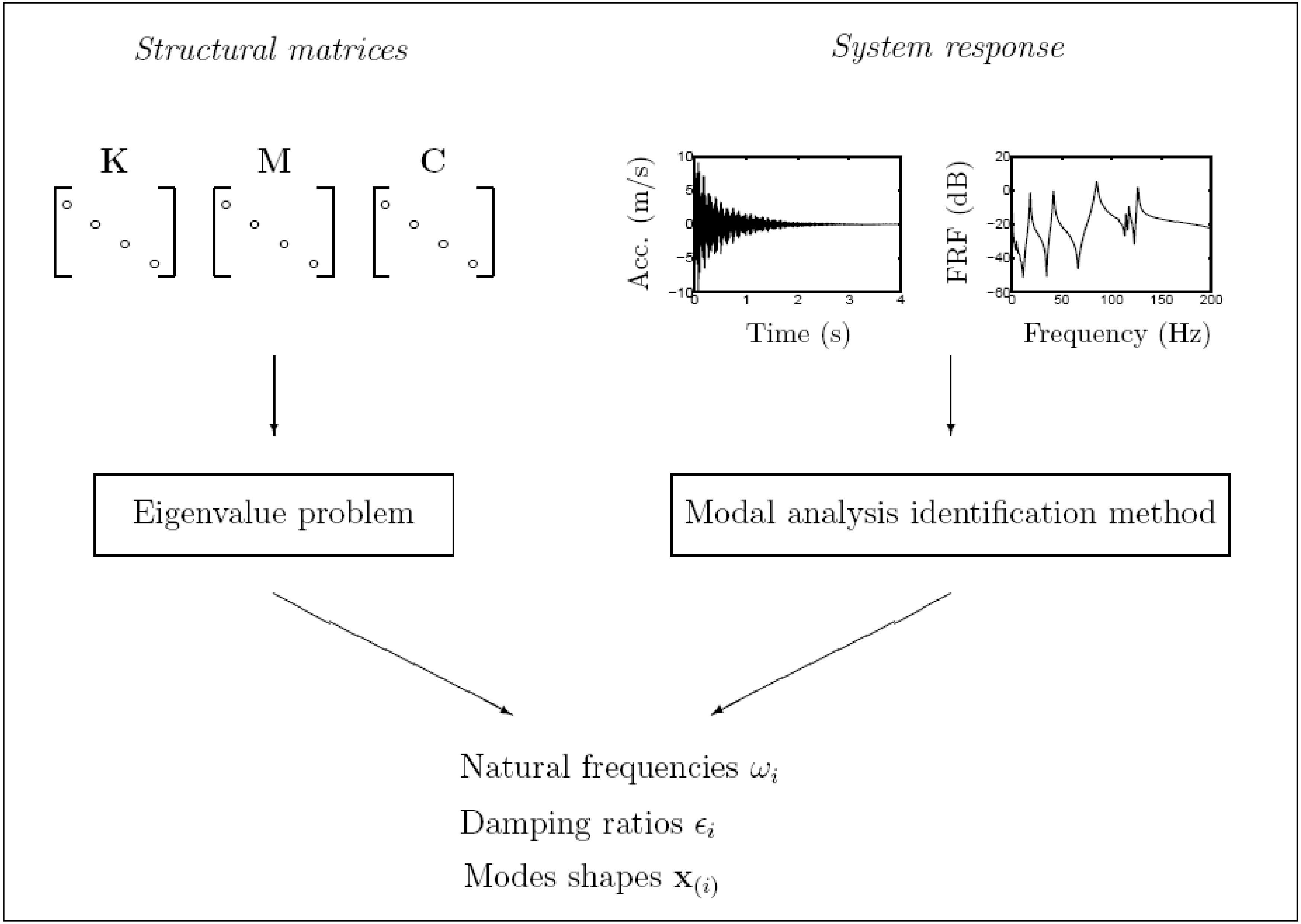
Phase Quadrature



Le *critère de la quadrature de phase* peut donc s'énoncer ainsi :

> *la structure vibre selon un des modes propres du système conservatif associé* si et seulement si *tous les points vibrent avec la même phase et sont déphasés de $\pi/2$ par rapport à l'excitation.*



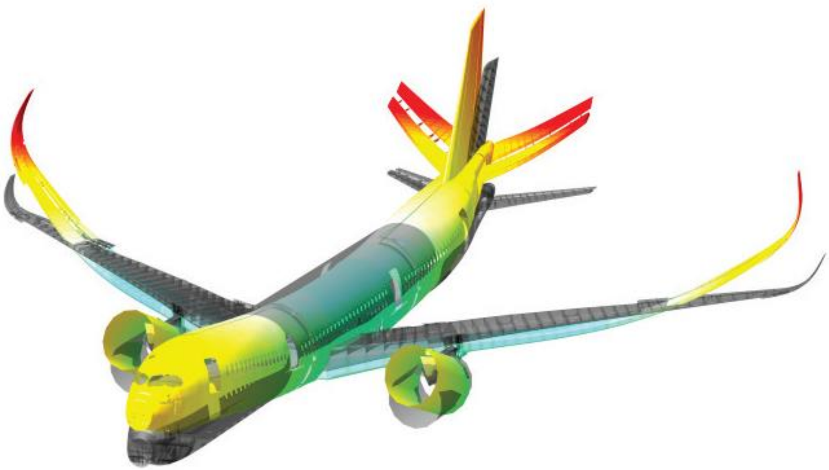


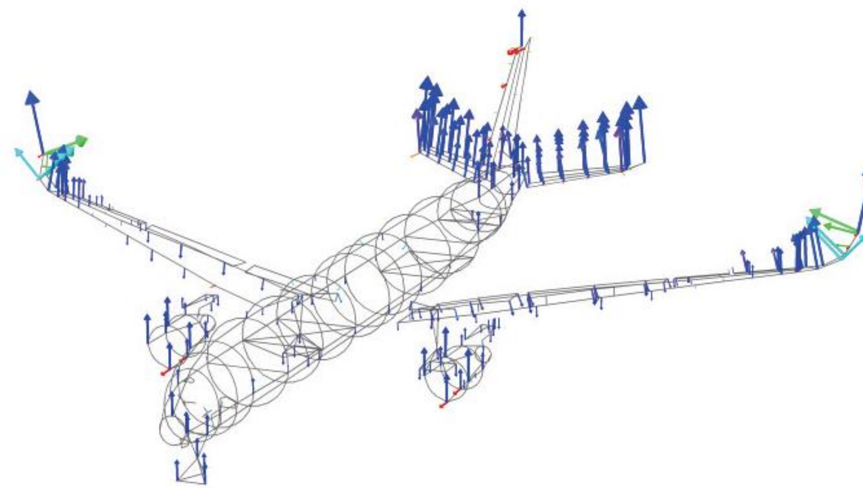
Theoretical and experimental modal analysis (TMA and EMA)

Industrial Use of the Concept of Modes

AIRBUS A350XWB

Govers et al., ISMA 2014.





- Design is based on shaping resonances

  (eigenvalue solver in all commercial FE packages).

- Certification is based on measuring resonances

  (stochastic subspace identification, eigensystem realization, PolyMAX, …).

Outline of this Lecture

What are nonlinear modes ?

What are their fundamental properties ?

Link between modes and resonance frequencies

A tutorial

Lyapunov: Cornerstone of Nonlinear Mode Theory

For n-DOF conservative systems with no internal resonances, there exist at least n different families of periodic solutions around the stable equilibrium point of the system.

Lyapunov: Cornerstone of Nonlinear Mode Theory

For n-DOF conservative systems with no internal resonances, there exist at least n different families of periodic solutions around the stable equilibrium point of the system.

At low energy, the periodic solutions of each family are in the neighborhood of a LNM of the linearized system. These n families define n NNMs that can be regarded as nonlinear extensions of the n LNMs of the underlying linear system.

Rosenberg (1960s): Nonlinear Normal Modes

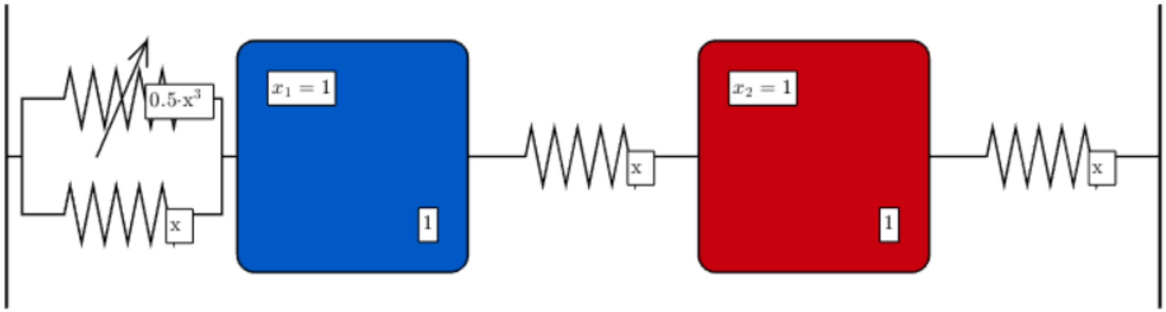
An NNM is a synchronous vibration of the system:

- All material points of the system reach their extreme values and pass through zero simultaneously.
- The system behaves like a nonlinear single-DOF system when it vibrates along an NNM.

