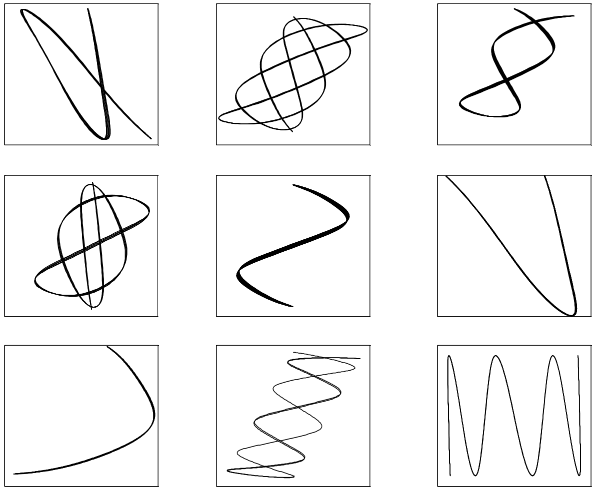
Nonlinear Vibrations of Aerospace Structures

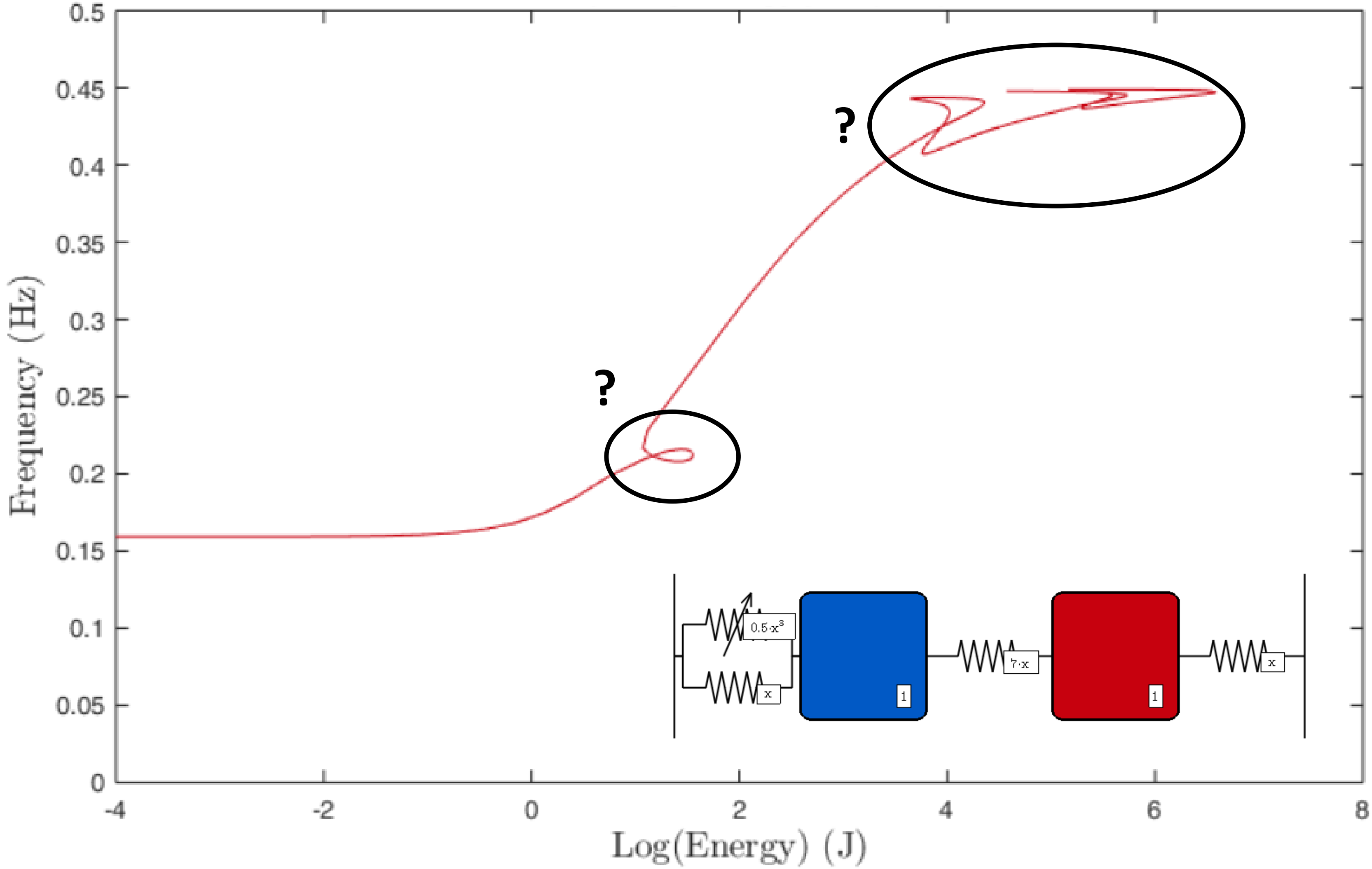
L04 | **Nonlinear modal analysis**

Modal interactions & new resonances

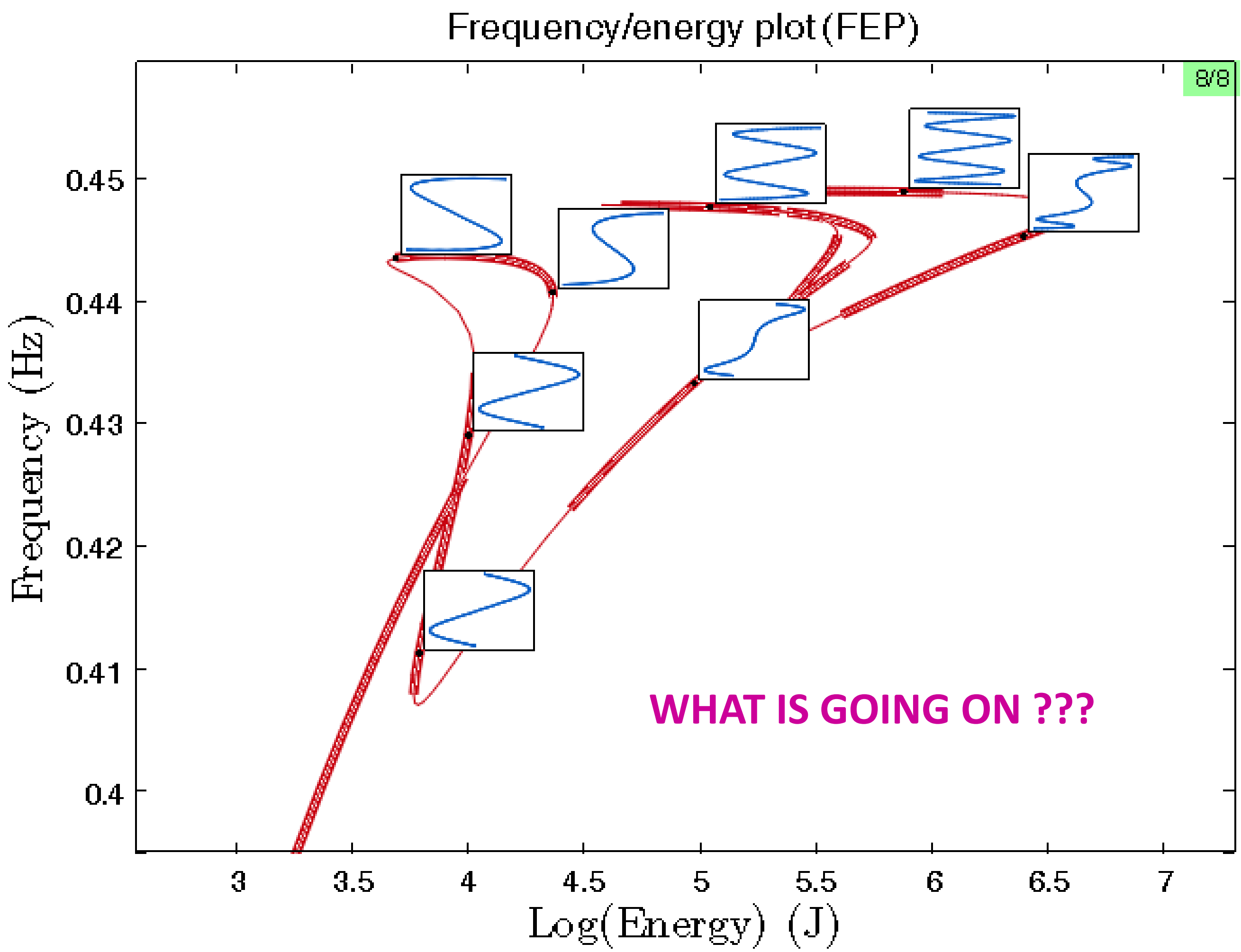
How to compute NNM ?



Last Lecture



Close-Up of the Different Loops



Outline of this Lecture

Discover additional properties of nonlinear modes

Numerical computation of nonlinear modes

Another puzzling result !

Our Objectives

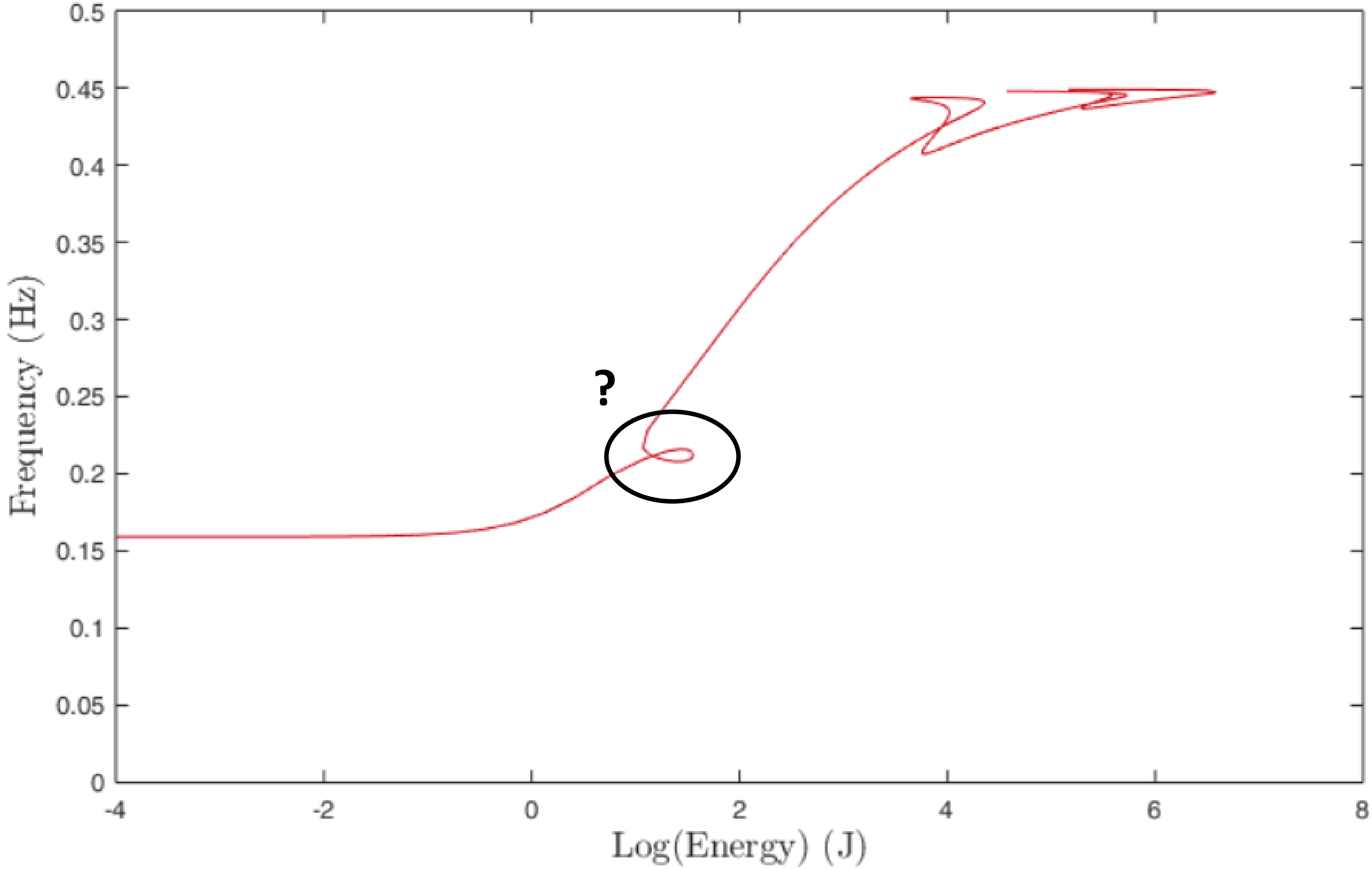
1. How to characterize the different loops ?

2. How they are created ?

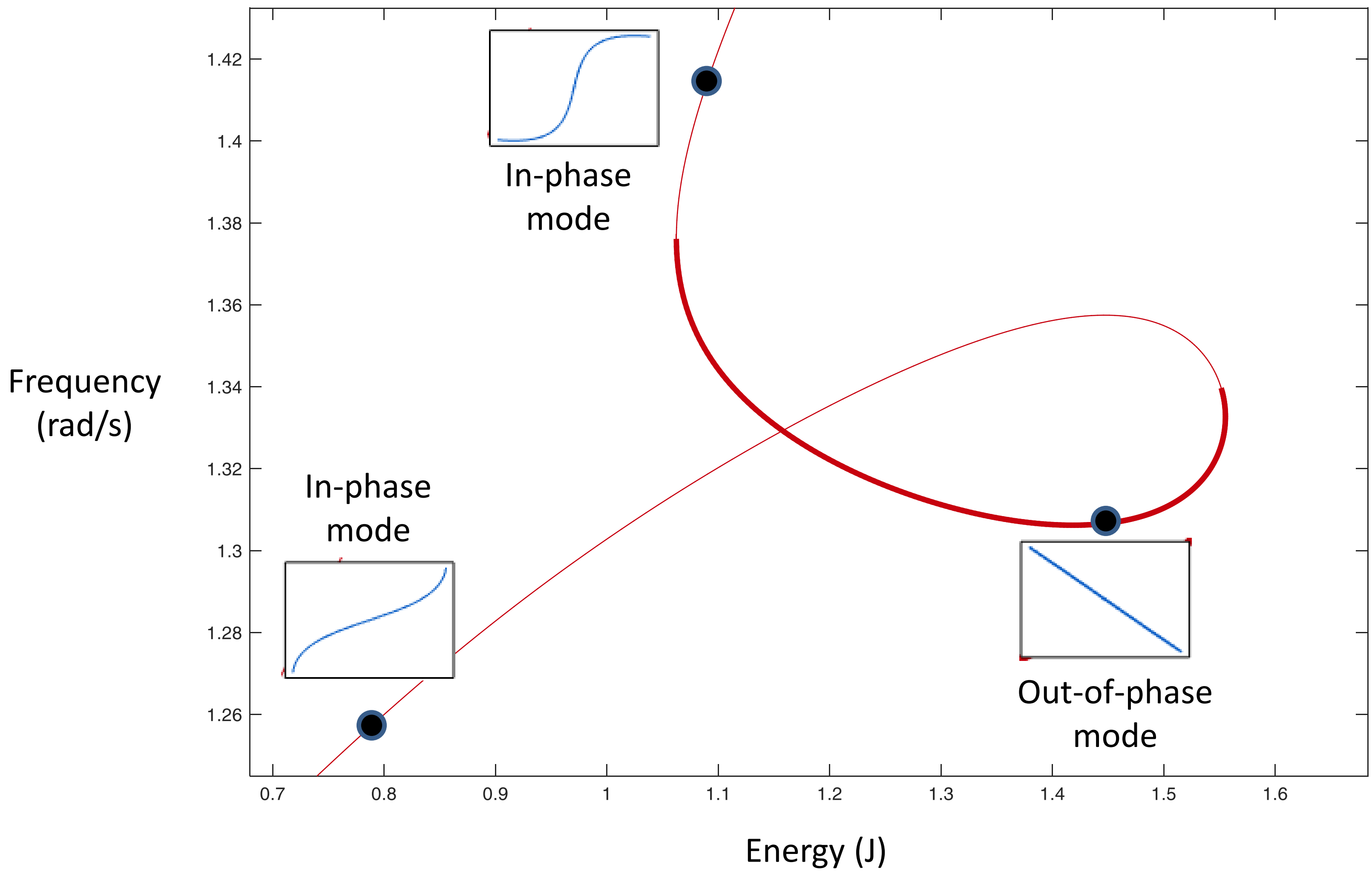
3. What is their impact on the theory of NNMs ?

4. What is their impact for engineering design ? Theoretical curiosity or real stuff ?

Let's Start with the First Loop

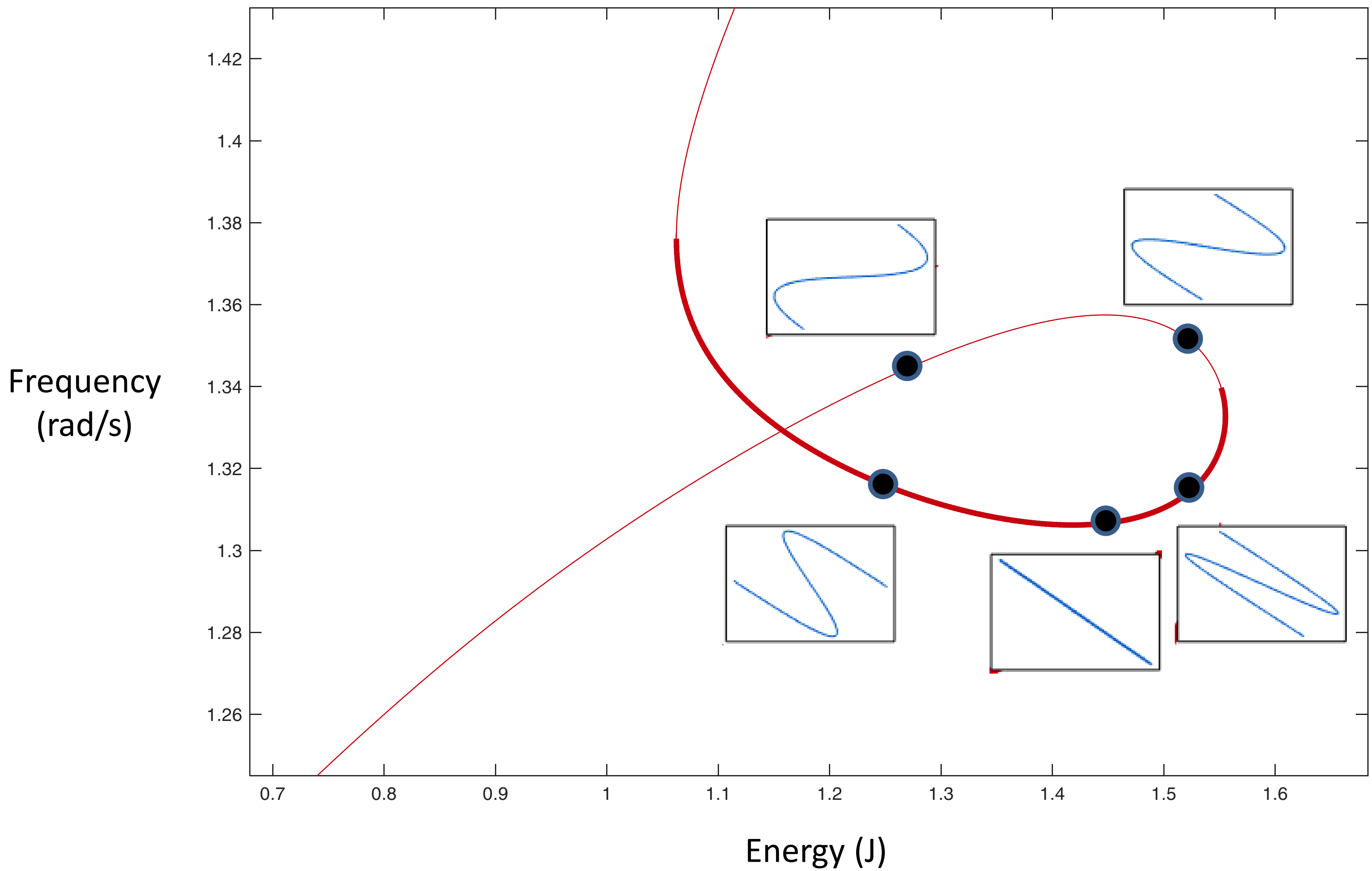


In-Phase and Out-of-Phase NNMs Are Connected



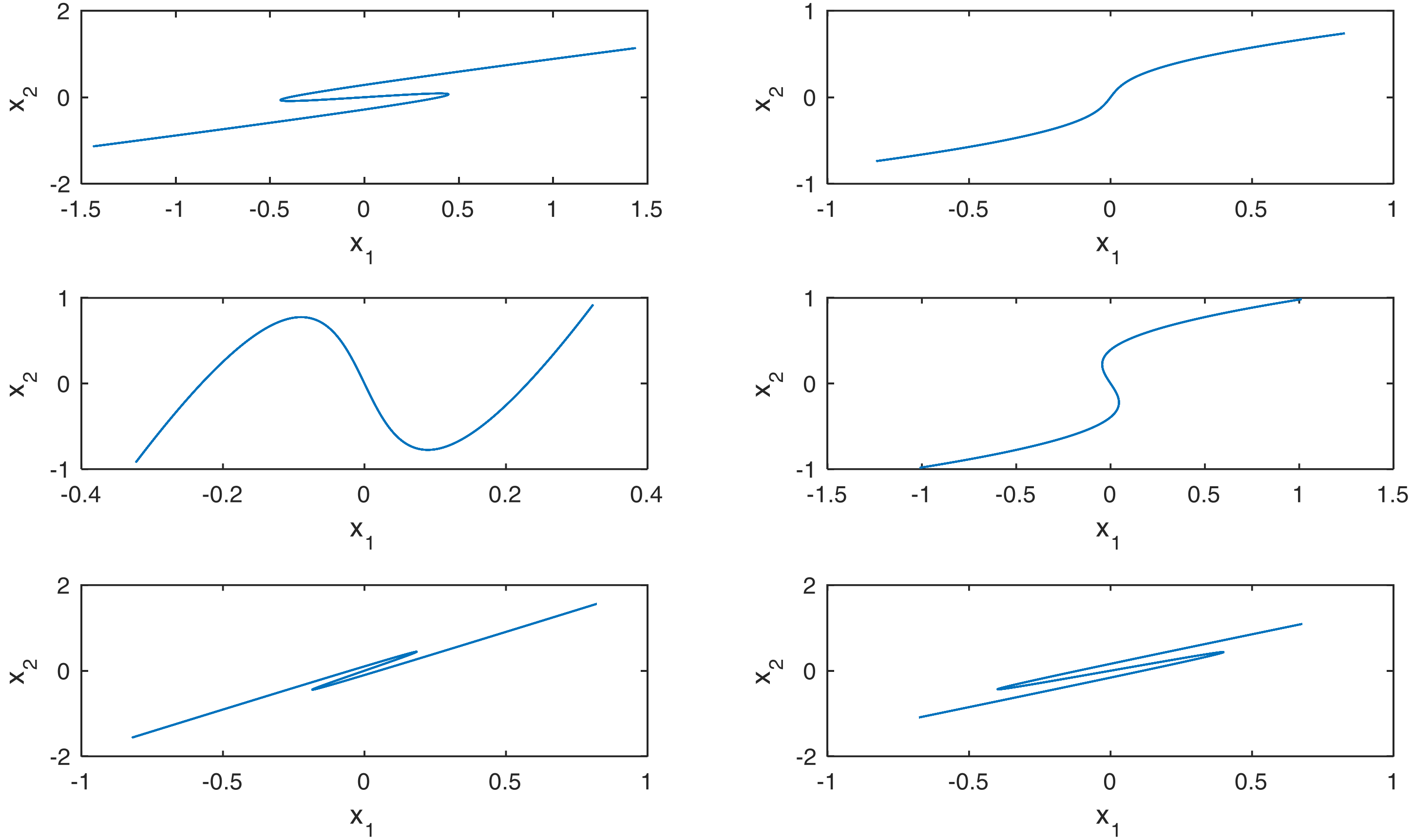
So there is **modal coupling/interaction** on this NNM branch

Modes on the Branch Present a Third Harmonics



Indeed, Remember Lissajous Curves…

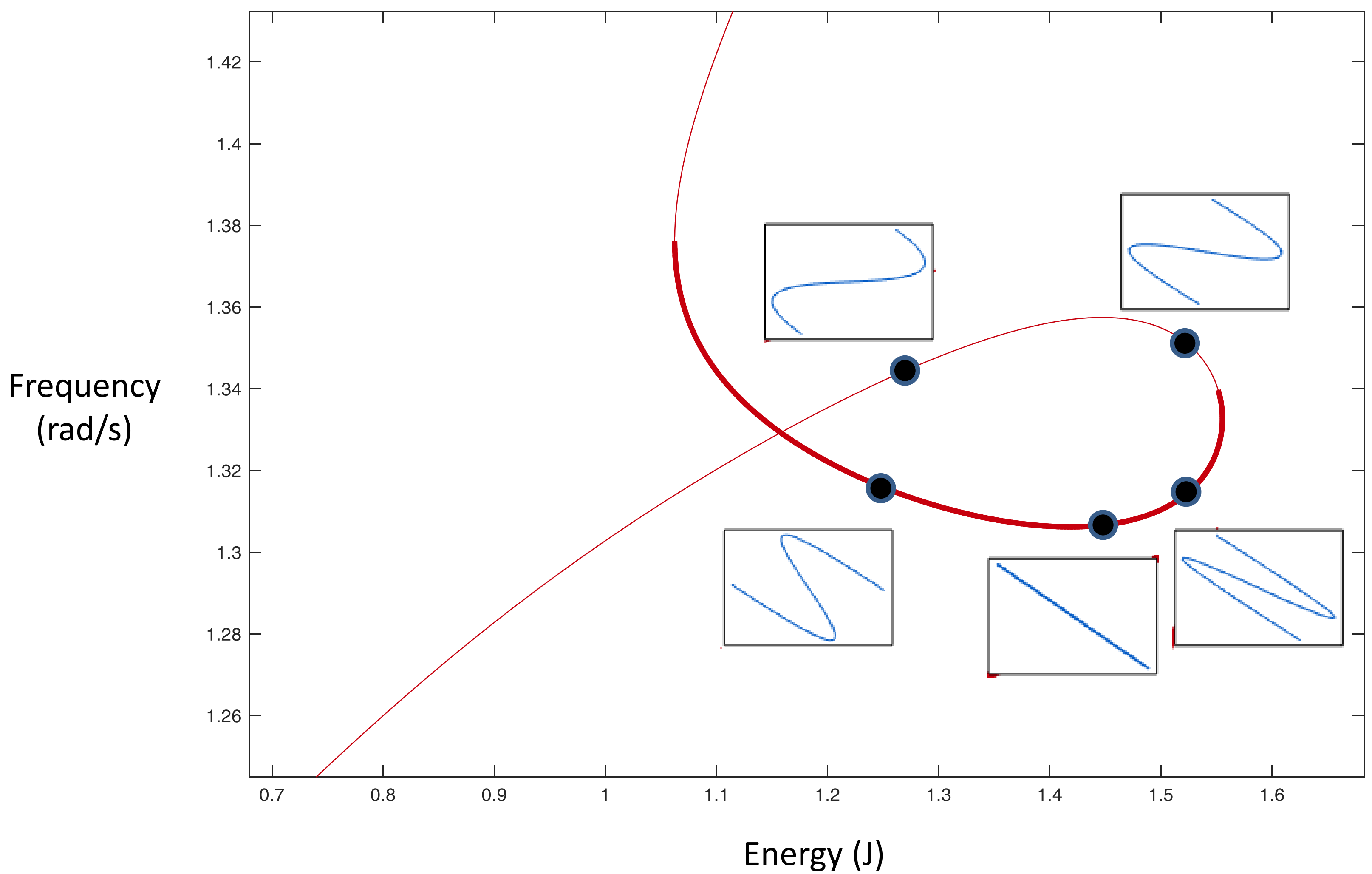
```
>> LissajousNNM
1=sin/sin 2=cos/cos: 2
Enter harmonics 1:1
Enter harmonics 2:3
```



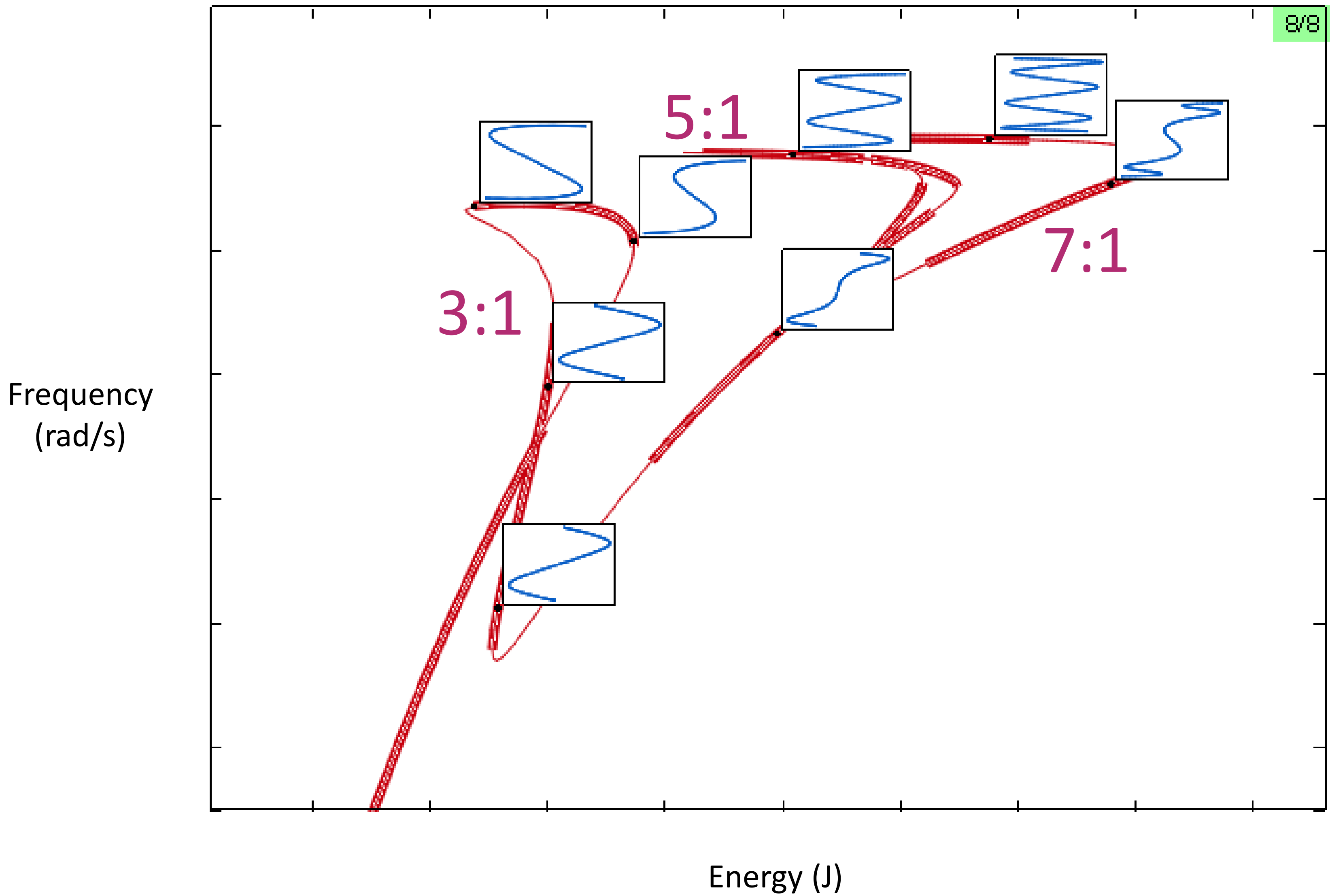
Matlab Code

```
f=input('1=sin/sin 2=cos/cos: ');
w1=input('Enter harmonics 1:');
w2=input('Enter harmonics 2:');

temps=[0:0.01:50];

if f==1
    for k=1:6
        x1=rand(1)*sin(w1*temps)+rand(1)*sin(w2*temps);
        x2=rand(1)*sin(w1*temps)+rand(1)*sin(w2*temps);
        subplot(3,2,k)
        plot(x1,x2)
        xlabel('x_1');
        ylabel('x_2');
        set(gcf,'uni','nor','pos',[0.2 0.2 0.6 0.6])
    end
else
    for k=1:6
        x1=rand(1)*cos(w1*temps)+rand(1)*cos(w2*temps);
        x2=rand(1)*cos(w1*temps)+rand(1)*cos(w2*temps);
        subplot(3,2,k)
        plot(x1,x2)
        xlabel('x_1');
        ylabel('x_2');
        set(gcf,'uni','nor','pos',[0.2 0.2 0.6 0.6])
    end
end
```