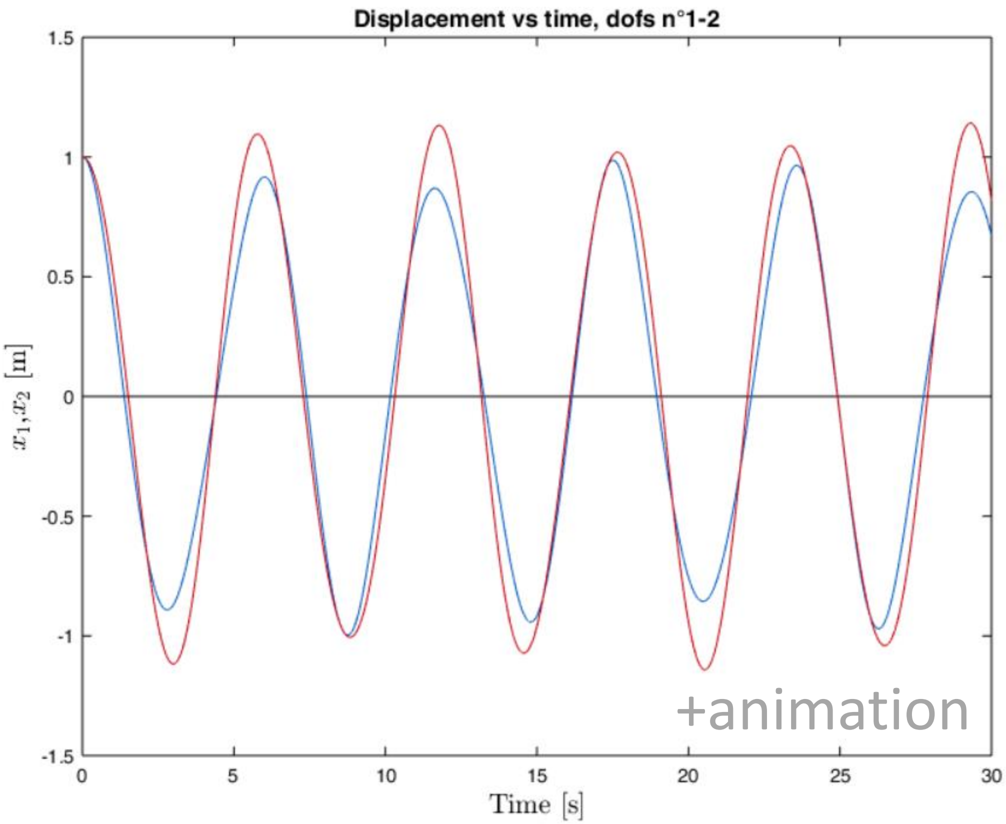
$$M\ddot{x}(t) + Kx(t) = 0 \qquad\qquad M\ddot{x}(t) + Kx(t) + \boxed{f_{NL}[x(t)]} = 0$$

LNM: synchronous periodic motion. ▶ NNM: synchronous periodic motion.

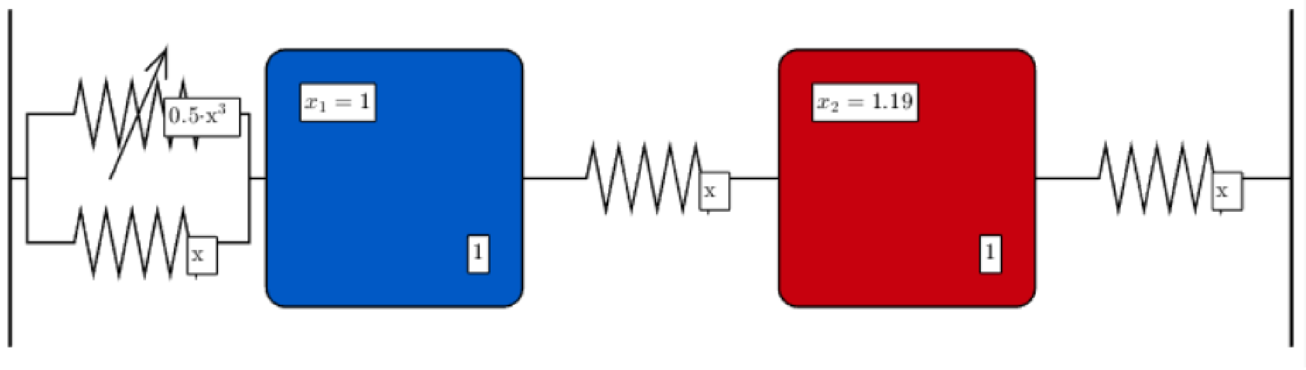
Is This a Nonlinear Mode ?

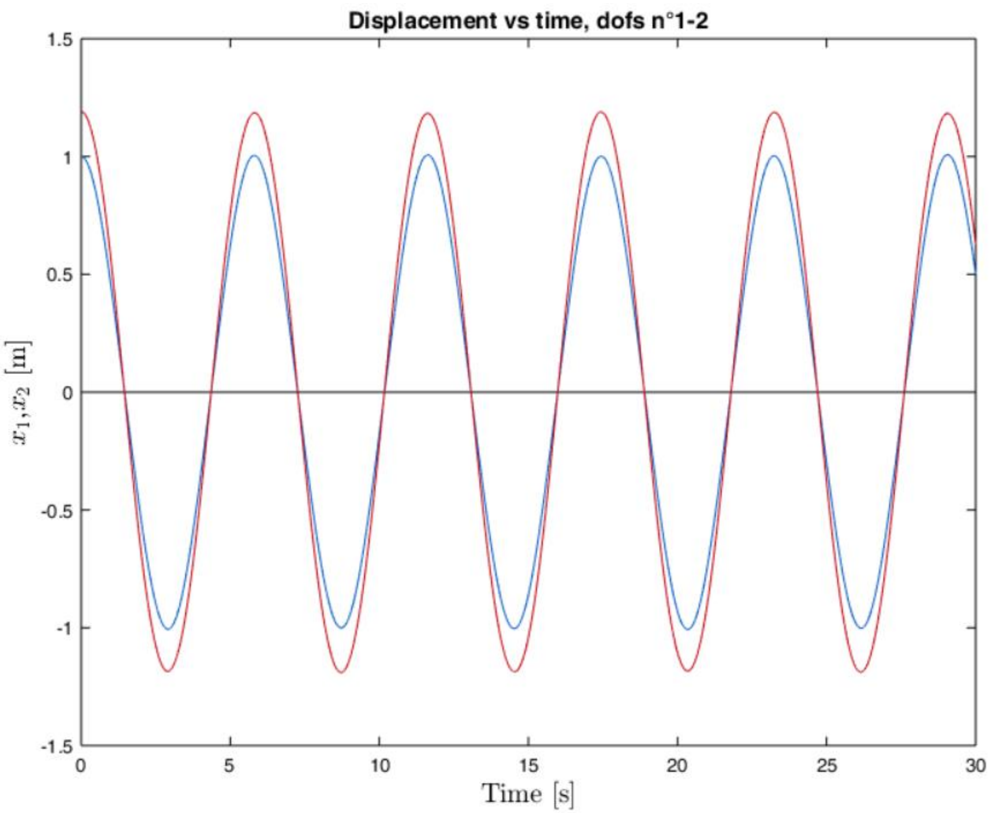


NI2D – 2DOF\NNM_ShawSystem

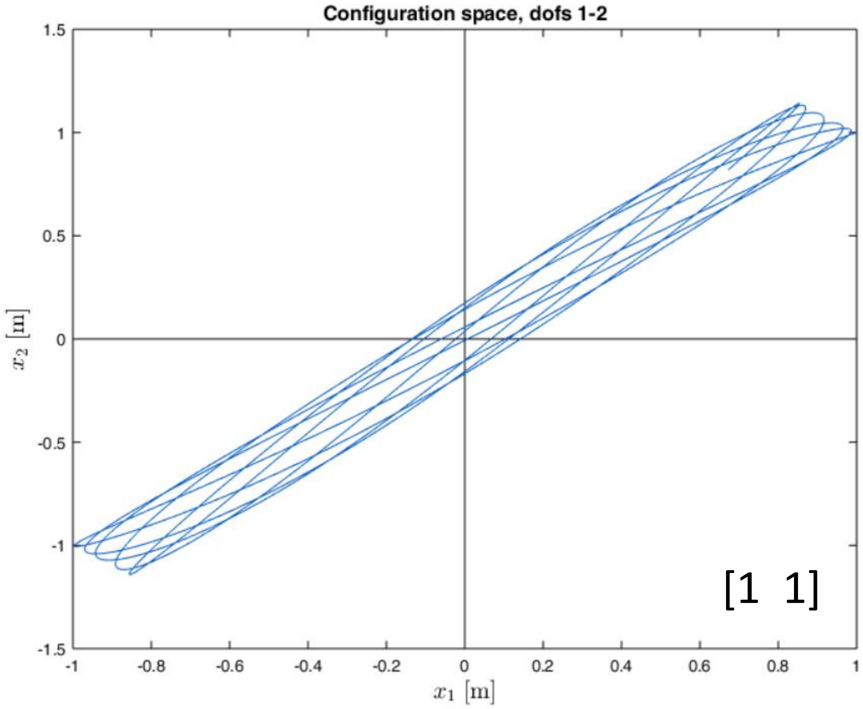


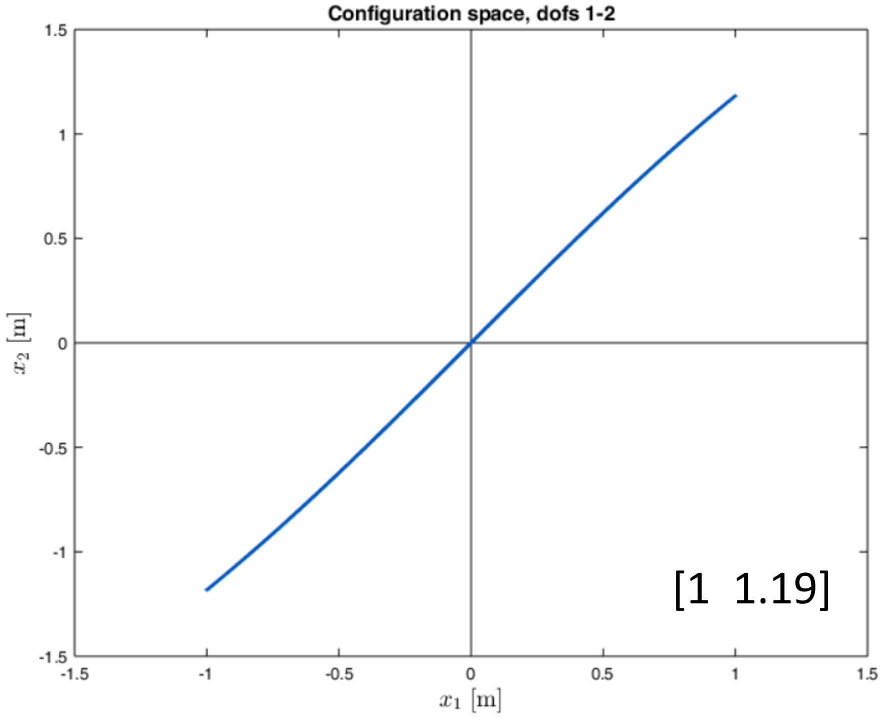
Is This a Nonlinear Mode ?