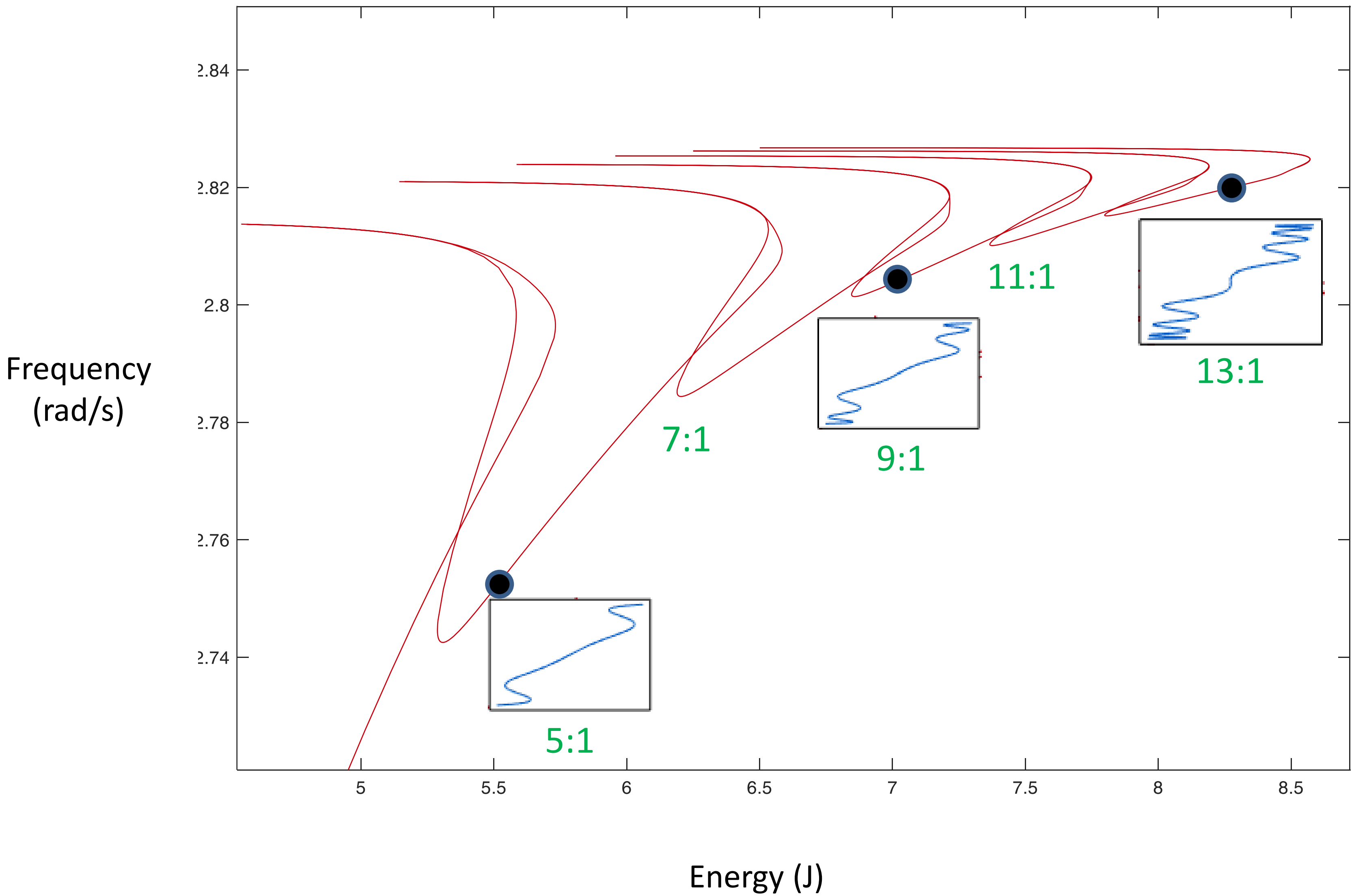
This Is Thus a 3:1 Modal Interaction



And Also…



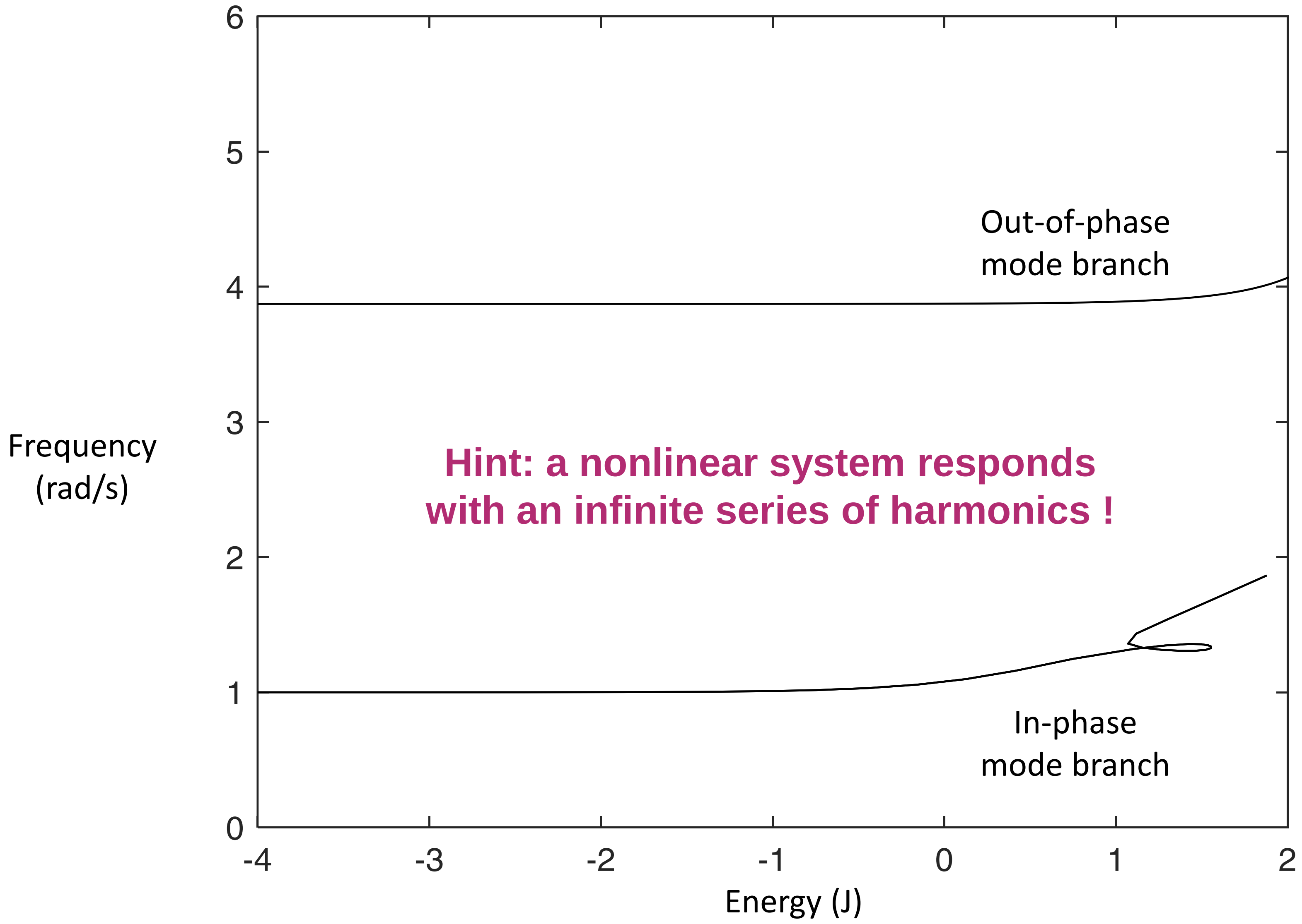
It Goes On Forever



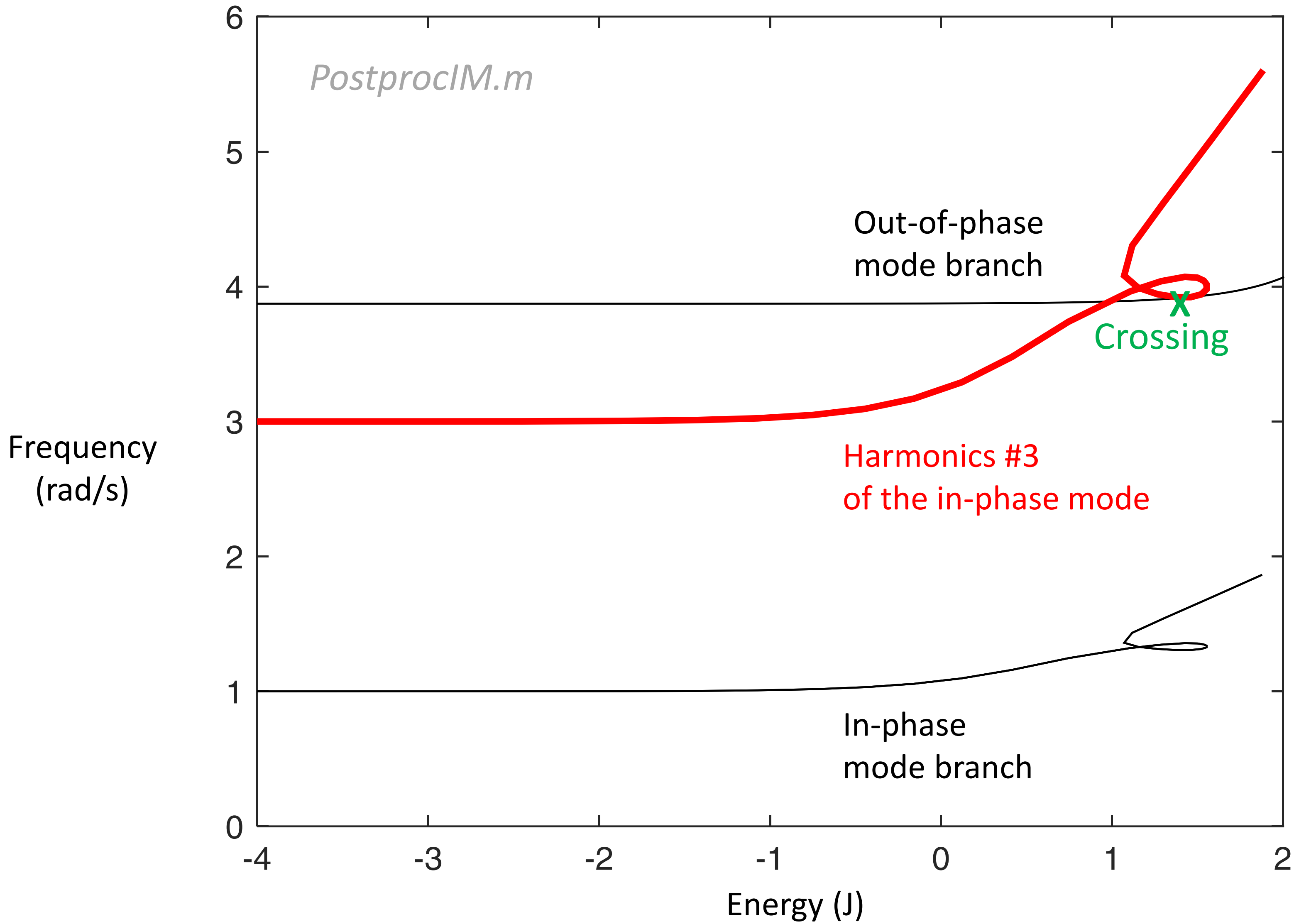
Our Objective

1. How to characterize the different loops ? **OK !**
2. How they are created ?
3. What is their impact on the theory of NNMs ?
4. What is their impact for engineering design ? Theoretical curiosity or real stuff ?

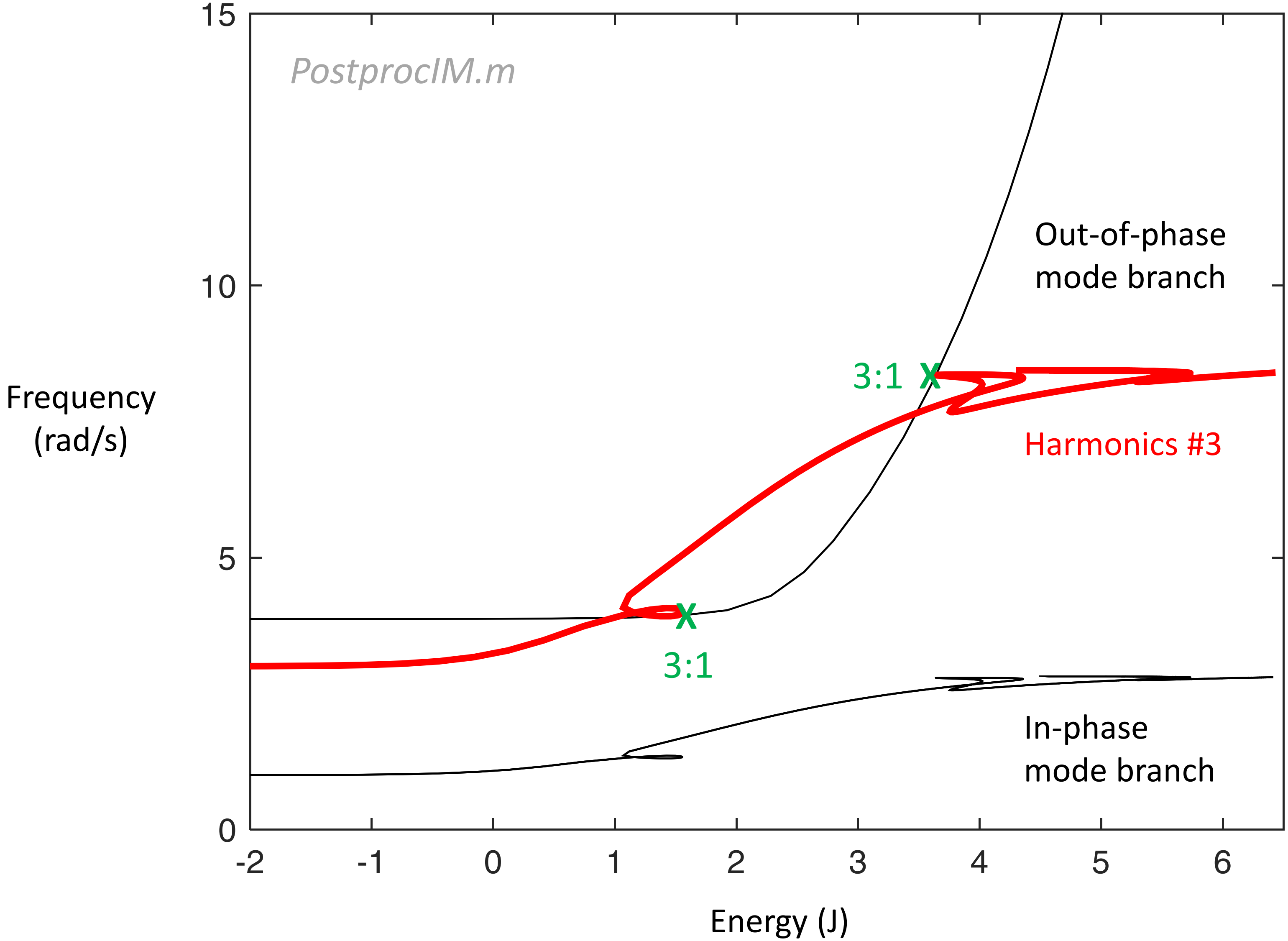
In-Phase and Out-of-Phase NNMs Are Disconnected



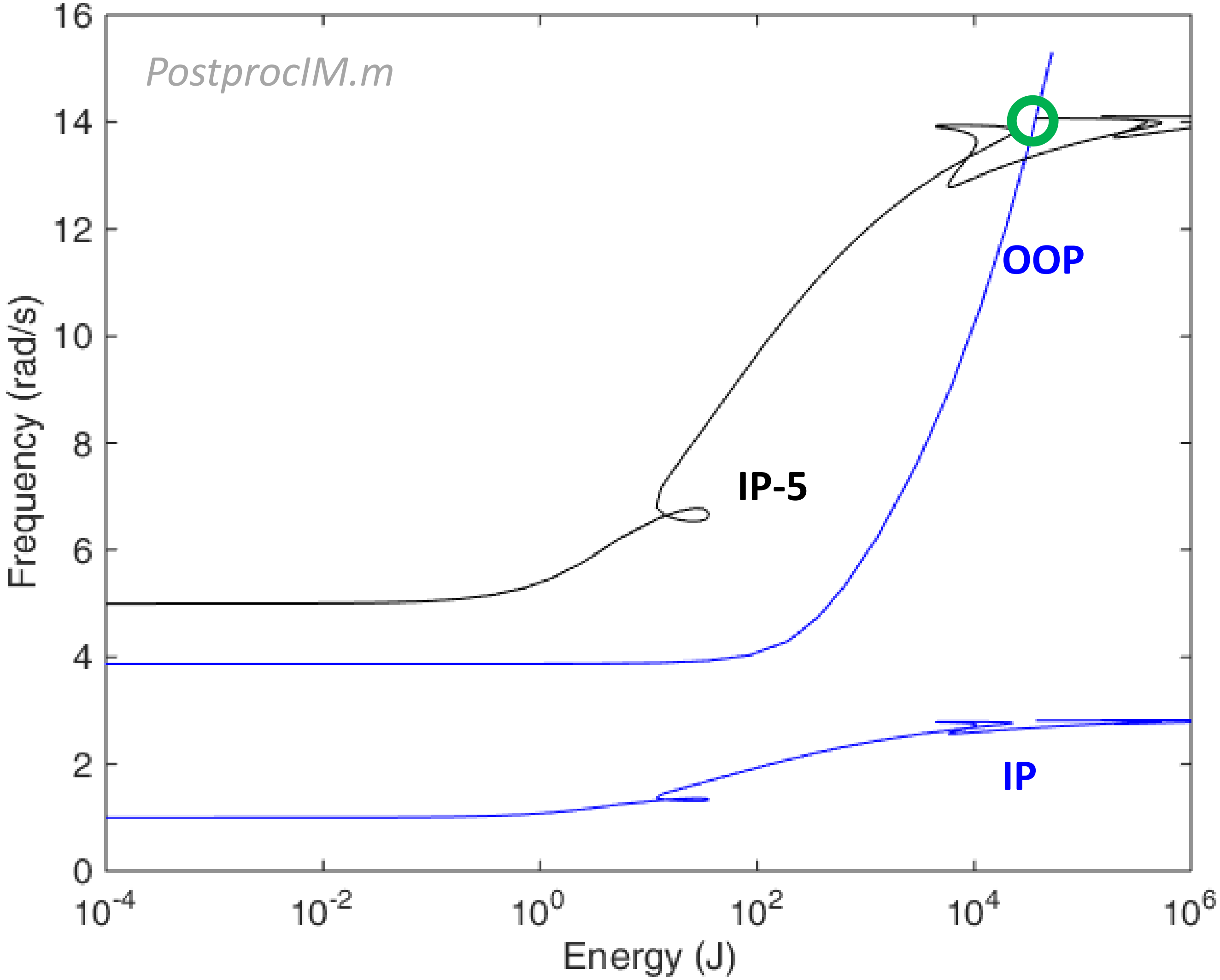
The Missing Piece of Info of the FEP: Harmonics



Evidence of the Second 3:1 Modal Interaction



There Is Also a 5:1 Modal Interaction



Another Viewpoint

$$\ddot{q}_1 + (2q_1 - q_2) + 0.5q_1^3 = 0$$
$$\ddot{q}_2 + (2q_2 - q_1) = 0$$

$q_{1,2} \cong A, B cos\omega t$

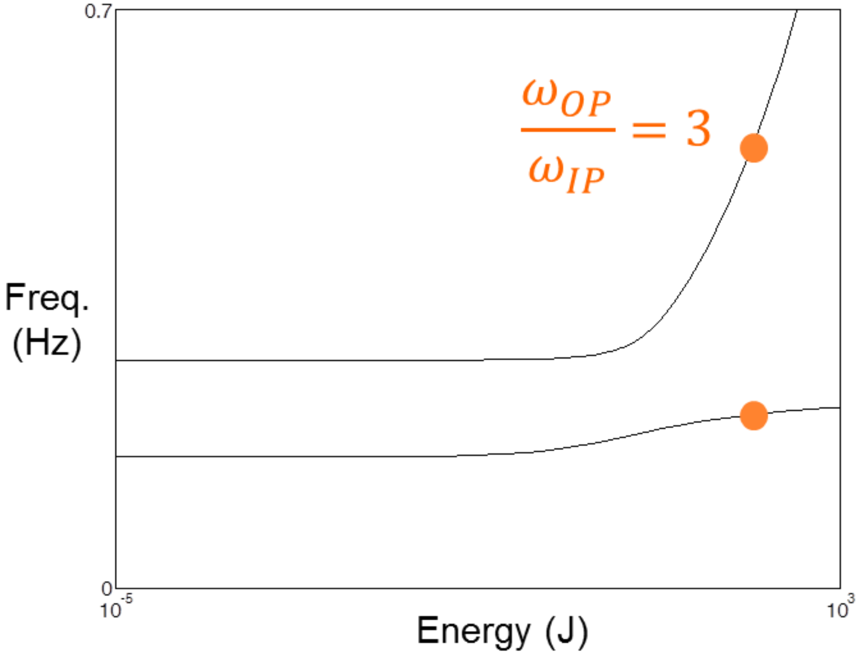
$$A = \pm\sqrt{\frac{8(\omega^2 - 2)(\omega^2 - 1)}{3(\omega^2 - 2)}}$$

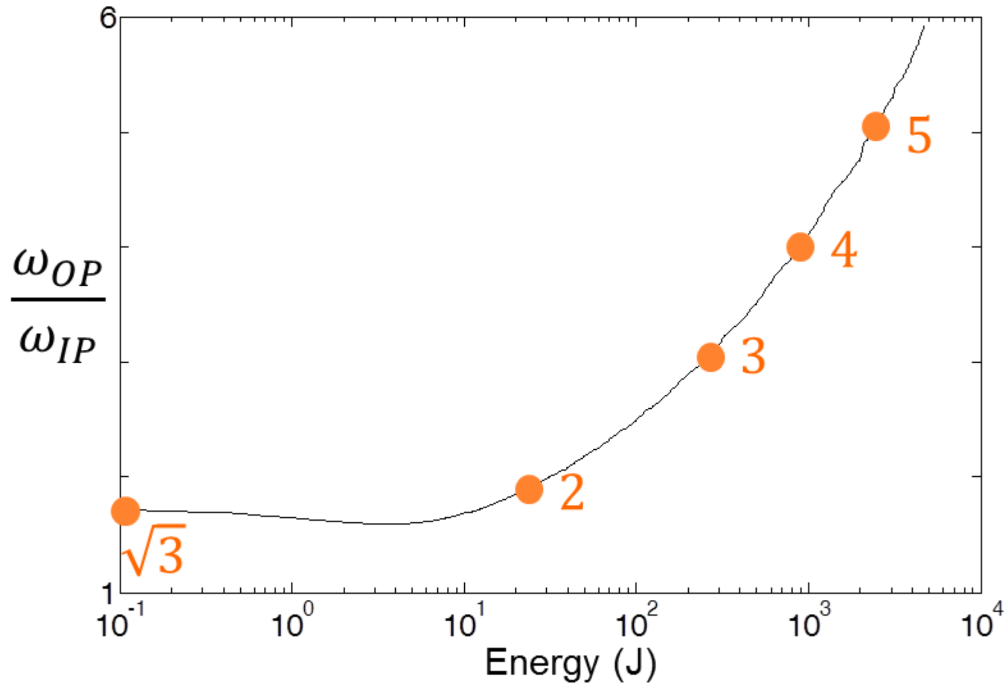
$$B = \frac{A}{2 - \omega^2}$$

$$\omega_1 \in \left[1, \sqrt{2}\right[\text{ rad/s}$$

$$\omega_2 \in \left[\sqrt{3}, +\infty\right[\text{ rad/s}$$

The Frequency Ratio: From $\sqrt{3}$ to $+\infty$



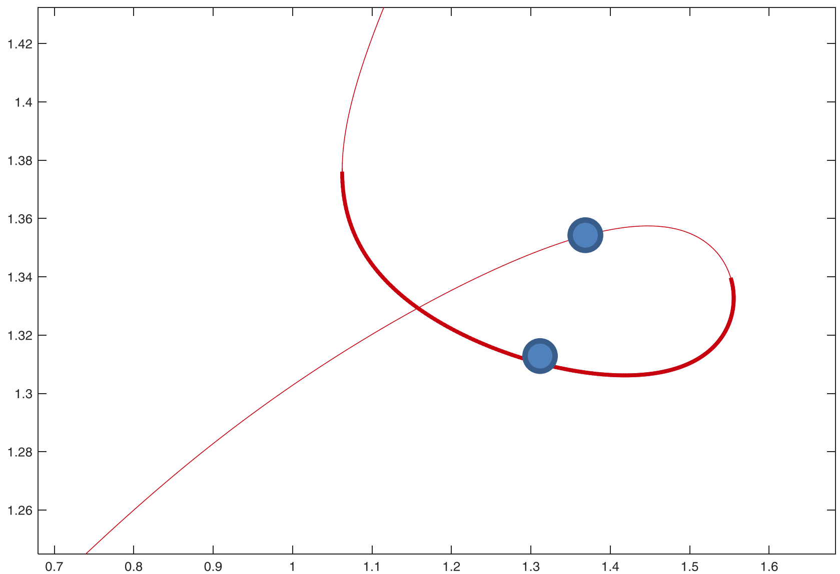


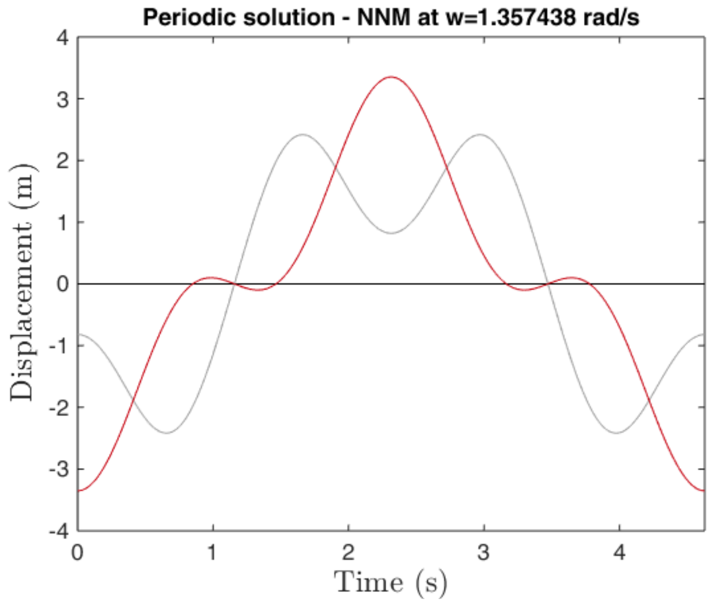
So there exists a **countable infinity of modal interactions** !

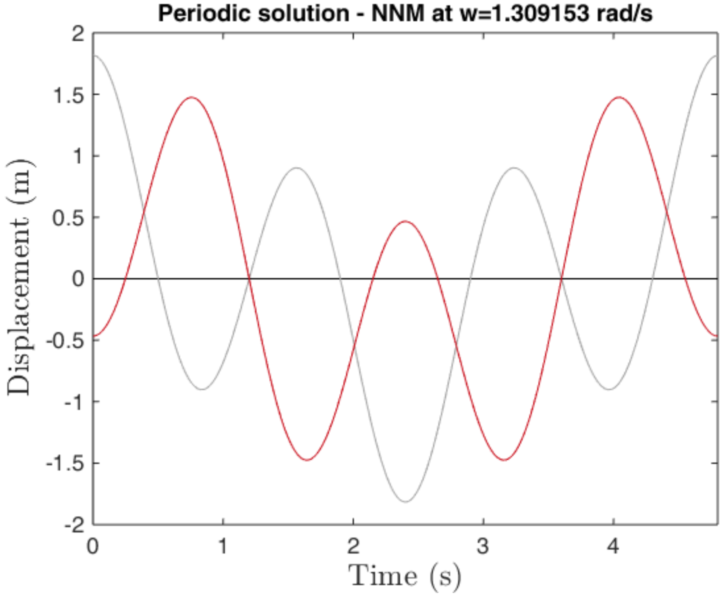
Our Objective

1. How to characterize the different loops ? **OK !**
2. How they are created ? **OK !**
3. What is their impact on the theory of NNMs ?
4. What is their impact for engineering design ? Theoretical curiosity or real stuff ?

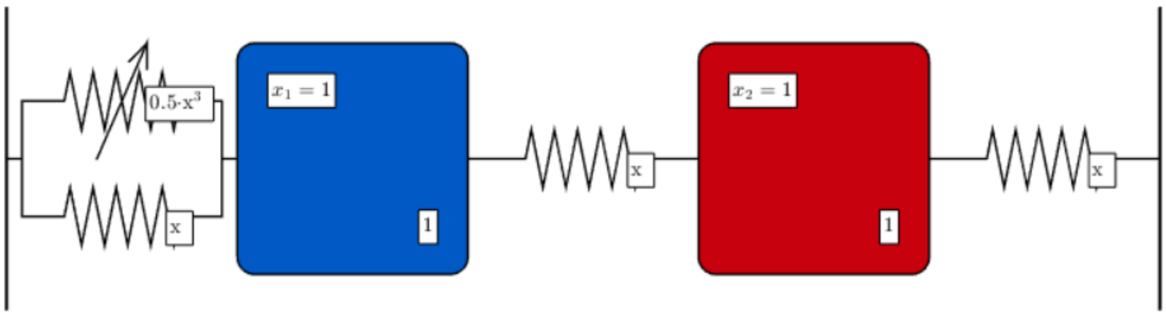
No Longer a Synchronous Motion





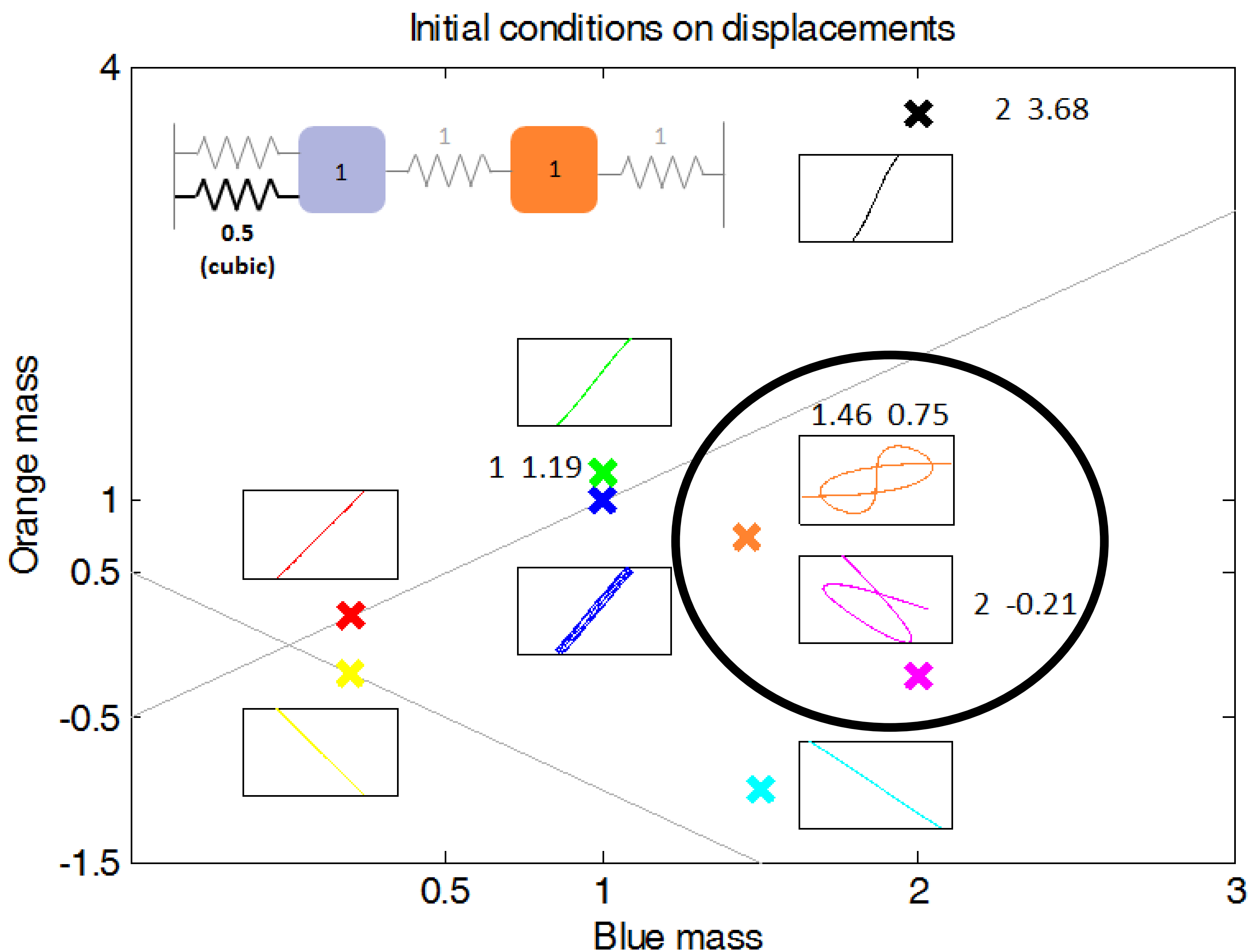


Let’s Revisit the System We Initially Considered



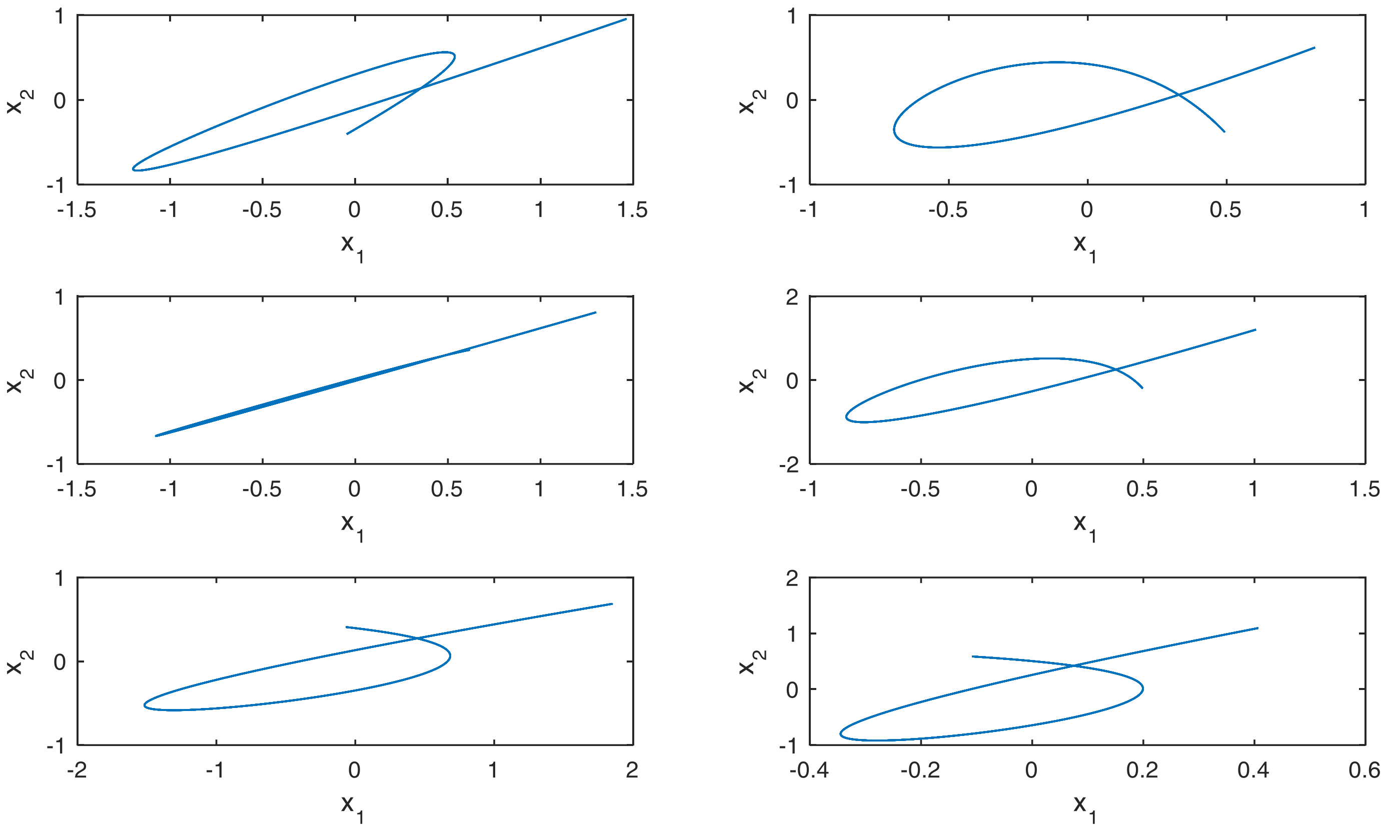
NI2D – 2DOF\NNM_ShawSystem

We Got in the Previous Lecture

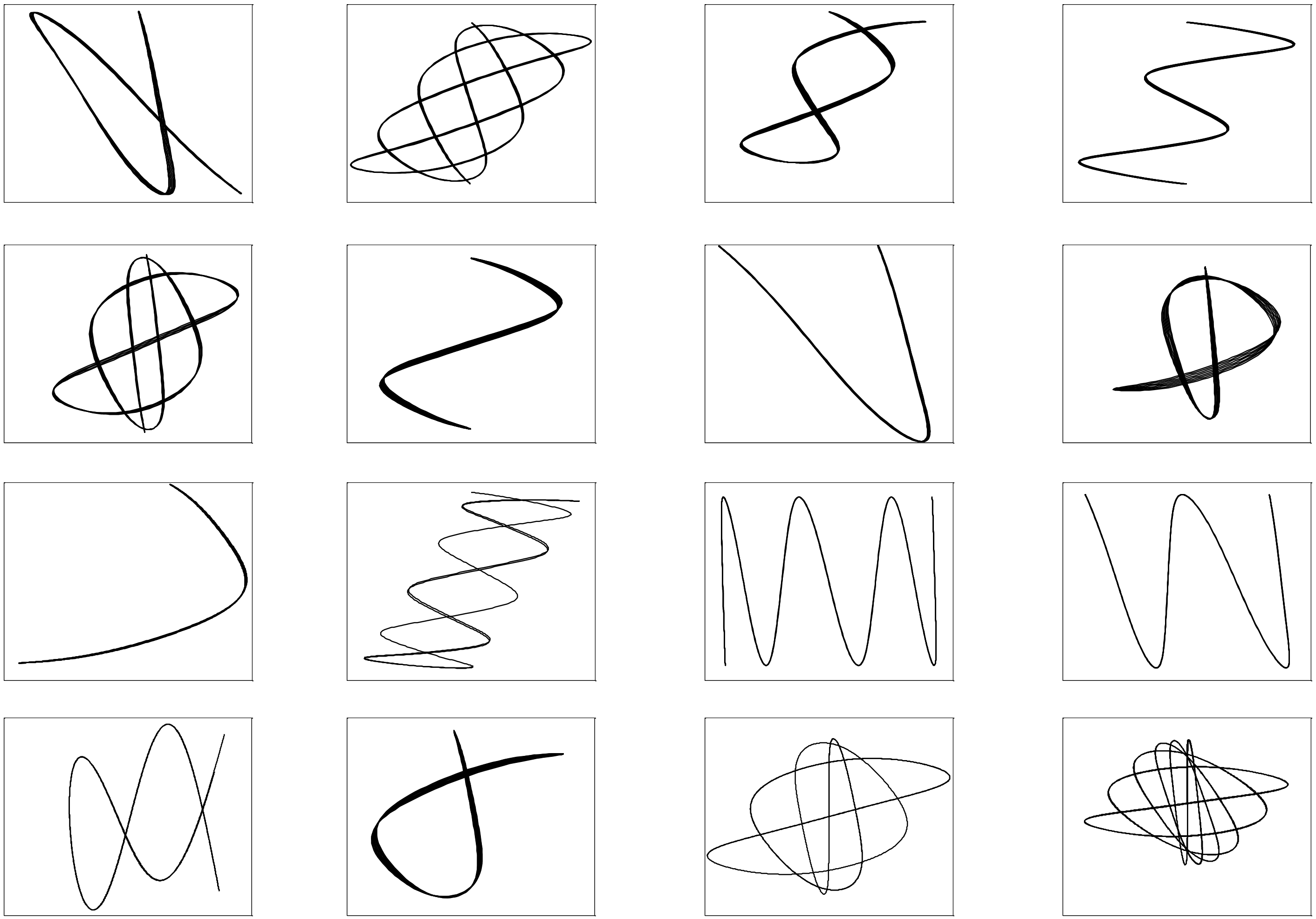


Again Lissajous Curves…

```
>> LissajousNNM
1=sin/sin 2=cos/cos: 2
Enter harmonics 1:2
Enter harmonics 2:3
```



Neither Abstract Nor a New Alphabet…



An Extended Definition of NNMs Is Needed

$$M\ddot{q}(t) + Kq(t) = 0$$

$$M\ddot{q}(t) + Kq(t) + f_{NL}[q(t)] = 0$$

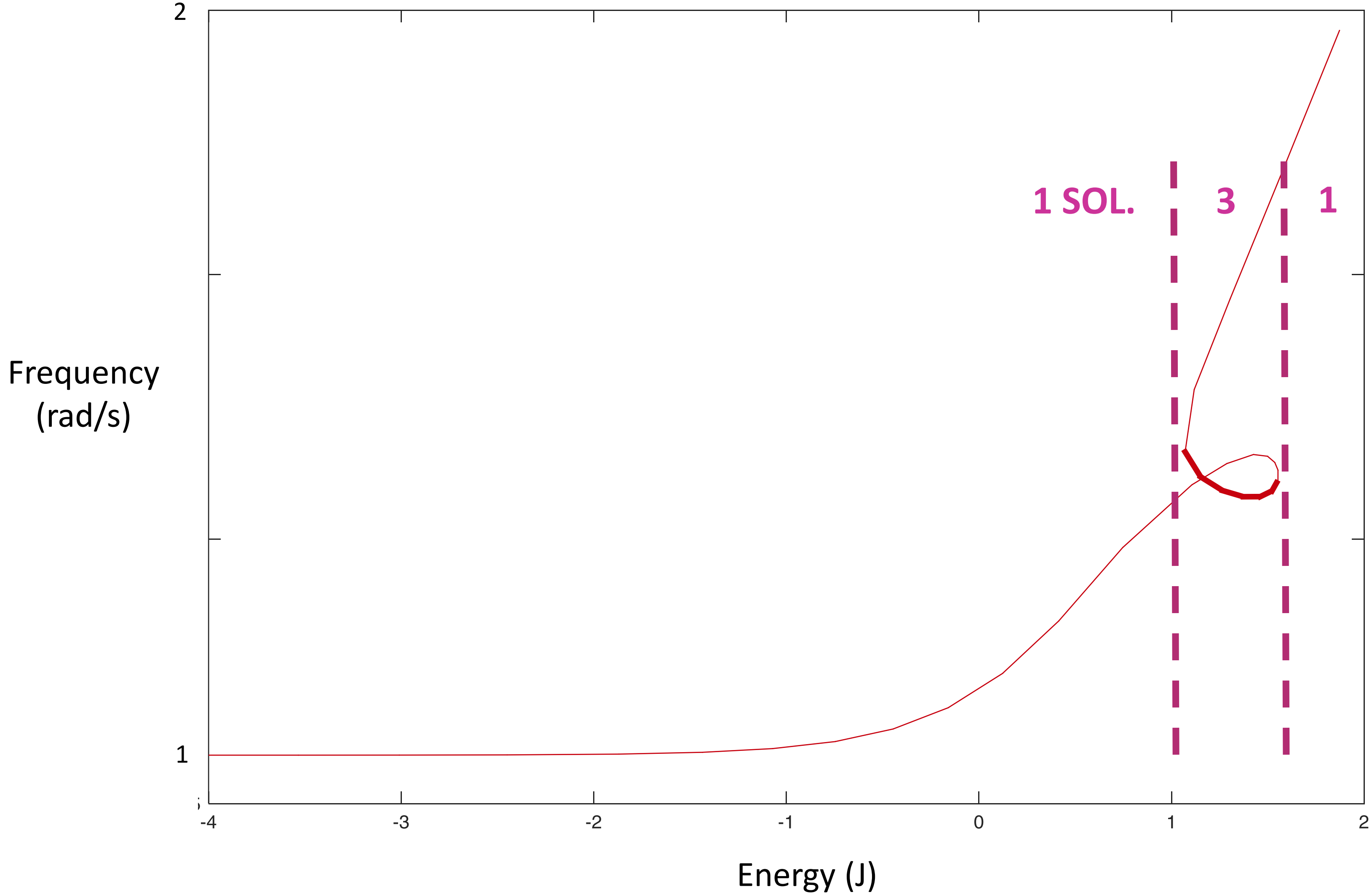
LNM: synchronous periodic motion.