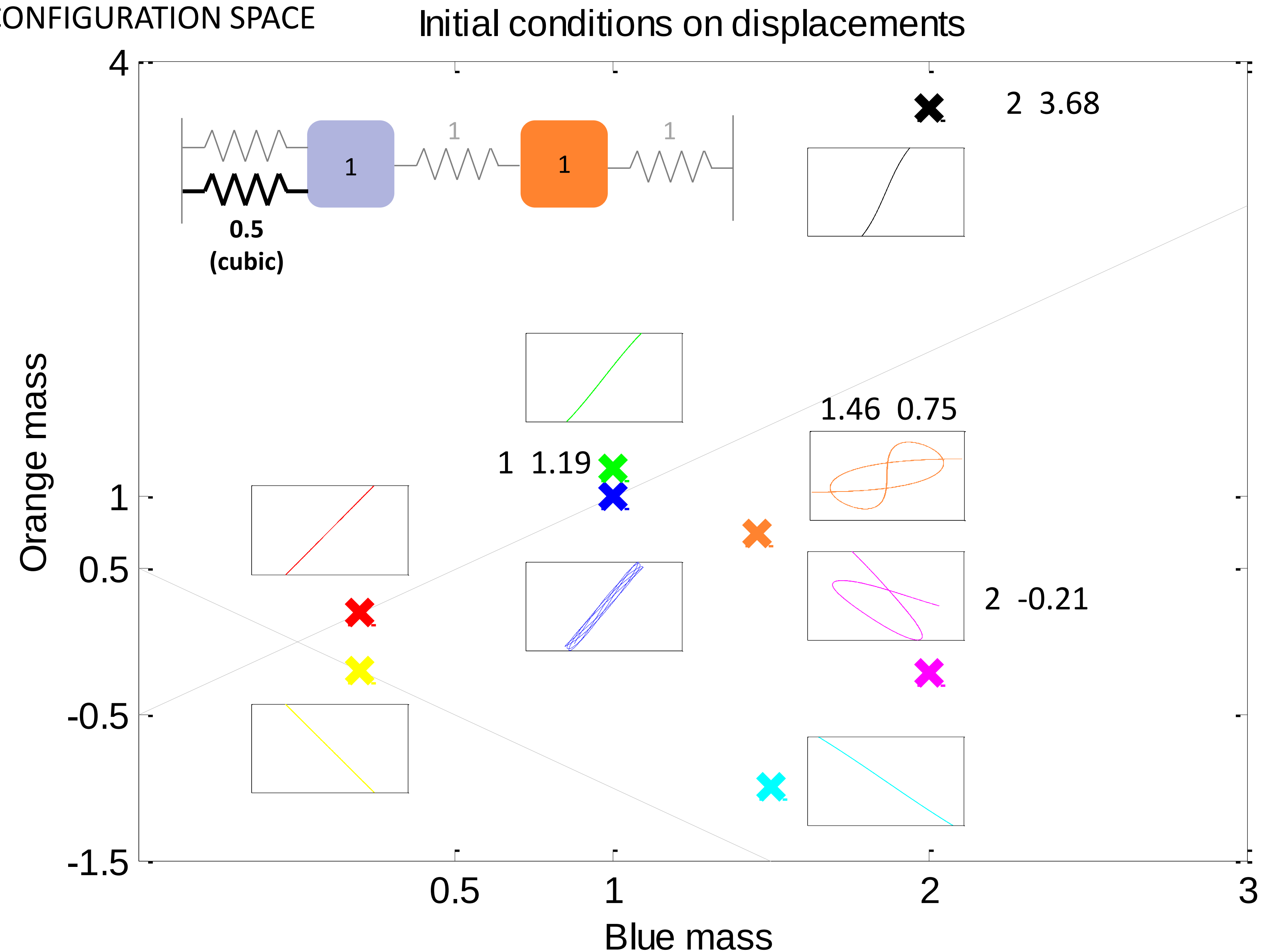


Is This a Nonlinear Mode ?





NI2D DEMO WITH CONFIGURATION SPACE



How Do We Calculate Nonlinear Normal Modes ?

$$\ddot{q}_1 + (2q_1 - q_2) + 0.5q_1^3 = 0$$
$$\ddot{q}_2 + (2q_2 - q_1) = 0$$

How Do We Calculate Nonlinear Normal Modes ?

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\Rightarrow \cos^3\theta = \frac{1}{4}\left(\cos 3\theta + 3\cos\theta\right)$$

$$\ddot{q}_1 + 2q_1 - q_2 + \frac{1}{2}q_1^3 = 0 \longrightarrow -\omega^2 A + 2A - B + \frac{3}{8}A^3 = 0 \quad (1)$$

$$\ddot{q}_2 + 2q_2 - q_1 = 0 \qquad -\omega^2 B + 2B - A = 0 \longrightarrow \boxed{B = \frac{A}{2-\omega^2}} \quad (2)$$

$$(1)\ \&\ (2) \Rightarrow -\omega^2 A + 2A - \frac{A}{2-\omega^2} + \frac{3}{8}A^3 = 0$$

$$\Rightarrow -\omega^2 + 2 - \frac{1}{2-\omega^2} + \frac{3}{8}A^2 = 0$$

$$\Rightarrow -2\omega^2 + \omega^4 + 4 - 2\omega^2 - 1 + \frac{3}{4}A^2 - \frac{3}{8}A^2\omega^2 = 0$$

$$\Rightarrow A^2\left(\frac{3}{4} - \frac{3}{8}\omega^2\right) = -\omega^4 + 4\omega^2 - 3$$

$$\Rightarrow \boxed{A = \pm\sqrt{\frac{8\left(\omega^2-1\right)\left(\omega^2-3\right)}{3\left(\omega^2-2\right)}}}$$

Fundamental Difference Between LNMs and NNMs

$$\ddot{q}_1 + (2q_1 - q_2) = 0$$
$$\ddot{q}_2 + (2q_2 - q_1) = 0$$

$q_{1,2} = A, B cos\omega t$

$$A = B,\ \ \omega_1 = 1\ \text{rad/s}$$

$$A = -B,\ \ \omega_2 = \sqrt{3}\ \text{rad/s}$$

$$\ddot{q}_1 + (2q_1 - q_2) + 0.5q_1^3 = 0$$
$$\ddot{q}_2 + (2q_2 - q_1) = 0$$

$q_{1,2} \cong A, B cos\omega t$

$$A = \pm\sqrt{\frac{8(\omega^2 - 3)(\omega^2 - 1)}{3(\omega^2 - 2)}}$$

$$B = \frac{A}{2 - \omega^2}$$

2 fundamental differences !

Which ones ?

Fundamental Difference Between LNMs and NNMs

$$\ddot{q}_1 + (2q_1 - q_2) = 0$$
$$\ddot{q}_2 + (2q_2 - q_1) = 0$$

$q_{1,2} = A, B cos\omega t$

$$A = B,\ \ \omega_1 = 1 \text{ rad/s}$$

$$A = -B,\ \ \omega_2 = \sqrt{3} \text{ rad/s}$$

$$\ddot{q}_1 + (2q_1 - q_2) + 0.5q_1^3 = 0$$
$$\ddot{q}_2 + (2q_2 - q_1) = 0$$

$q_{1,2} \cong A, B cos\omega t$

$$A = \pm\sqrt{\frac{8(\omega^2 - 3)(\omega^2 - 1)}{3(\omega^2 - 2)}}$$

$$B = \frac{A}{2 - \omega^2}$$

1. Modal shapes depend on frequency

Fundamental Difference Between LNMs and NNMs

$$\ddot{q}_1 + (2q_1 - q_2) = 0$$
$$\ddot{q}_2 + (2q_2 - q_1) = 0$$

$q_{1,2} = A, B cos\omega t$

$$A = B,\ \ \omega_1 = 1\ \mathrm{rad/s}$$

$$A = -B,\ \ \omega_2 = \sqrt{3}\ \mathrm{rad/s}$$

$$\ddot{q}_1 + (2q_1 - q_2) + 0.5q_1^3 = 0$$
$$\ddot{q}_2 + (2q_2 - q_1) = 0$$

$q_{1,2} \cong A, B cos\omega t$

$$A = \pm\sqrt{\frac{8(\omega^2 - 3)(\omega^2 - 1)}{3(\omega^2 - 2)}}$$

$$B = \frac{A}{2 - \omega^2}$$

2. The natural frequency is not fixed (but existence conditions !)

Fundamental Difference Between LNMs and NNMs

$$\ddot{q}_1 + (2q_1 - q_2) = 0$$
$$\ddot{q}_2 + (2q_2 - q_1) = 0$$

$q_{1,2} = A, B cos\omega t$

$$A = B,\ \omega_1 = 1 \text{ rad/s}$$

$$A = -B,\ \omega_2 = \sqrt{3} \text{ rad/s}$$

$$\ddot{q}_1 + (2q_1 - q_2) + 0.5q_1^3 = 0$$
$$\ddot{q}_2 + (2q_2 - q_1) = 0$$

$q_{1,2} \cong A, B cos\omega t$

$$A = \pm\sqrt{\frac{8(\omega^2 - 3)(\omega^2 - 1)}{3(\omega^2 - 2)}}$$

$$B = \frac{A}{2 - \omega^2}$$

$$\omega_1 \in \left[1, \sqrt{2}\right[\text{ rad/s}$$

$$\omega_2 \in \left[\sqrt{3}, +\infty\right[\text{ rad/s}$$

Existence conditions for NNM

Useful Graphical Representation

$$\ddot{q}_1 + (2q_1 - q_2) + 0.5q_1^3 = 0$$
$$\ddot{q}_2 + (2q_2 - q_1) = 0$$

Initial conditions: $[q_1(0) \quad q_2(0) \quad \dot{q}_1(0) \quad \dot{q}_2(0)] = [A \quad B \quad 0 \quad 0]$

Total energy =
initial potential energy : $\mathrm{E} = \mathrm{V} = \dfrac{A^2}{2} + \dfrac{(B-A)^2}{2} + \dfrac{B^2}{2} + \dfrac{0.5A^4}{4}$