NNM: synchronous periodic motion.

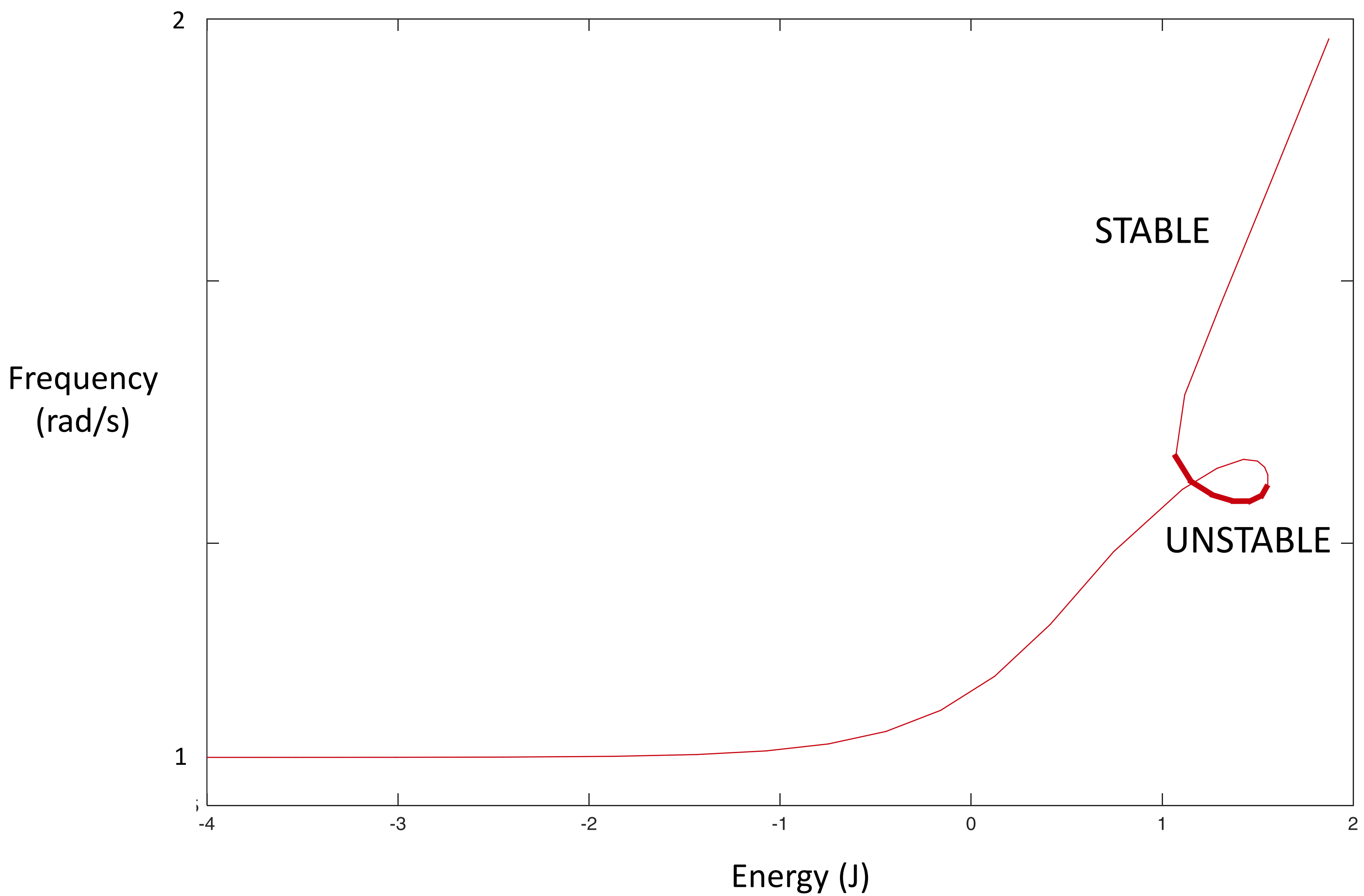
Modal interactions

NNM: periodic motion (nonnecessarily synchronous).

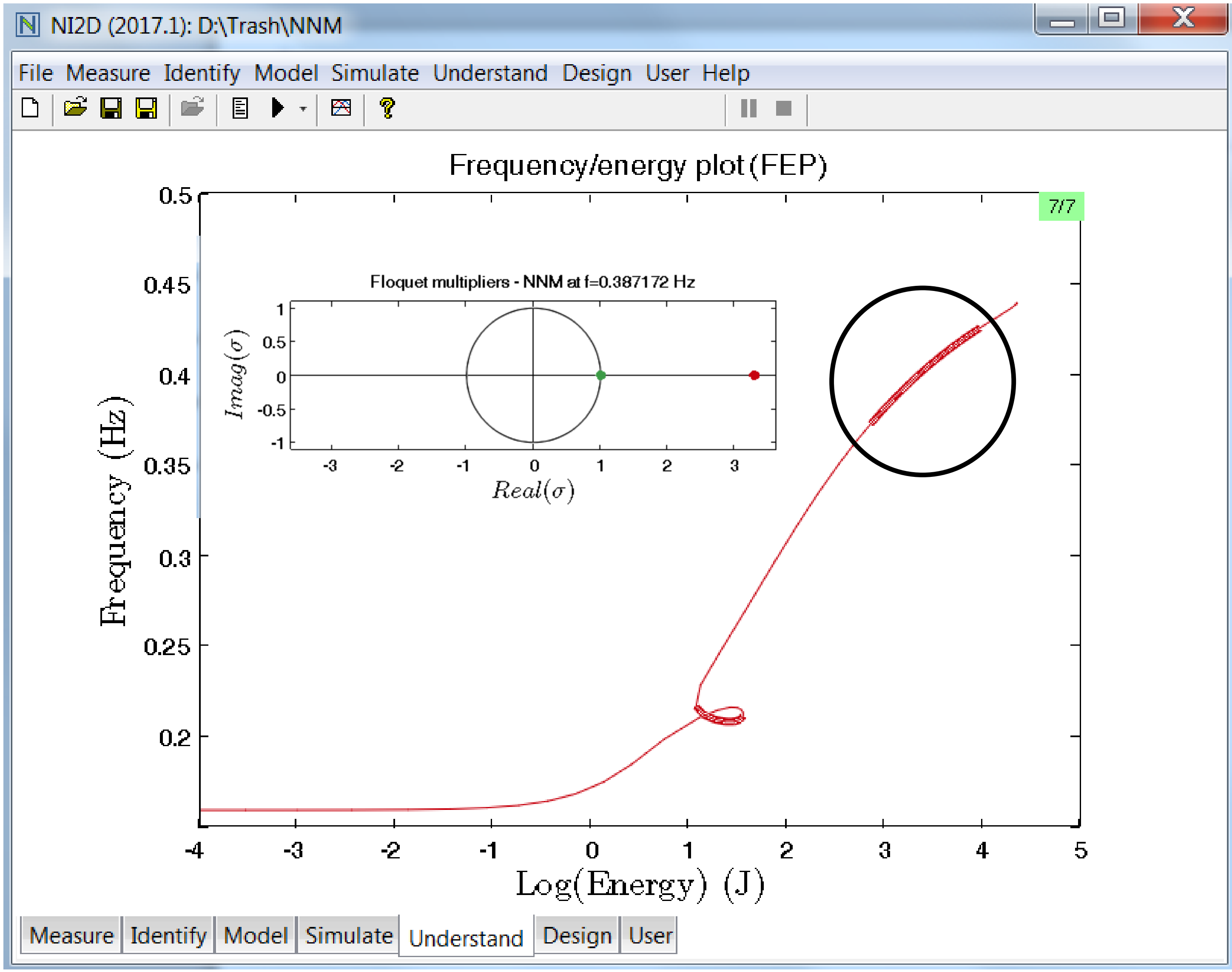
Another Impact: The In-Phase Mode Bifurcates



Remember Bifurcation/Stability Are Tight Together

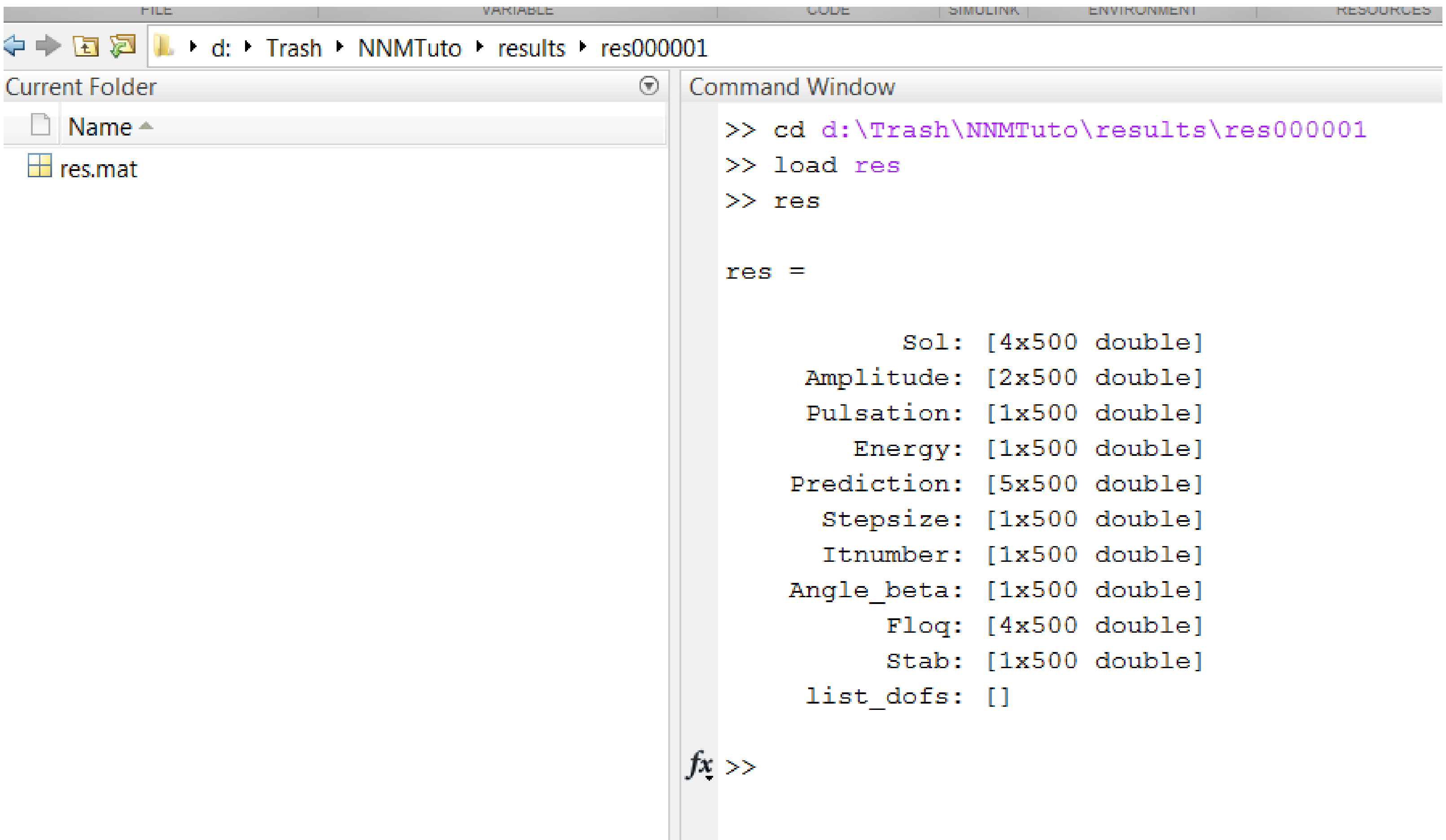


An Unstable Part of the Branch



Let’s Try to Investigate What’s Going On

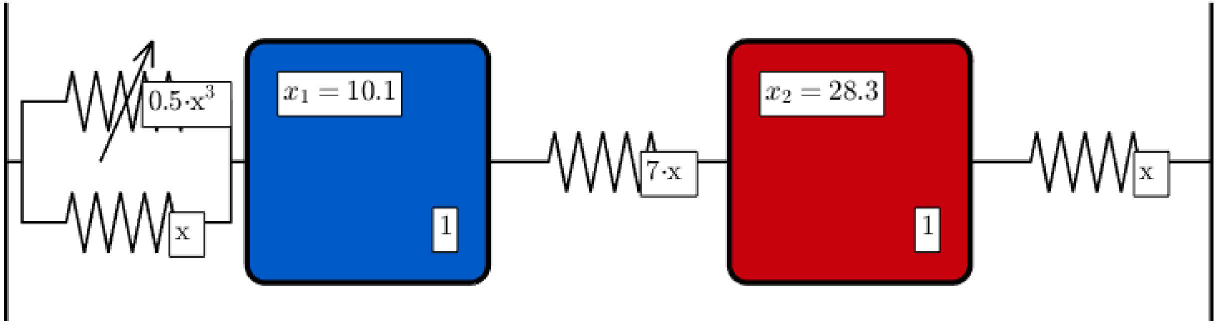
Open Matlab and load the file that contains the results



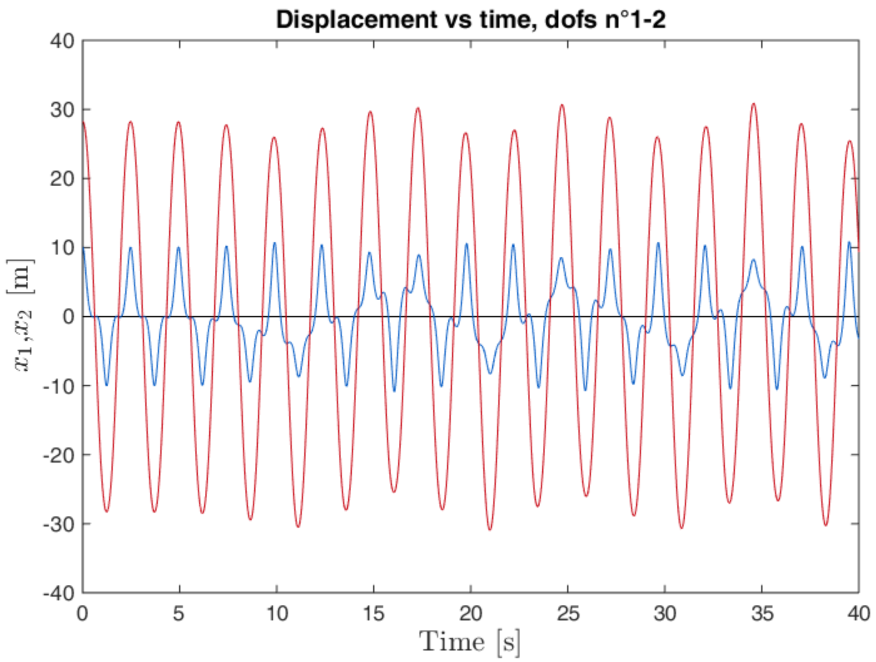
Initial Conditions for an Unstable NNM

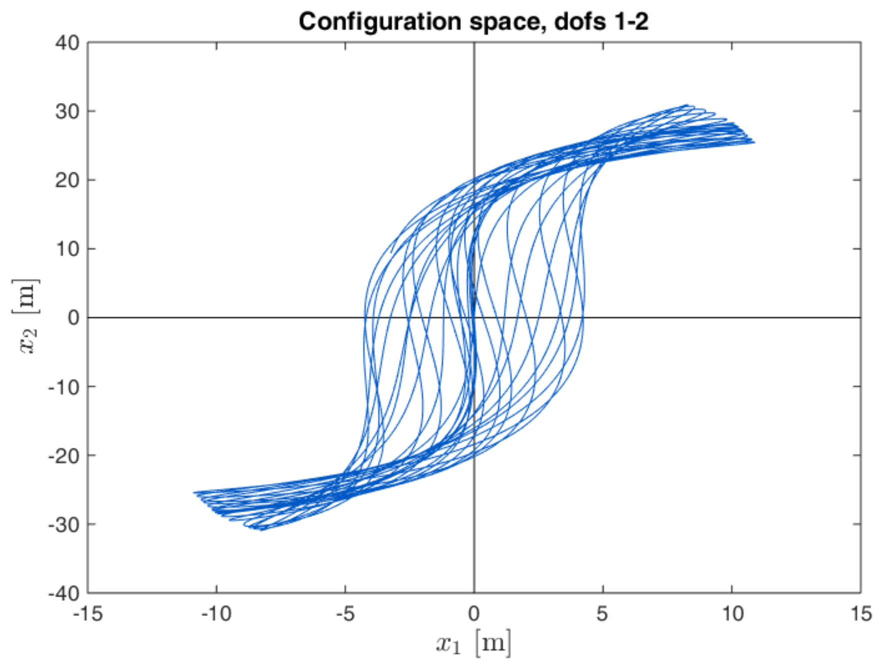
```
>> log10(res.Energy)

ans =

  Columns 1 through 10

   -4.0000   -3.5355   -3.0114   -2.4511   -1.8710   -1.4370   -1.0721   -0.7467   -0.4459   -0.1596

  Columns 11 through 20

    0.1225    0.4141    0.7460    1.1054    1.2869    1.4251    1.4983    1.5359    1.5526    1.5525

  Columns 21 through 30

    1.5215    1.4604    1.3731    1.2651    1.1511    1.0705    1.1176    1.3035    1.5701    1.8719

  Columns 31 through 40

    2.1140    2.3121    2.5284    2.7065    2.8575    2.9883    3.1036    3.2386    3.3569    3.4621

  Columns 41 through 50

    3.5862    3.6958    3.8244    3.9723    4.1396    4.2933    4.3316    4.3422    4.3508    4.3539
```

```
>> res.Sol(:,40)

ans =

   10.0700
   28.2892
         0
         0
```



An Unstable NNM (Newmark Simulation)

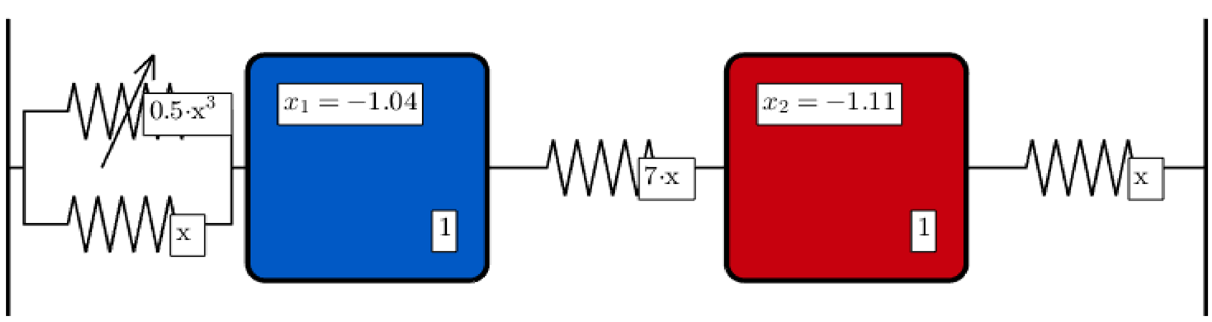




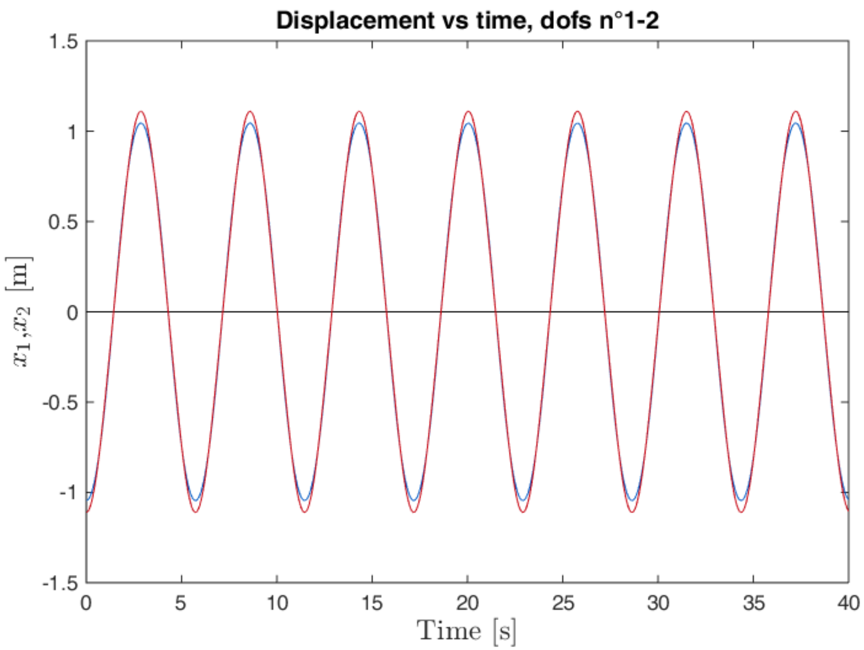
Initial Conditions for a Stable NNM

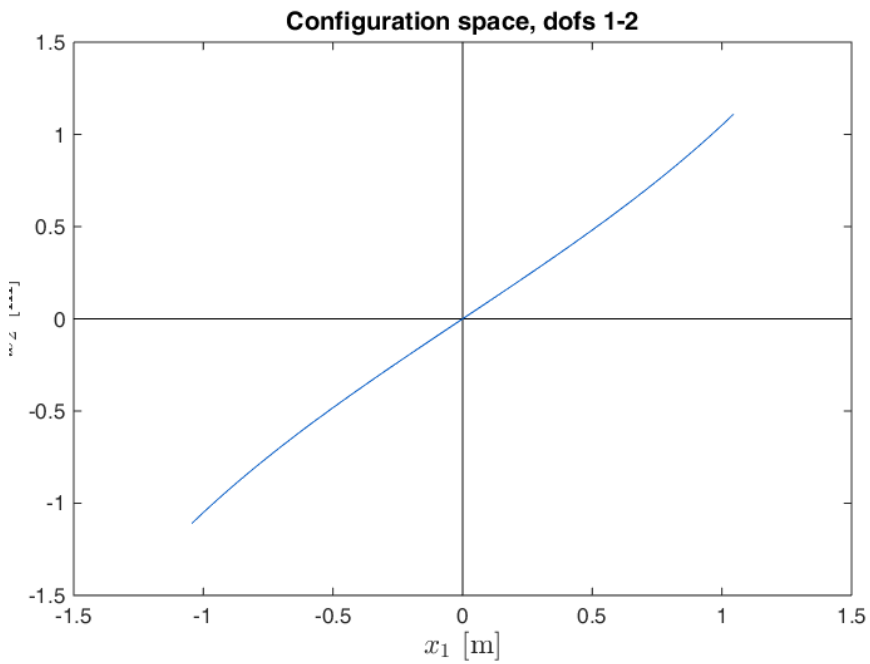
```
>> log10(res.Energy)

ans =

  Columns 1 through 10

   -4.0000   -3.5355   -3.0114   -2.4511   -1.8710   -1.4370   -1.0721   -0.7467   -0.4459   -0.1596

  Columns 11 through 20

    0.1225    0.4141    0.7460    1.1054    1.2869    1.4251    1.4983    1.5359    1.5526    1.5525

  Columns 21 through 30

    1.5215    1.4604    1.3731    1.2651    1.1511    1.0705    1.1176    1.3035    1.5701    1.8719

  Columns 31 through 40

    2.1140    2.3121    2.5284    2.7065    2.8575    2.9883    3.1036    3.2386    3.3569    3.4621

  Columns 41 through 50

    3.5862    3.6958    3.8244    3.9723    4.1396    4.2933    4.3316    4.3422    4.3508    4.3539
```

```
>> res.Sol(:,11)

ans =

   -1.0449
   -1.1101
         0
         0
```



A Stable NNM (Newmark Simulation)





Our Objective

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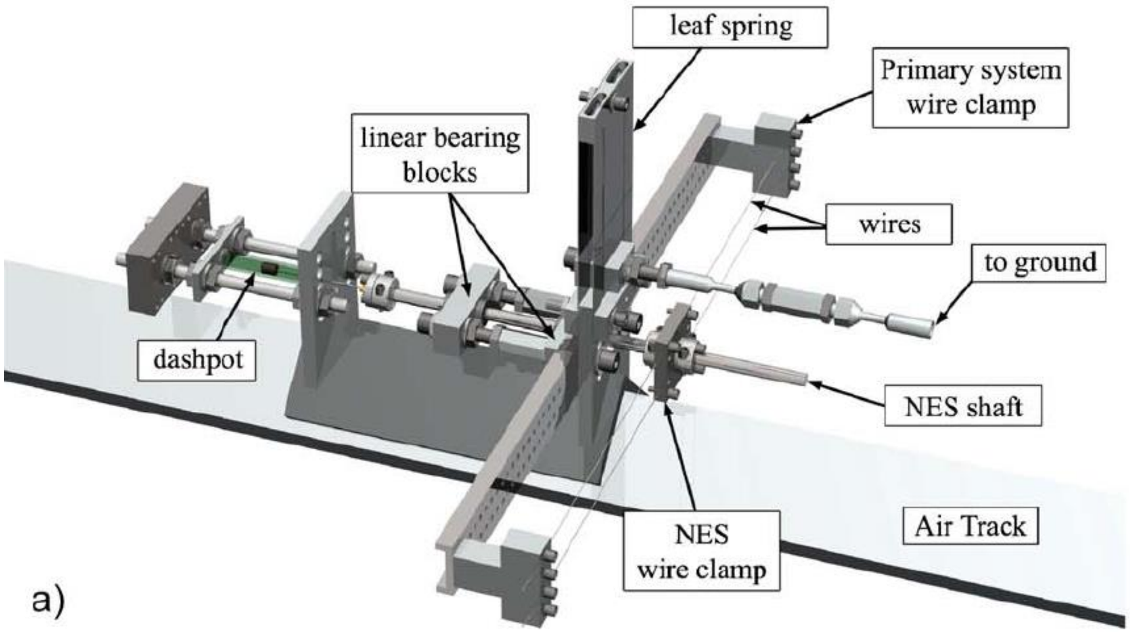
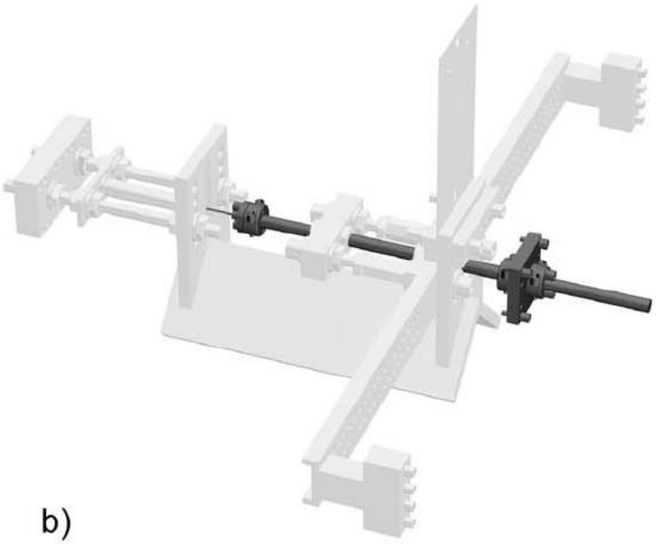
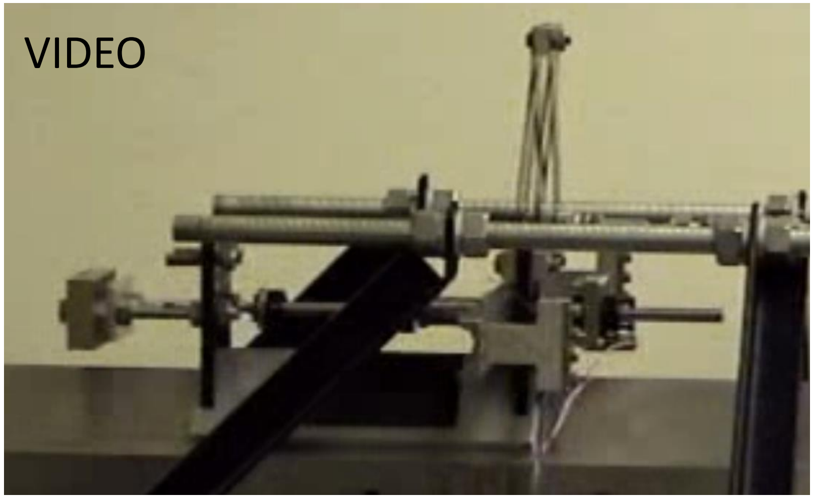
Do Modal Interactions Exist In Reality ?



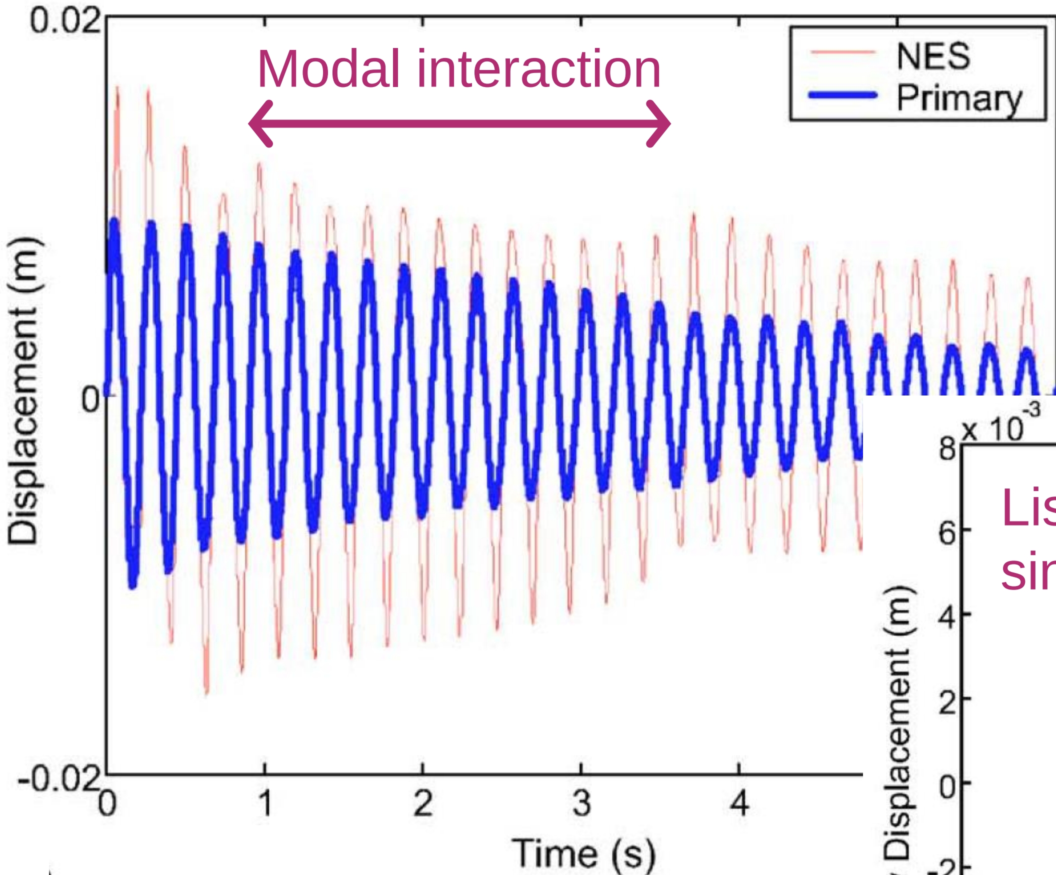
a)

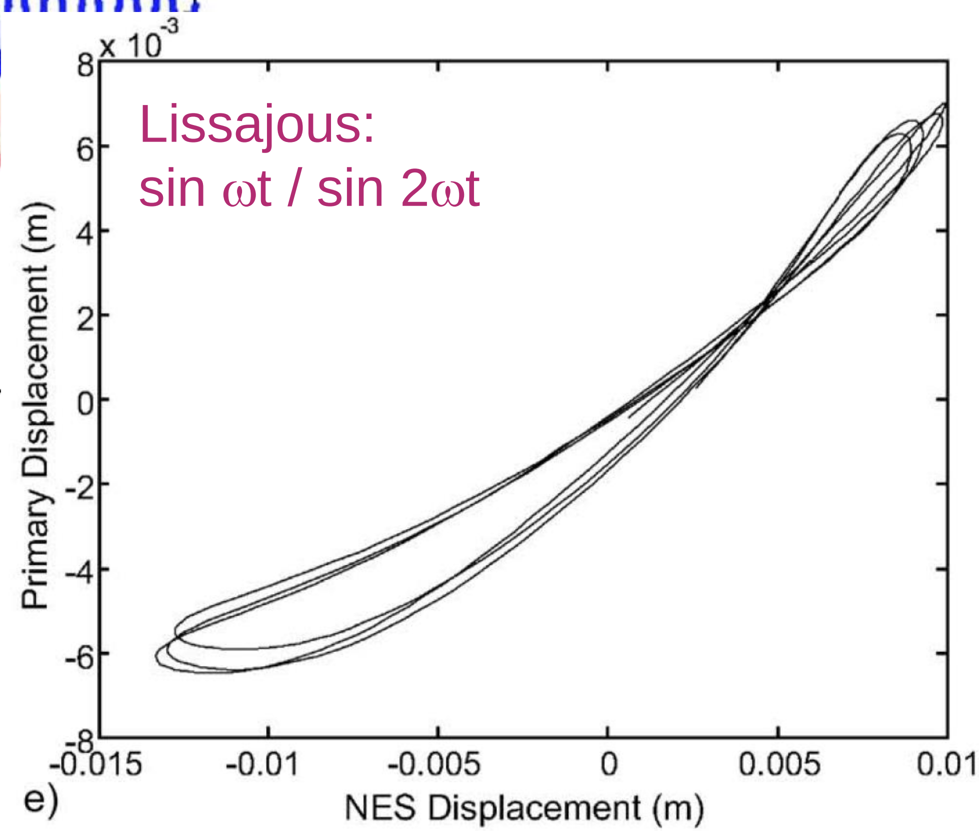


b)



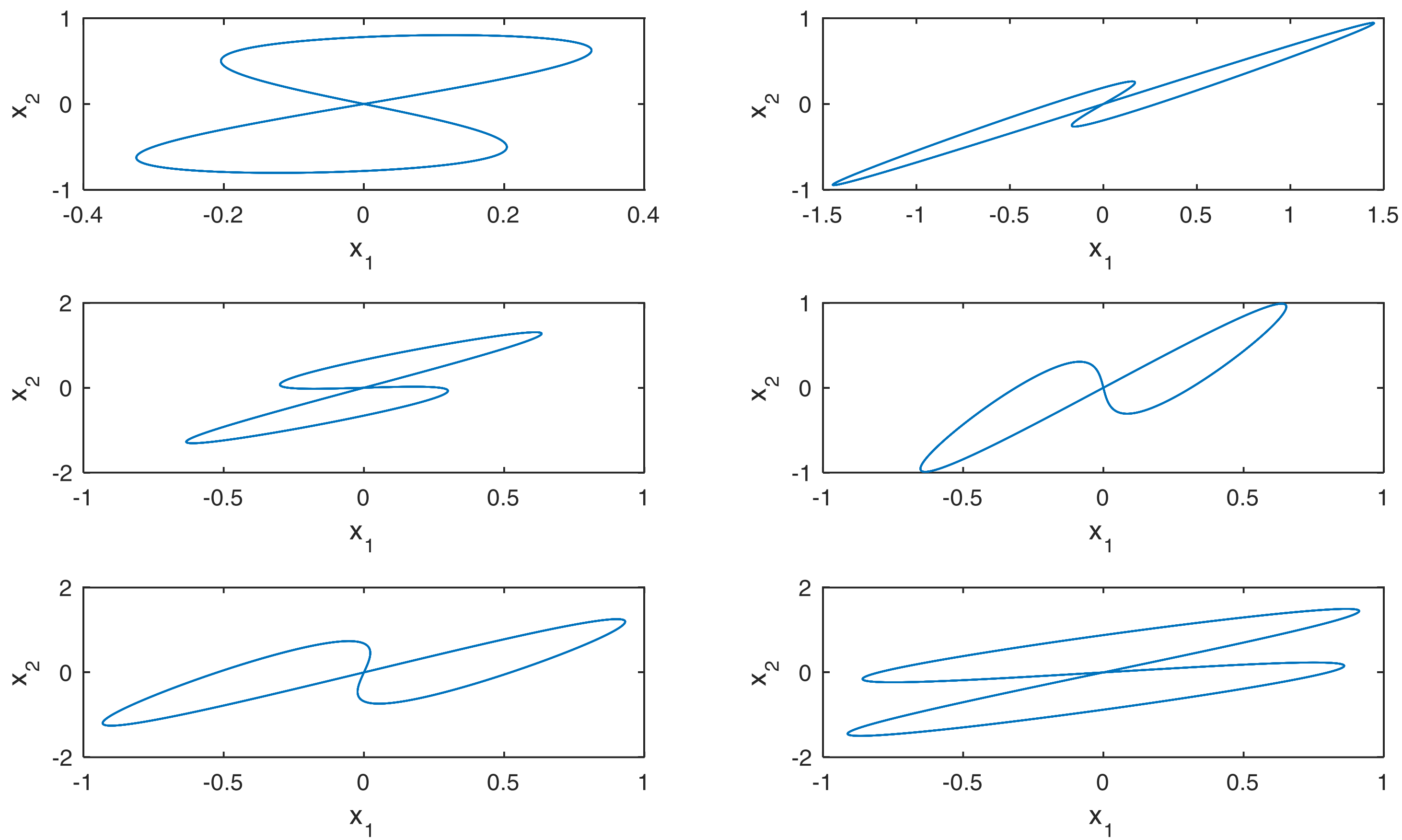
It Seems


0.02
0
-0.02
Modal interaction
NES
Primary
Displacement (m)
0
1
2
3
4
Time (s)


x 10^-3
8
6
4
2
0
-2
-4
-6
-8
Lissajous:
sin ωt / sin 2ωt
Primary Displacement (m)
-0.015
-0.01
-0.005
0
0.005
0.01
NES Displacement (m)
e)