A frequency-energy plot is calculated by

- Selecting a frequency in the interval provided by the existence conditions,
- Calculating A and B according to the analytical formulas
- Calculating the corresponding total energy
- Representing the frequency as a function of the total energy

In Matlab

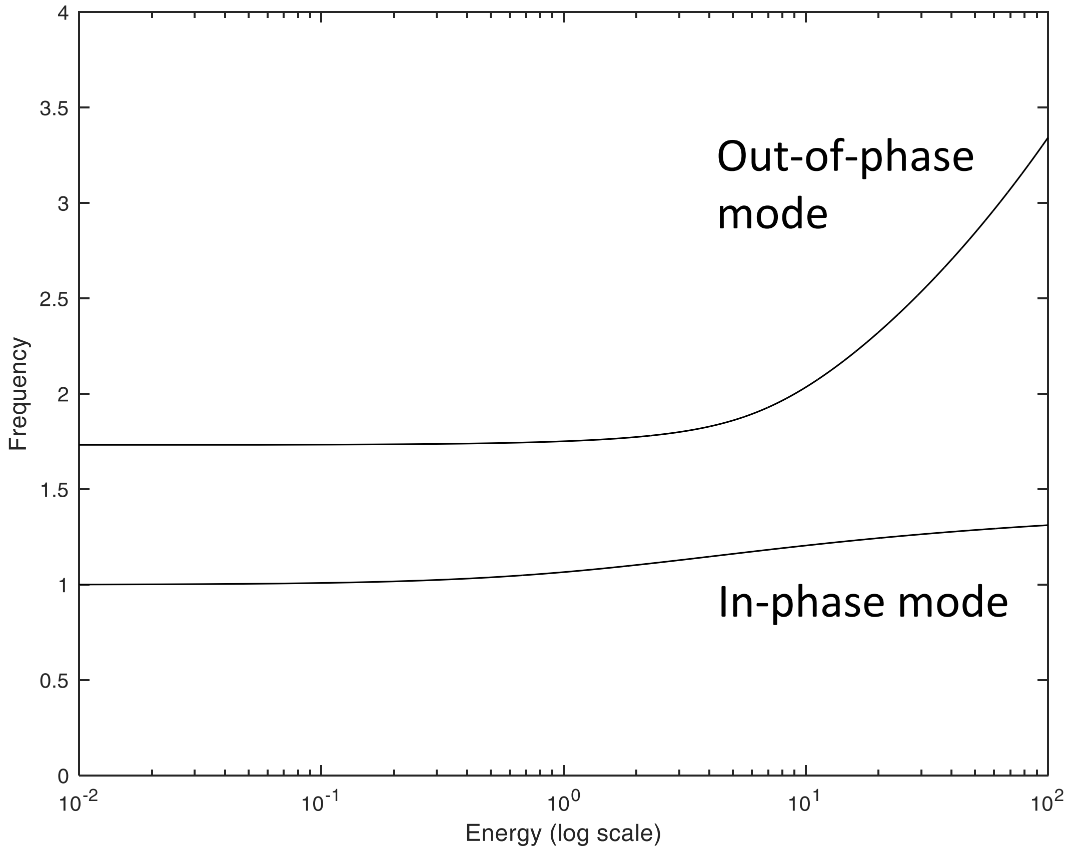
HB1_2DOF_FEP.m

```
clear all
close all

cpt=1;
for omeg=1.00001:0.001:sqrt(2)
    A=sqrt(8*(omeg^2-3)*(omeg^2-1)/3/((omeg^2-2)));
    B=A/(2-omeg^2);
    NRJ(cpt)=(A-B)^2/2+A^2/2+B^2/2+0.5*A^4/4;
    freq(cpt)=omeg;
    AIP(cpt)=A;
    BIP(cpt)=B;
    cpt=cpt+1;
end
semilogx(NRJ,freq,'k')

cpt=1;
for omeg=sqrt(3)+0.0000001:0.001:4
    A=sqrt(8*(omeg^2-3)*(omeg^2-1)/3/((omeg^2-2)));
    B=A/(2-omeg^2);
    NRJ2(cpt)=(A-B)^2/2+A^2/2+B^2/2+0.5*A^4/4;
    freq2(cpt)=omeg;
    AOP(cpt)=A;
    BOP(cpt)=B;
    cpt=cpt+1;
end
hold on
semilogx(NRJ2,freq2,'k')
```

A Frequency-Energy Plot Is a Convenient Depiction



Limitation of Analytical Calculations

$$\ddot{q}_1 + (2q_1 - q_2) + 0.5q_1^3 = 0$$
$$\ddot{q}_2 + (2q_2 - q_1) = 0$$

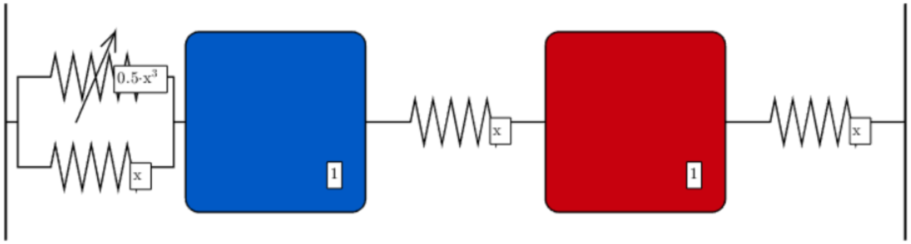
$q_{1,2} \cong A, B cos\omega t$

$$A = \pm\sqrt{\frac{8(\omega^2 - 3)(\omega^2 - 1)}{3(\omega^2 - 2)}}$$

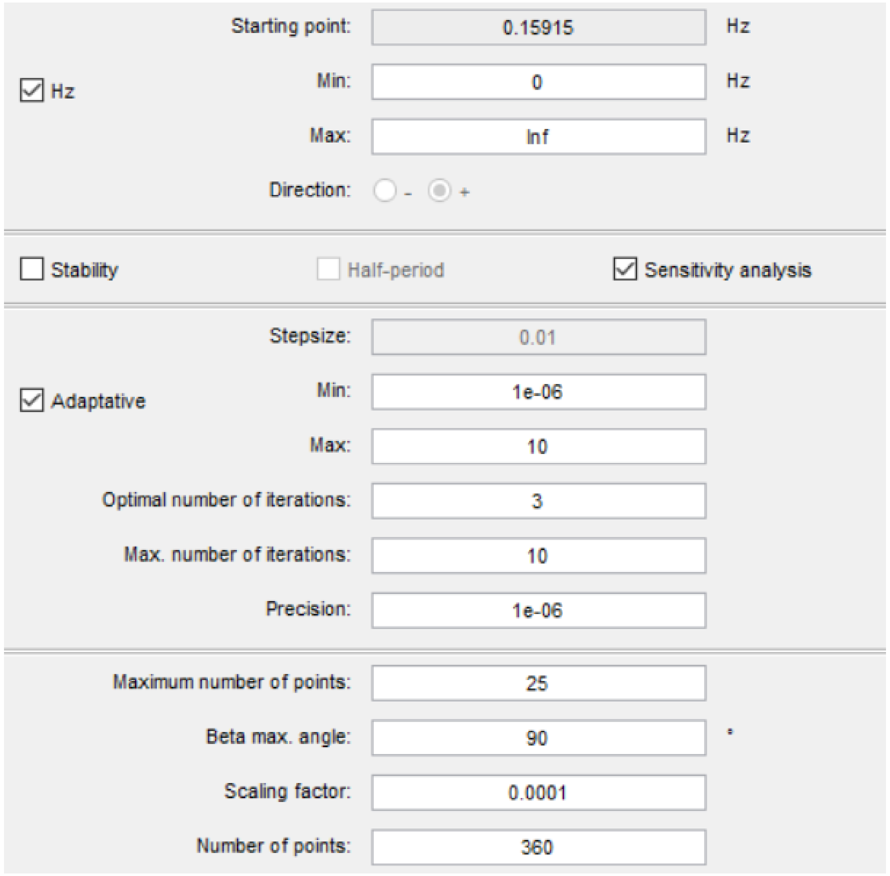
$$B = \frac{A}{2 - \omega^2}$$

A 1-term harmonic balance approximation cannot calculate the curvature of nonlinear modes