Lissajous Curves…

```
>> LissajousNNM
1=sin/sin 2=cos/cos: 1
Enter harmonics 1:1
Enter harmonics 2:2
```



More Details

Experimental investigation of targeted energy transfer in strongly and nonlinearly coupled oscillators

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Jeffrey J. Kowtko, Young S. Lee, and Lawrence A. Bergman[c)]
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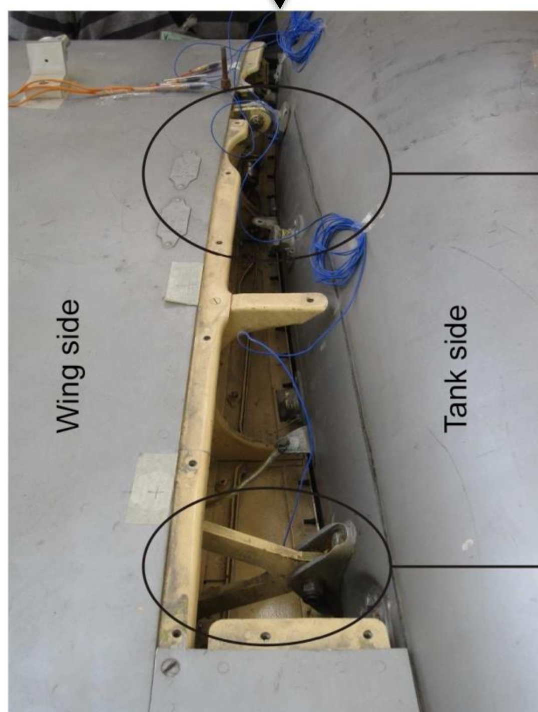
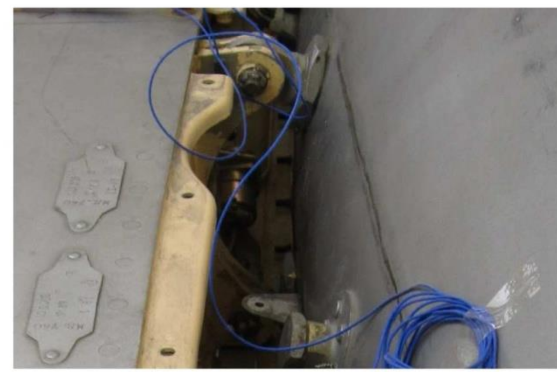
J. Acoust. Soc. Am., Vol. 118, No. 2, August 2005

Do Modal Interactions Exist In Complex Systems ?

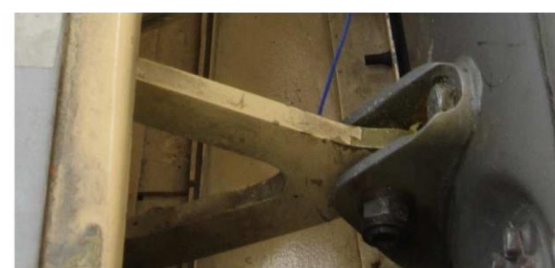
Bolted connections between external fuel tank and wing tip



Front connection



Rear connection



The Testing Campaign

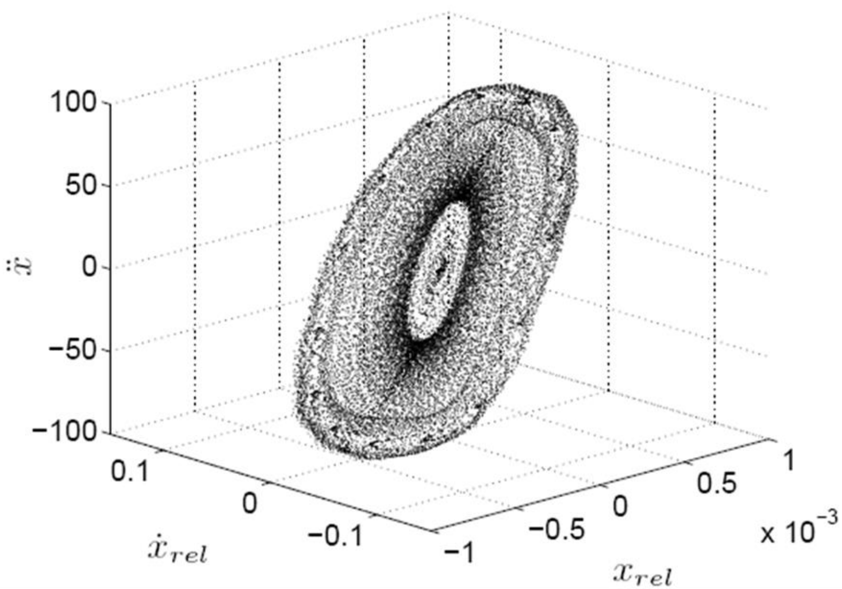
IMG_1679.avi

IMG_1680.MOV

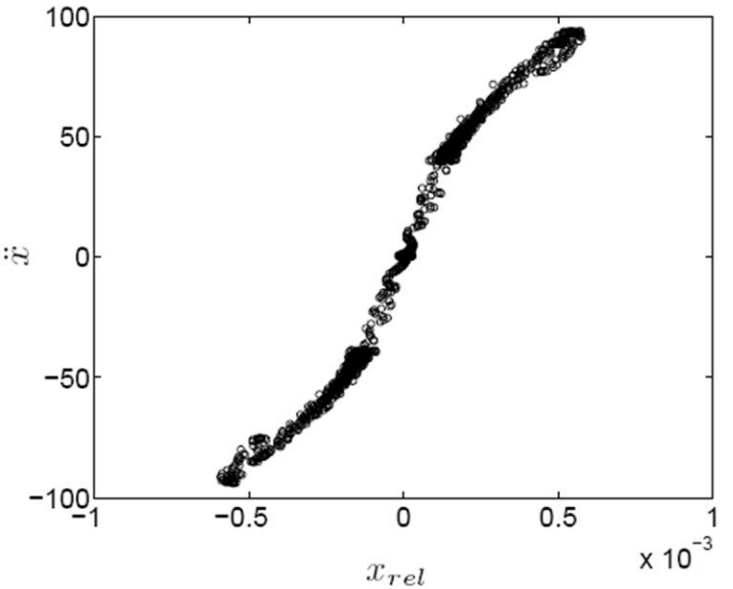
Softening Nonlinearity in the Bolted Connections

Swept sine testing

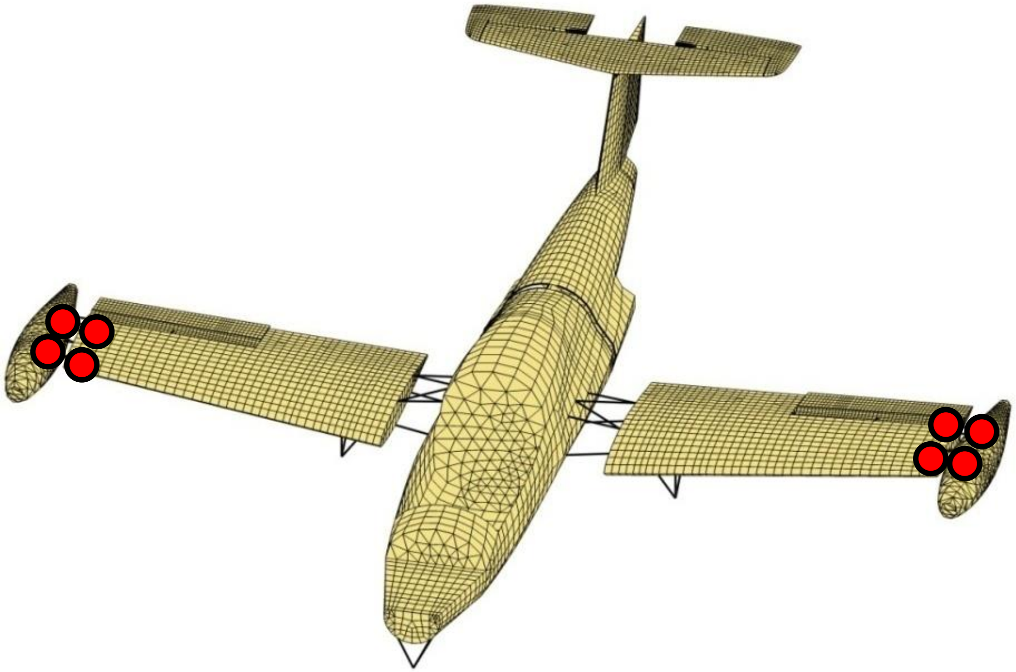
Acceleration surface method



Stiffness curve



Finite Element Model Reduction



Finite element model
(2D shells and beams, 85000 DOFs)

Condensation of the linear components of the model

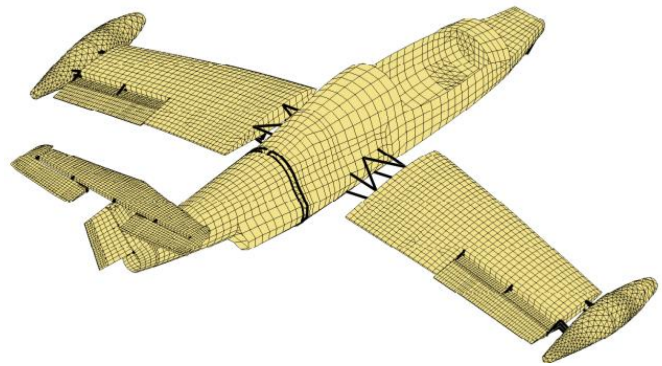
Craig-Bampton technique

- 8 remaining nodes
- 500 internal modes

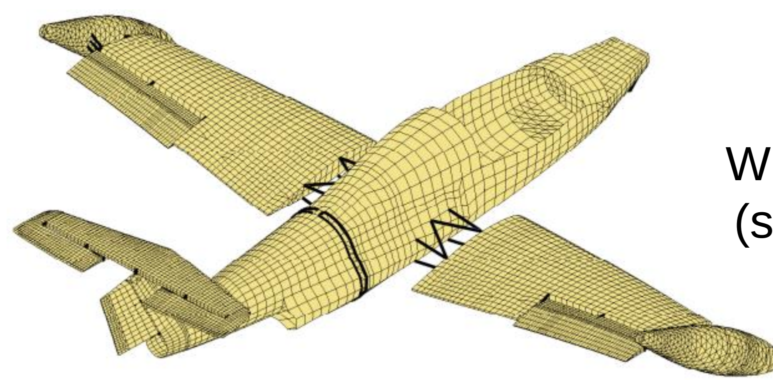
Reduced model accurate in [0-100] Hz, 548 DOFs

A Close Look at Two Modes

Mode	Freq. (Hz)	Mode	Freq. (Hz)
1	0.0936	13	21.2193
2	0.7260	14	22.7619
3	0.9606	15	23.6525
4	1.2118	16	25.8667
5	1.2153	17	28.2679
6	1.7951	18	29.3309
7	2.1072	19	31.0847
8	2.5157	20	34.9151
9	3.5736	21	39.5169
10	8.1913	22	40.8516
11	9.8644	23	47.3547
12	16.1790	24	52.1404

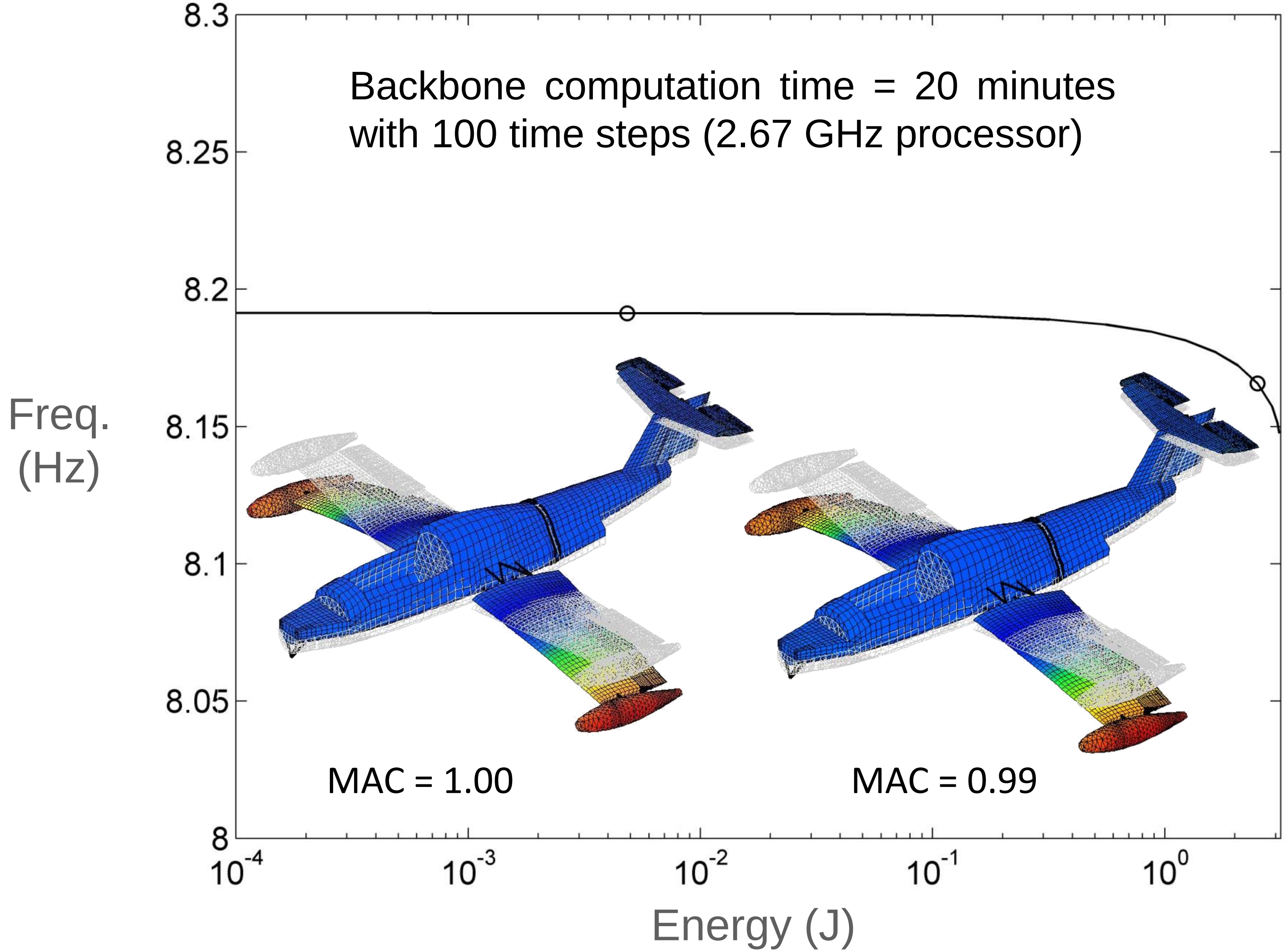


Wing bending

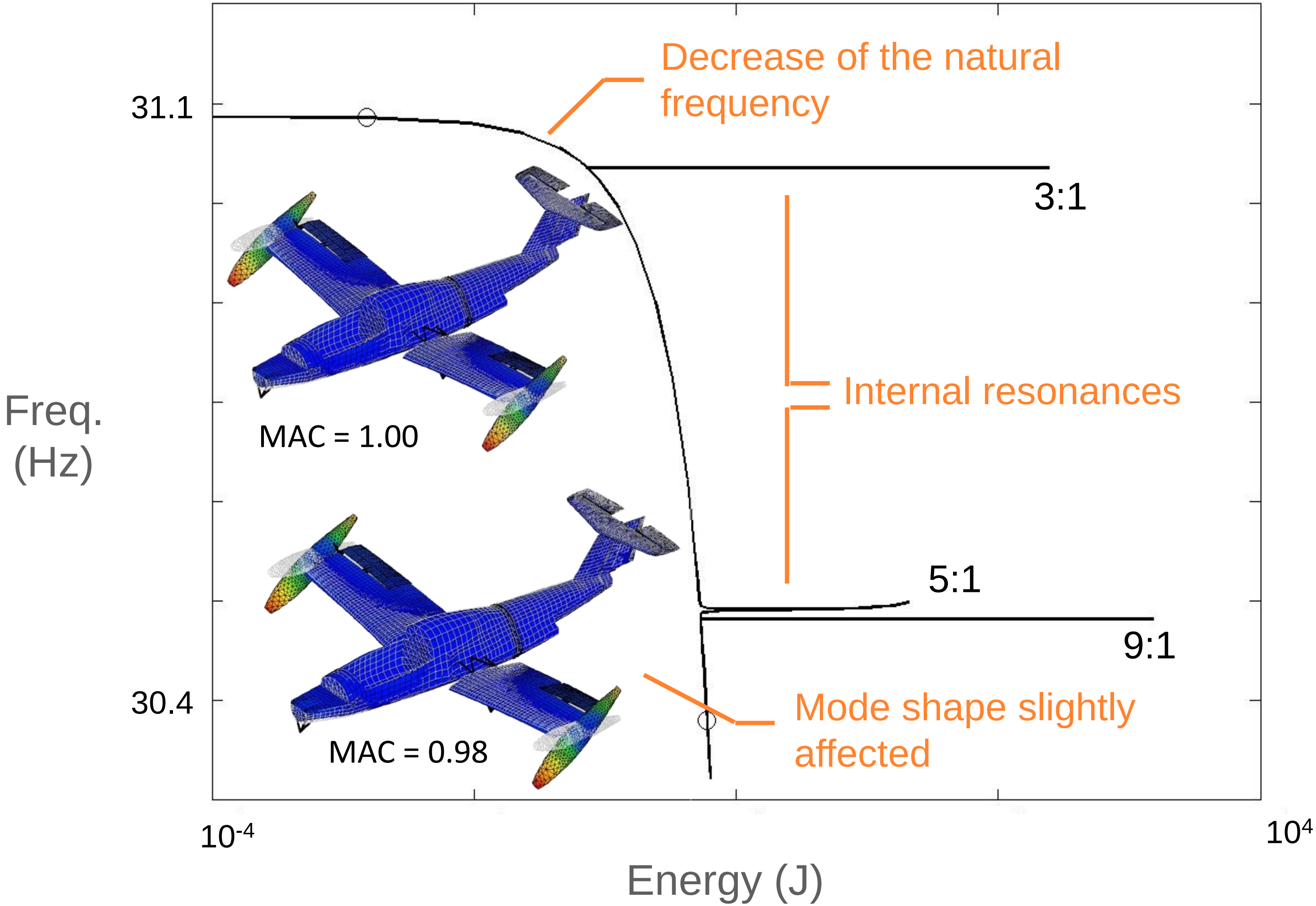


Wing torsion
(symmetric)