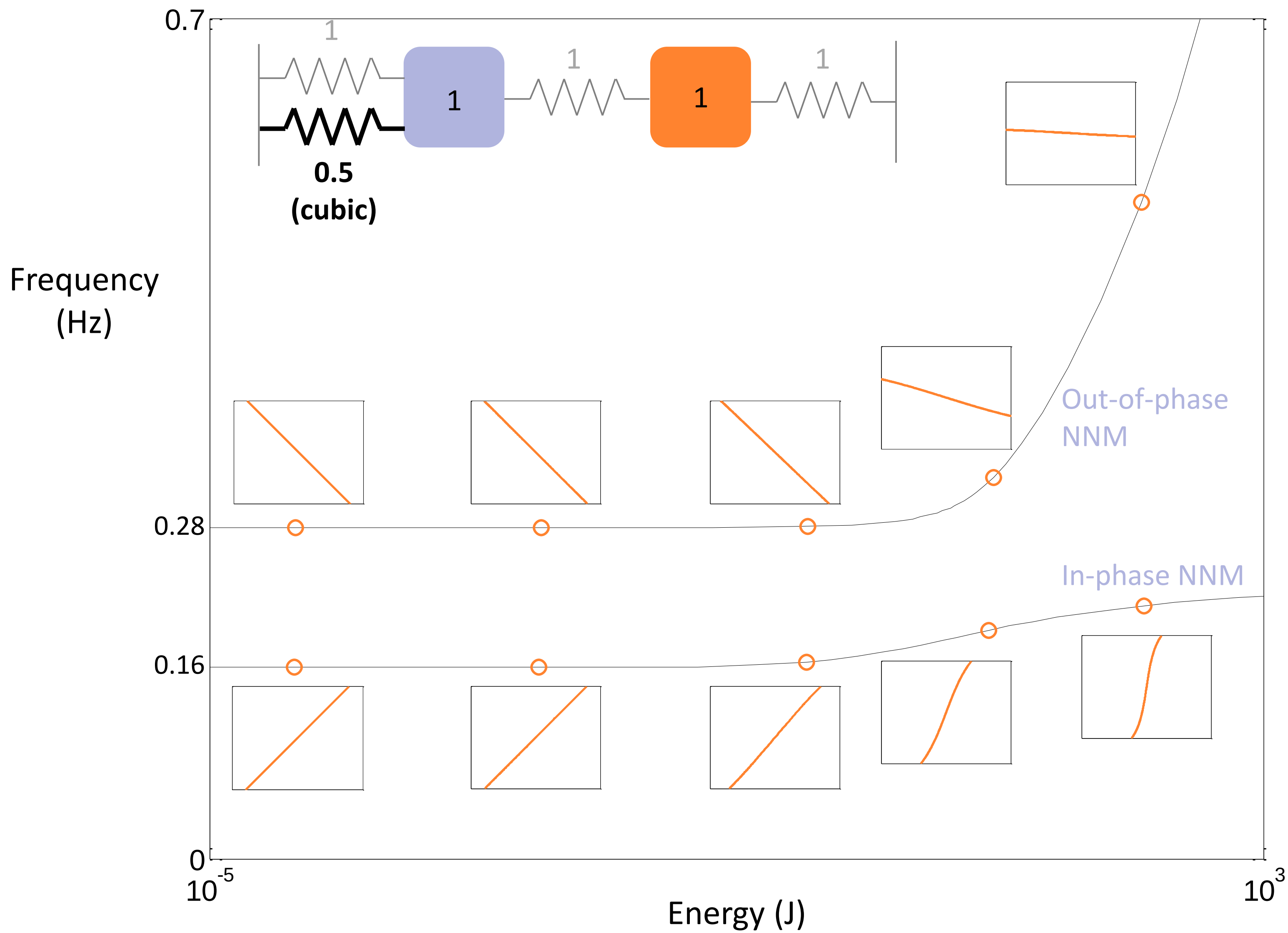
Numerical Calculation



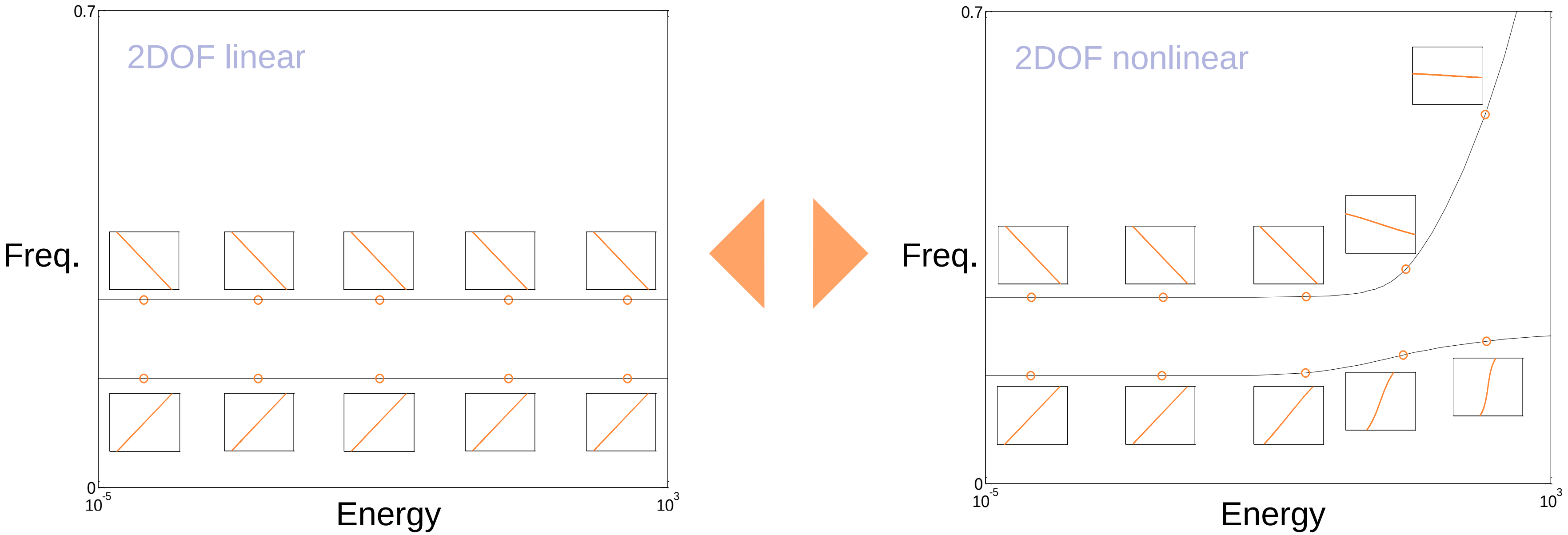
NI2D – 2DOF\NNM_ShawSystem



« Curved » Nonlinear Modes Are Now Obtained

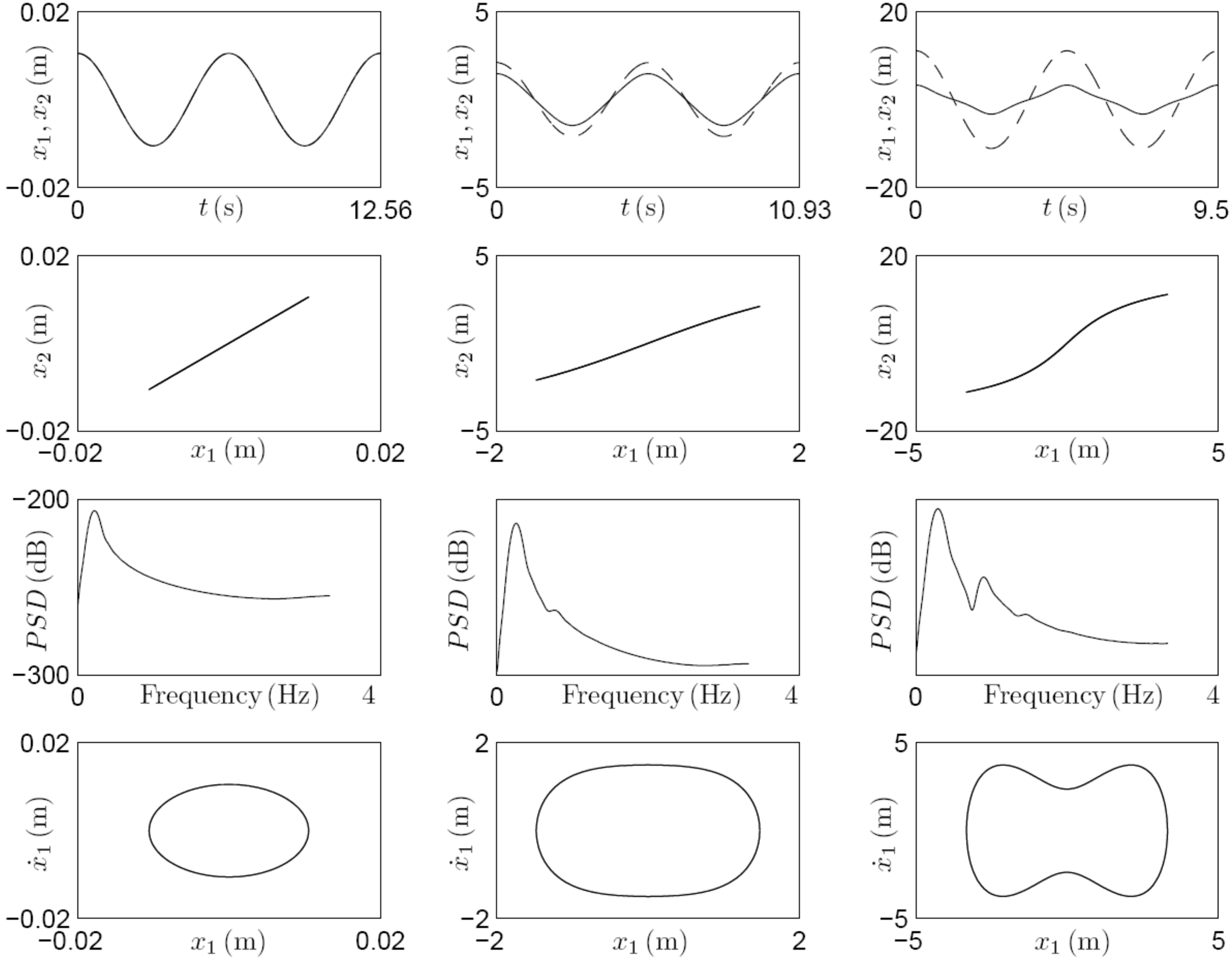


Linear Modes vs. Nonlinear Modes



But …
- Frequency-energy dependence
- And other important differences

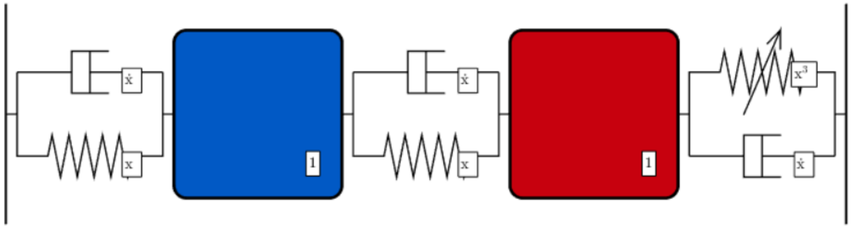
The In-Phase NNM for Increasing Energies

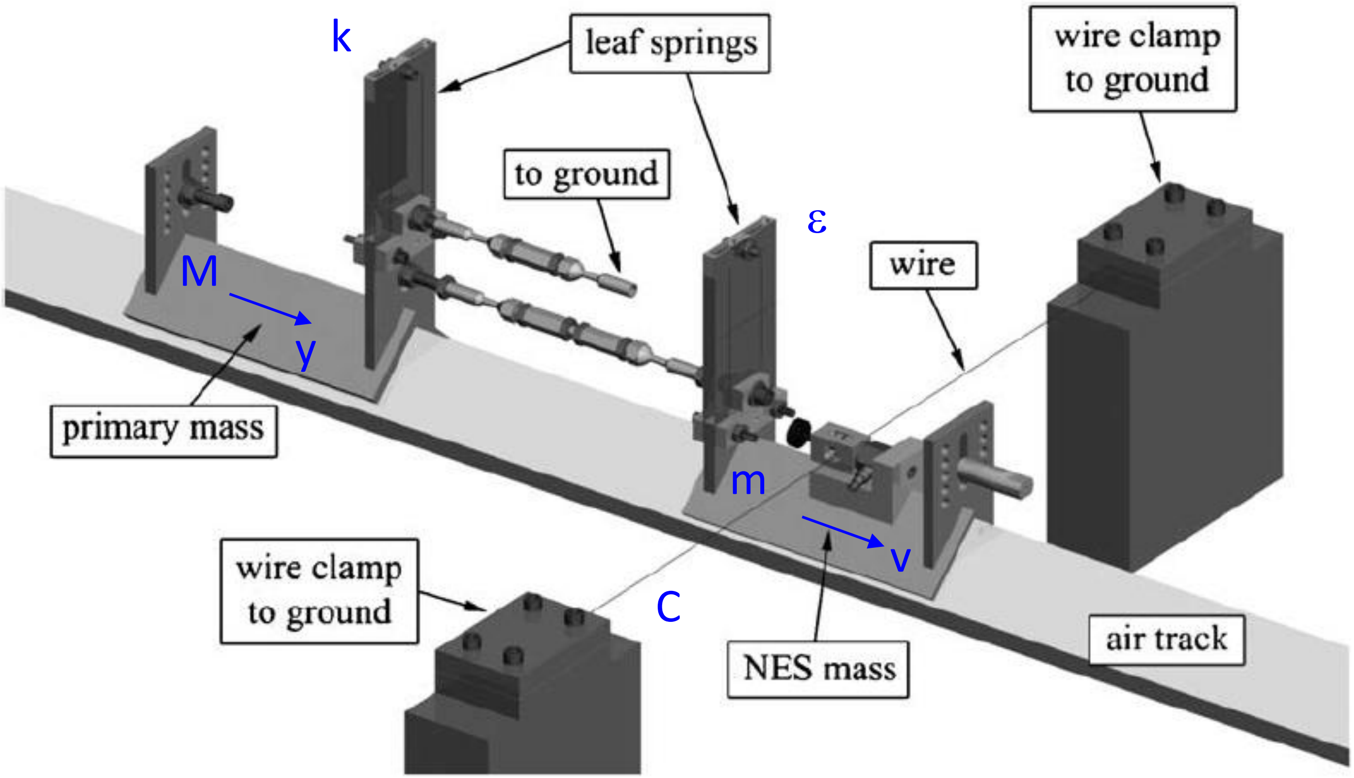


Experimental Demonstration

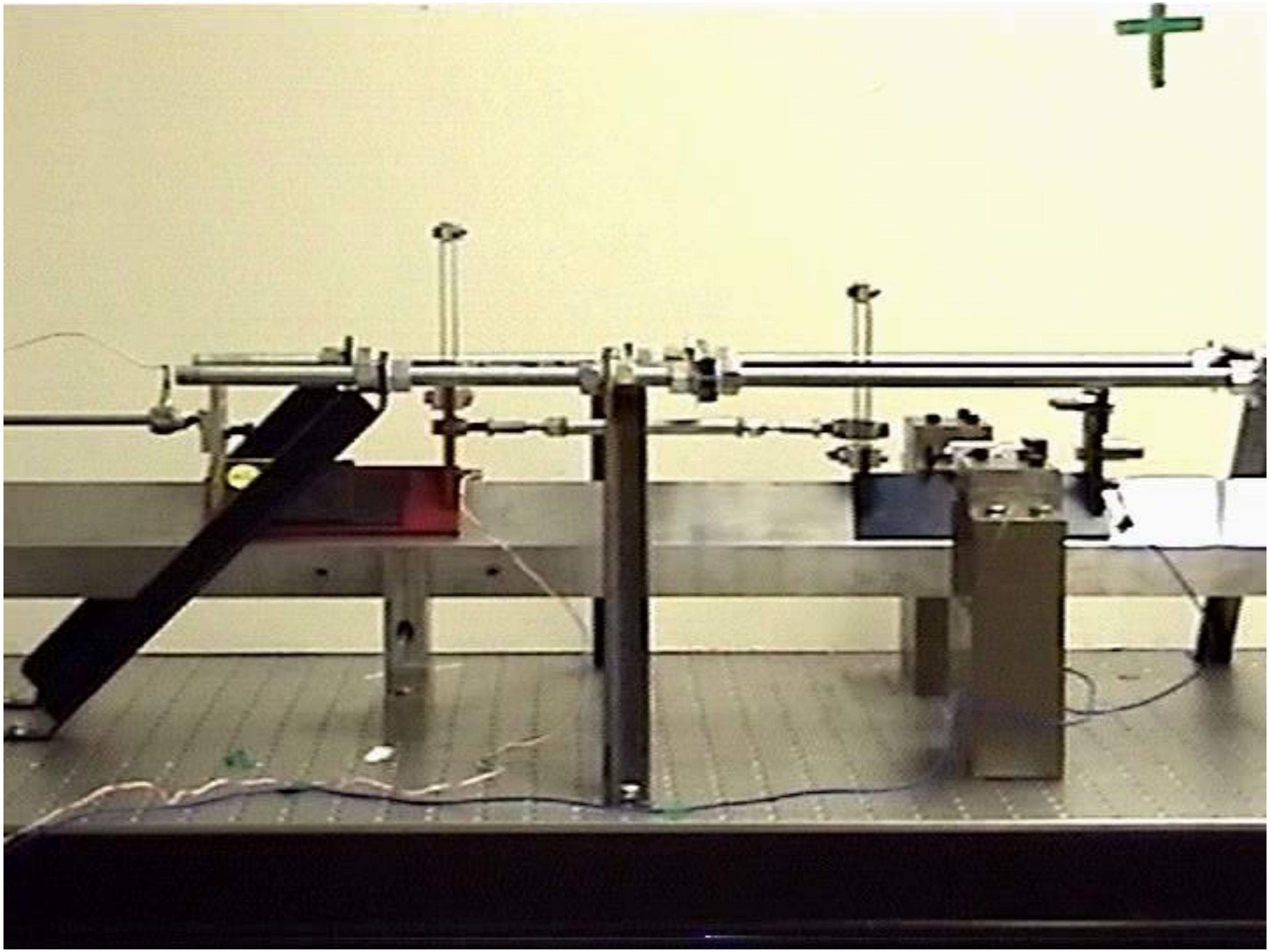
$$M\ddot{y} + \epsilon\lambda_1\dot{y} + \epsilon\lambda(\dot{y} - \dot{v}) + \epsilon(y - v) + ky = 0$$

$$m\ddot{v} + \epsilon\lambda_2\dot{v} + \epsilon\lambda(\dot{v} - \dot{y}) + \epsilon(v - y) + Cv^3 = 0$$

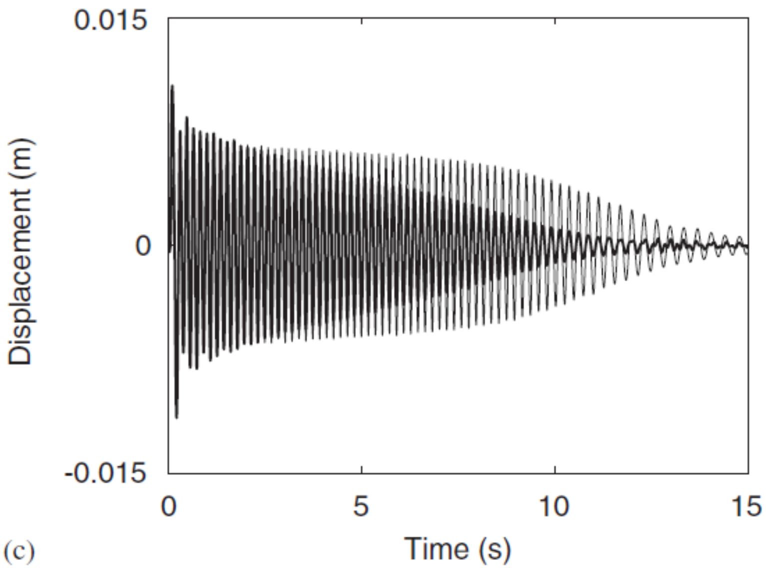




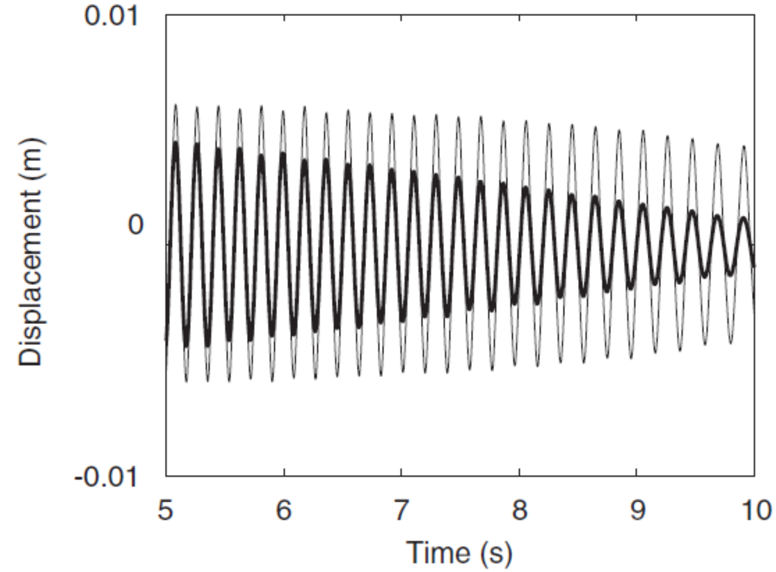
What You See Is a Nonlinear Normal Mode

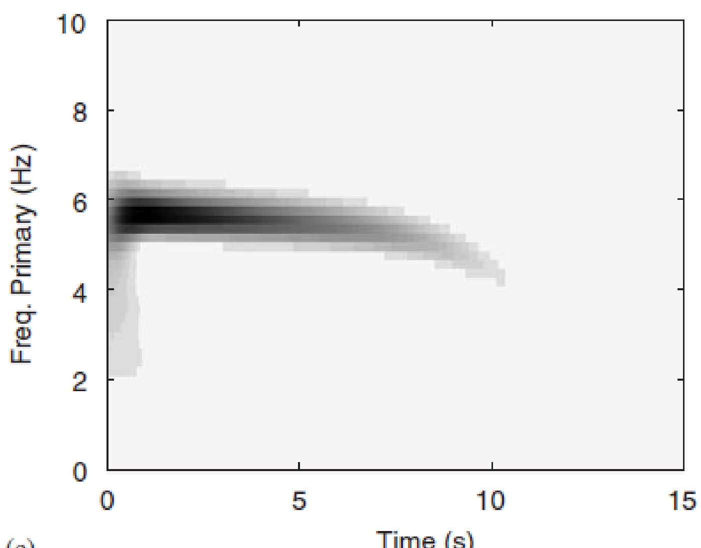


Experimental Demonstration

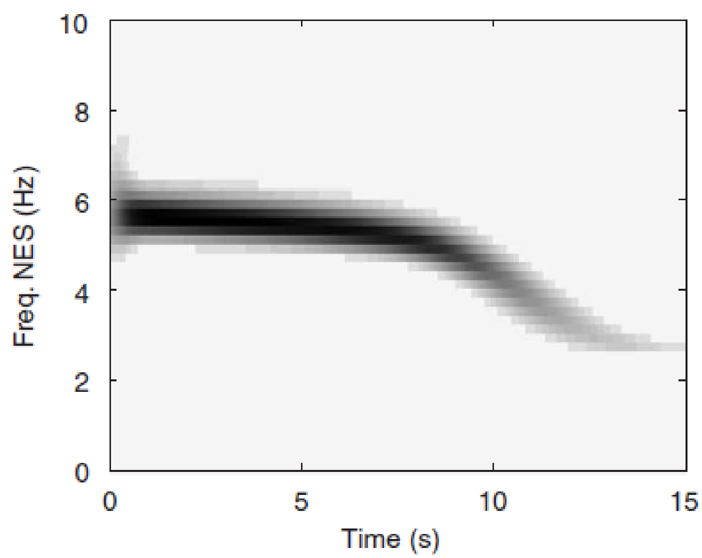


(c)

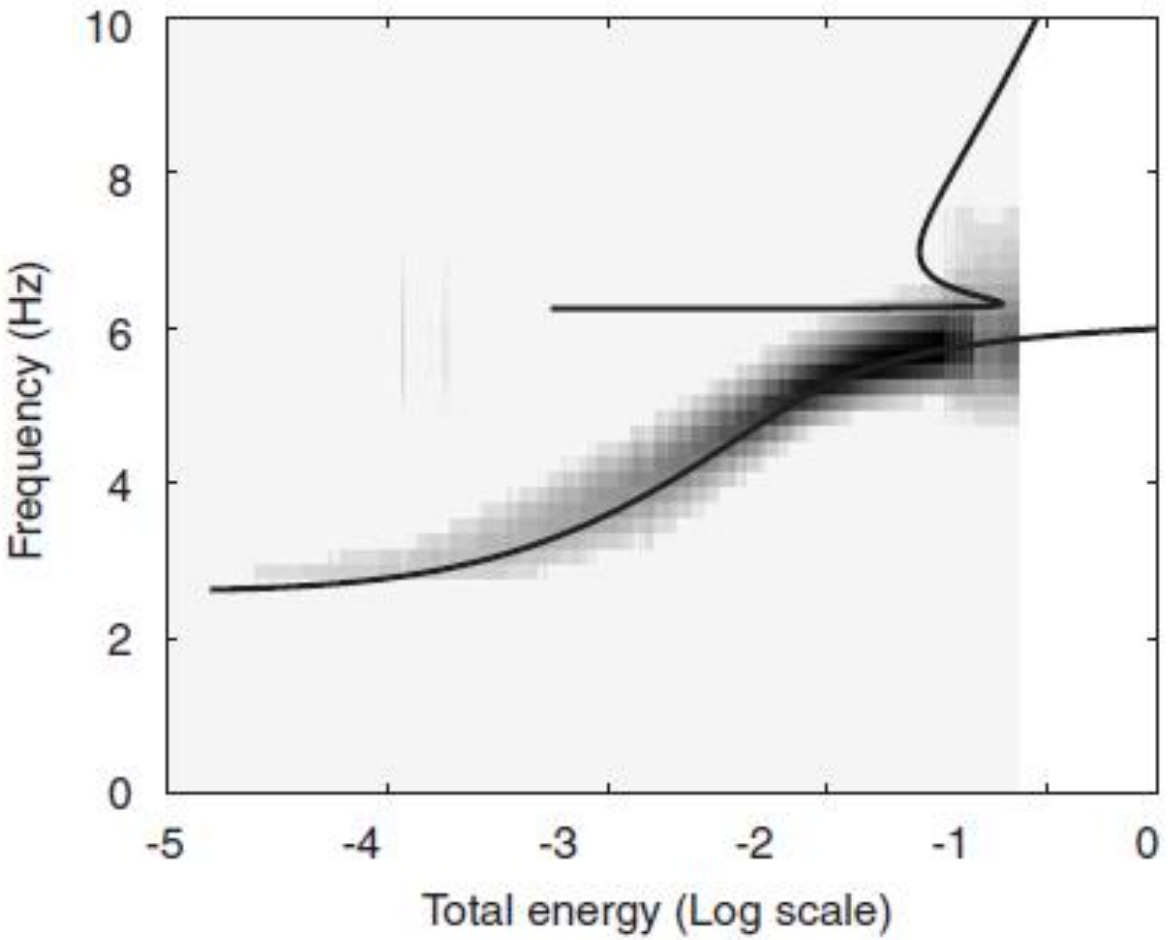




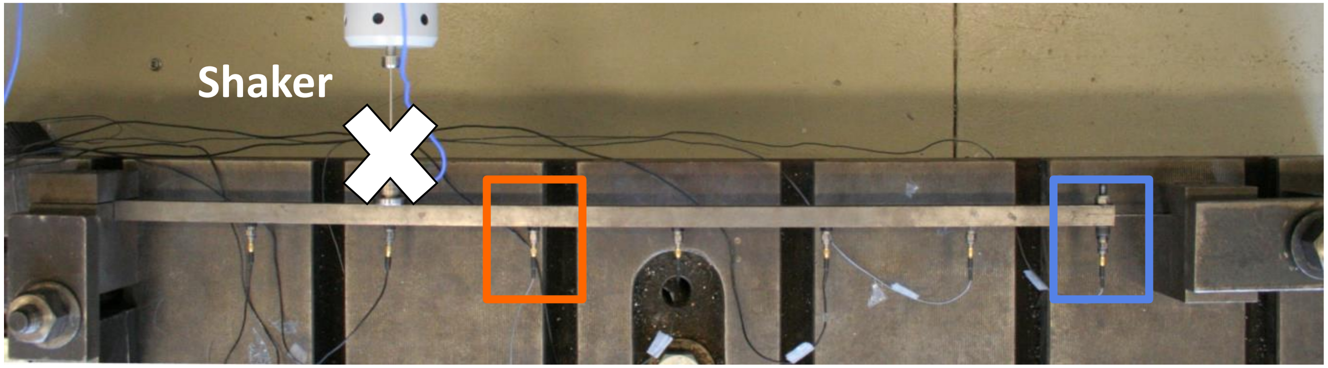
(c)

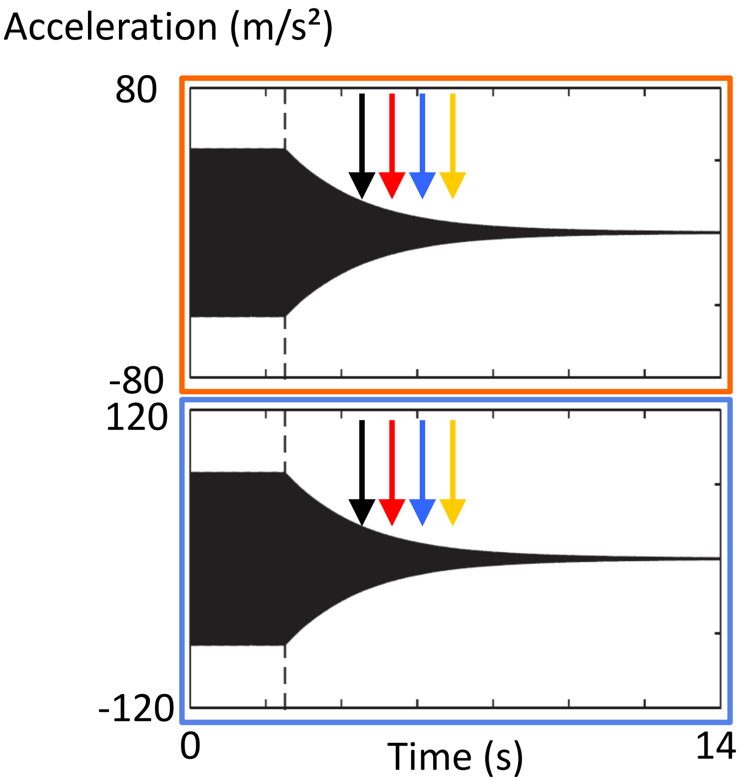


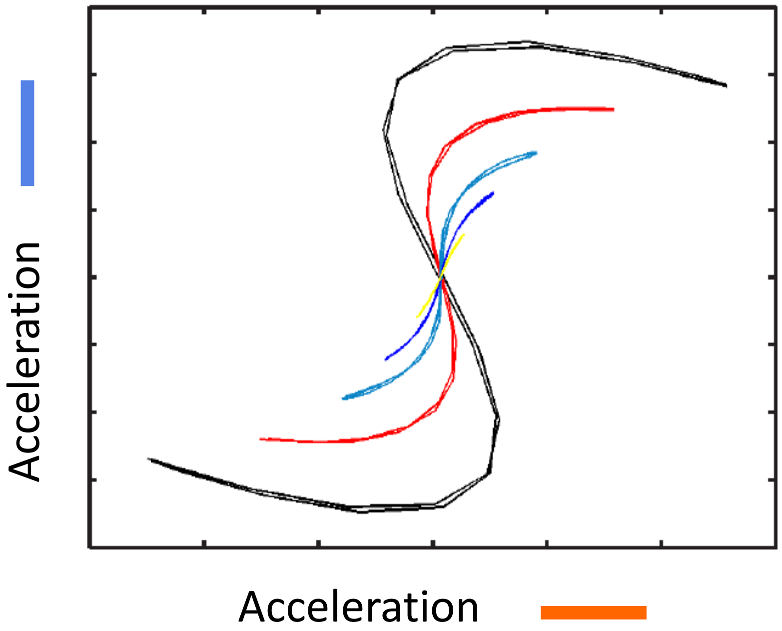
What You Have Just Seen



What You See Is a Nonlinear Normal Mode







Outline of this Lecture

What are nonlinear modes ?

What are their fundamental properties ?

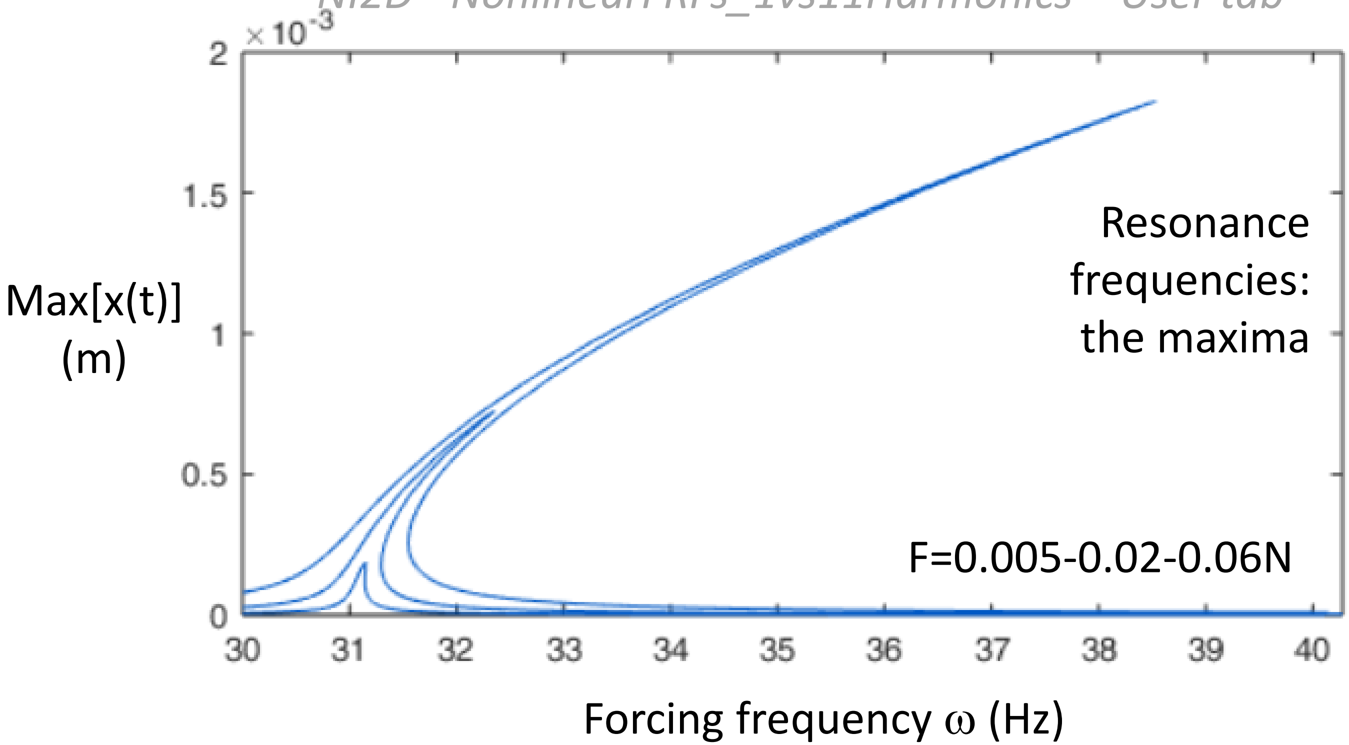
Link between modes and resonance frequencies

A tutorial

Forced/Damped Response of the Beam (L02)

$$0.289\ddot{x} + 0.1357\dot{x} + 11009x + 2.37.10^9\,x^3 = Fsin\omega t$$

NI2D - NonlinearFRFs_1vs11Harmonics – User tab



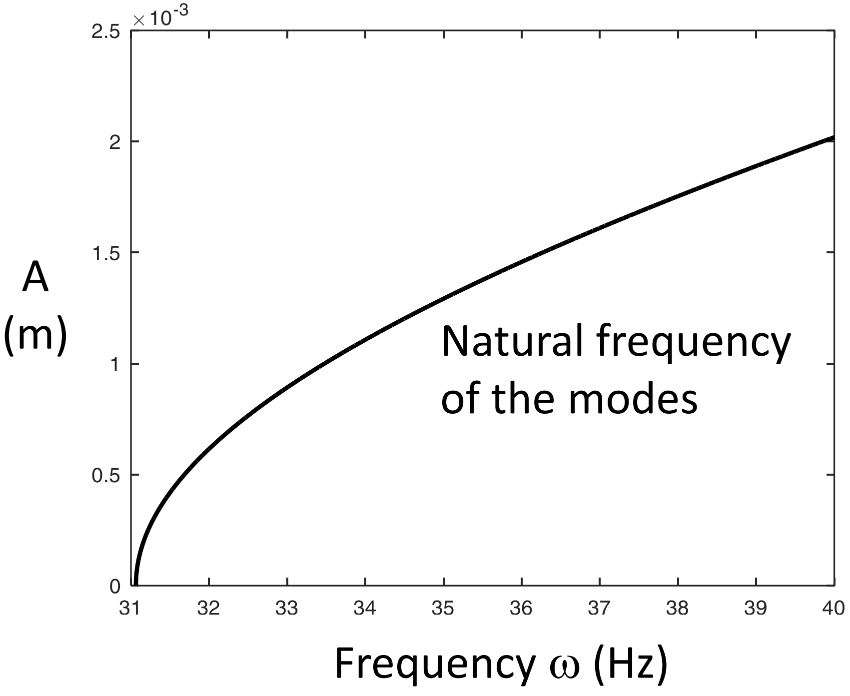
Free/Undamped Response of the Beam

$$0.289\ddot{x} + 11009x + 2.37.10^9\, x^3 = 0$$

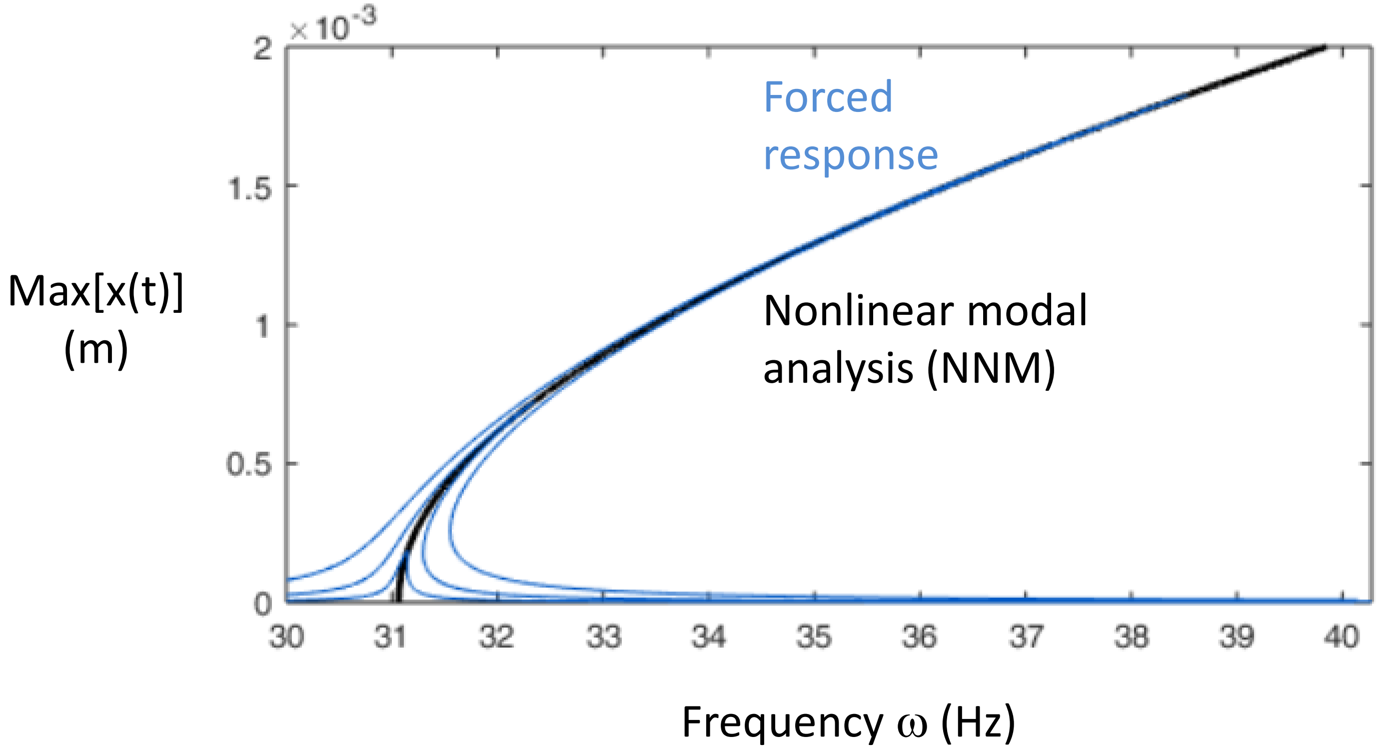
$$x = A\cos\omega t$$

$$A = \sqrt{\frac{-11009 + 0.289\omega^2}{0.75 \times 2.37e9}}$$

LinkModeFRF.m



Link between Natural/Resonance Frequencies ?



NNM can predict the locus of resonance frequencies for various forcing amplitudes !

In Summary

	LNMs	NNMs
Clear physical meaning		
Structural deformation at resonance	**YES**	**YES**
Synchronous vibration of the structure	**YES**	**YES, BUT…**
Important mathematical properties		
Orthogonality	**YES**	**NO**
Modal superposition	**YES**	**NO**
Invariance	**YES**	**YES**