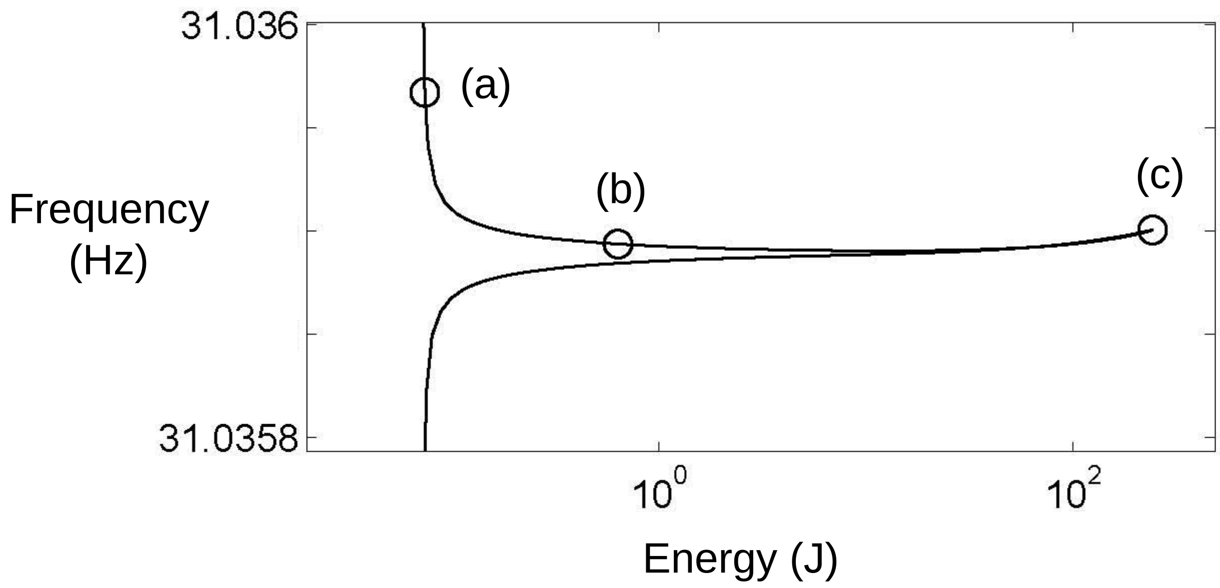
The First Wing Bending Mode Is Not Affected

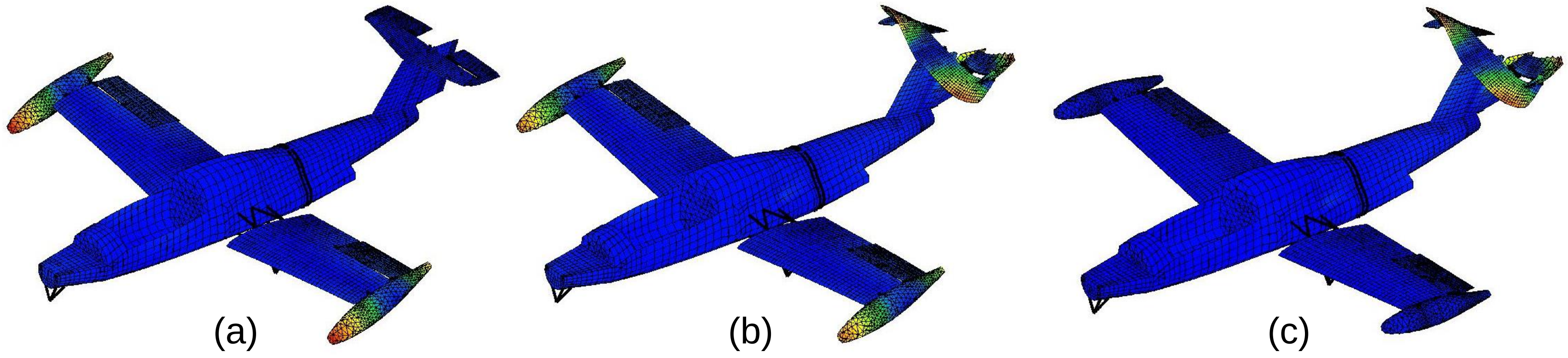


The First Wing Torsional Mode Is Nonlinear

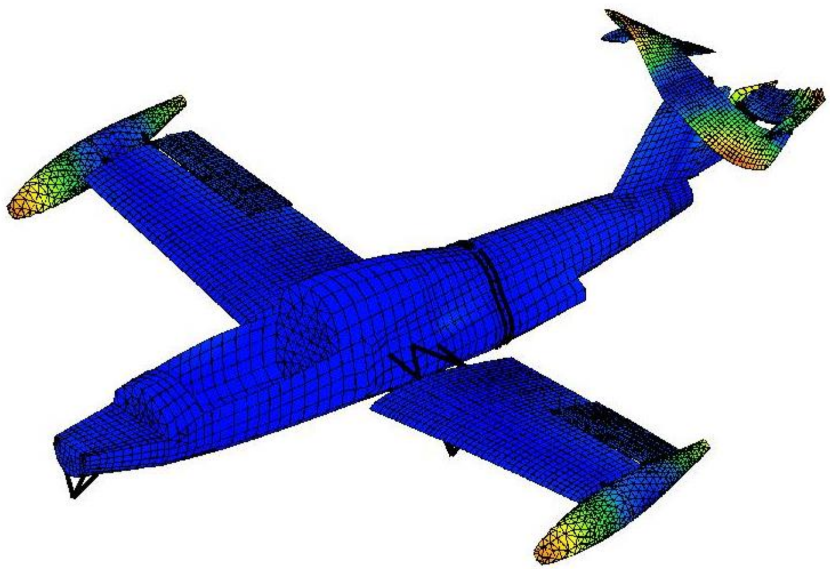


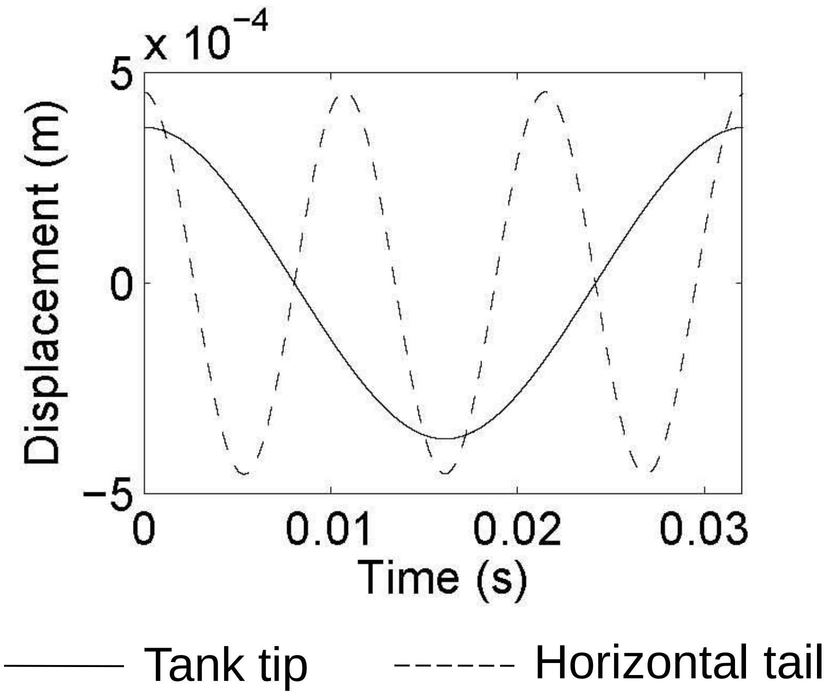
Close-Up of the 3:1 Modal Interaction





No Resemblance With Any LNMs





More Details

Nonlinear Modal Analysis of a Full-Scale Aircraft

G. Kerschen,* M. Peeters,† and J. C. Golinval*
University of Liège, 4000 Liège, Belgium
and
C. Stéphan‡
ONERA–The French Aerospace Lab, 92320 Châtillon, France

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Modal Interactions @ SAB & Airbus

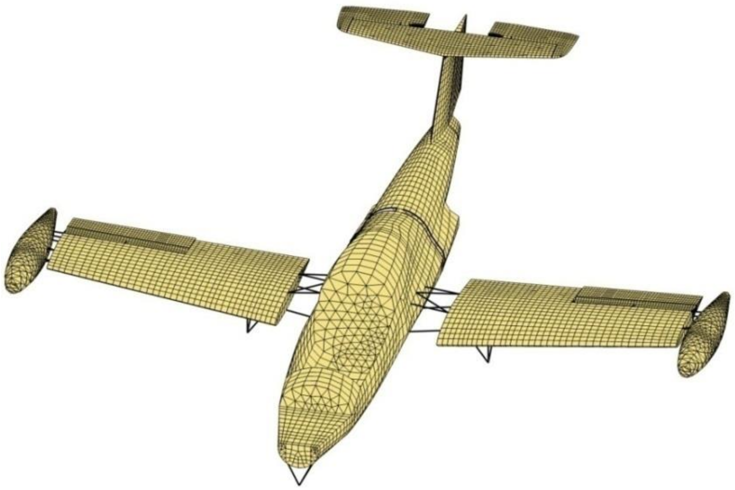
Outline of this Lecture

Discover the properties of nonlinear modes

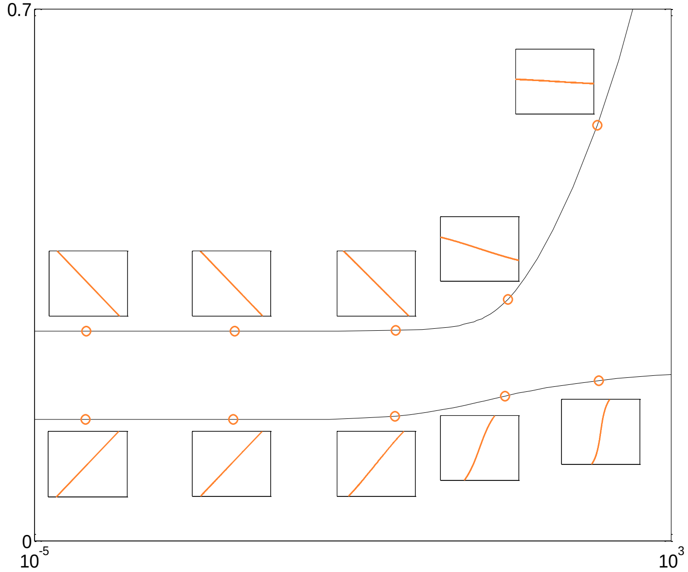
Numerical computation of nonlinear modes

Another puzzling result !

How To Calculate NNMs ?



Finite element model



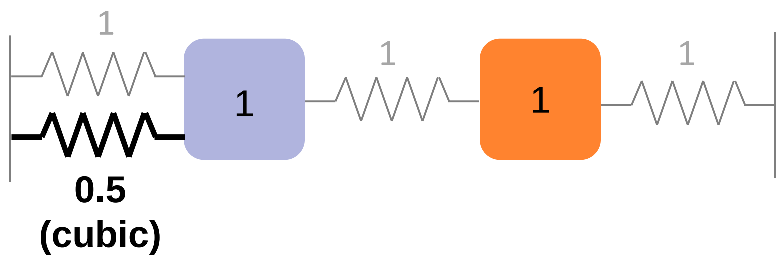
Frequency-energy plot

Basic Idea

An NNM is a periodic motion of a nonlinear system.

1. Look for periodic solutions !

Naive Approach



$z_0 \to z(t, z_0)$ using numerical time integration

Case 1: $x_1(0) = 1$ and $x_2(0) = 1$ $\longrightarrow$ ~~Periodic solution~~

Case 2: $x_1(0) = 1$ and $x_2(0) = 1.08$ $\longrightarrow$ ~~Periodic solution~~

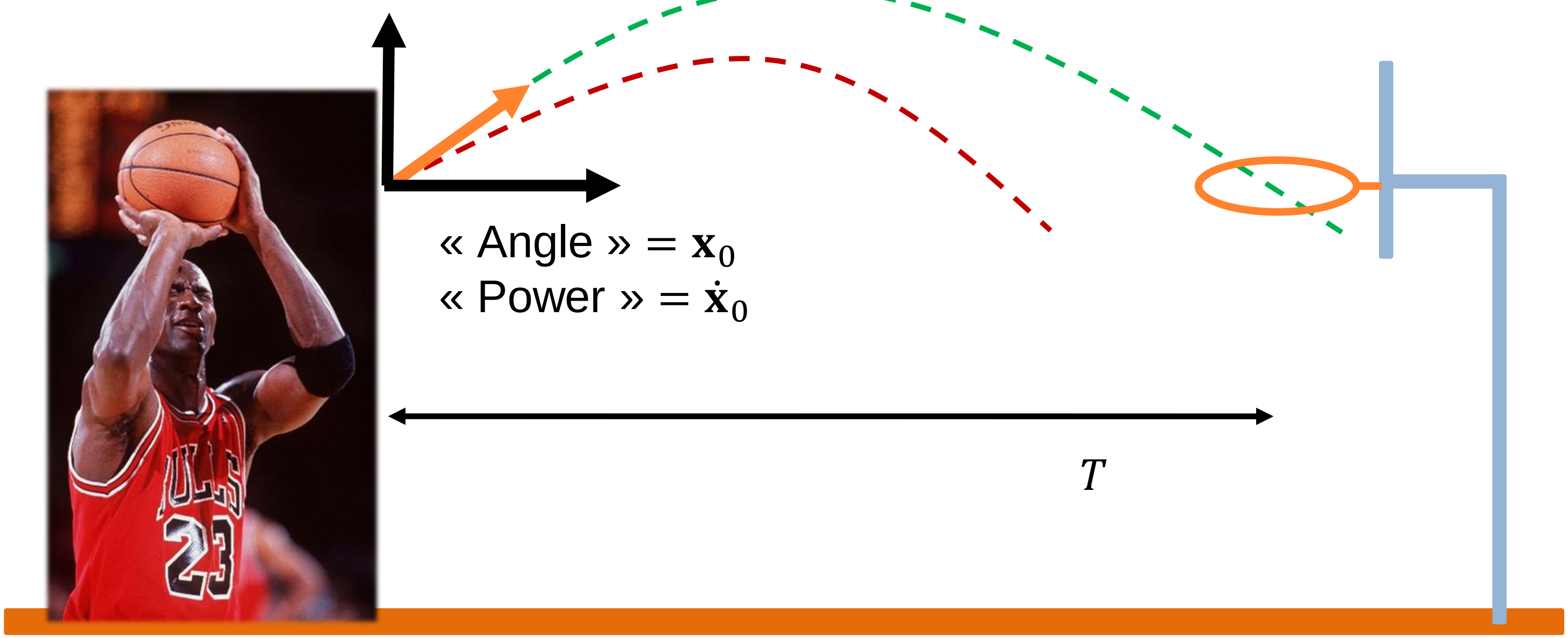
Case 3: $x_1(0) = 1$ and $x_2(0) = 1.16$ $\longrightarrow$ ~~Periodic solution~~

Case 4: $x_1(0) = 1$ and $x_2(0) = 1.24$ $\longrightarrow$ ~~Periodic solution~~

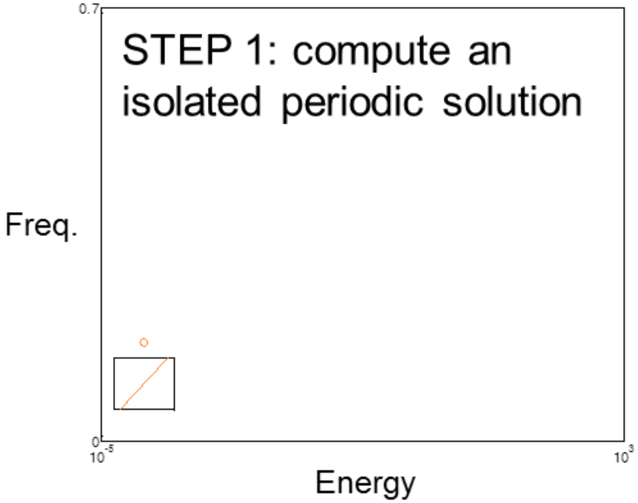
Case 5: $x_1(0) = 1$ and $x_2(0) = 1.19$ $\longrightarrow$ Periodic solution !

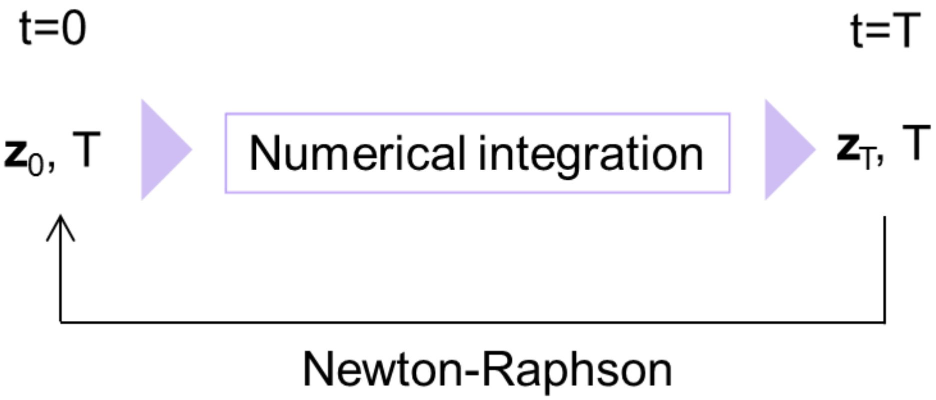
Shooting Technique

Optimisation of the initial state of a system $[\mathbf{x}_0 \quad \dot{\mathbf{x}}_0]^T$ to obtain a periodic solution after time integration over a period T.



A More Robust Approach





Newton-Raphson

Example: find the zero of $f(x) = \frac{1}{2}(x-1)^2$

$$f(x_1) = f(x_0) + \left.\frac{df(x)}{dx}\right|_{x=x_0}(x_1 - x_0) = 0$$

$$\frac{1}{2}(x_0 - 1)^2 + (x_0 - 1)(x_1 - x_0) = 0$$

$$x_1 = x_0 + \frac{-\frac{1}{2}(x_0 - 1)^2}{(x_0 - 1)} = x_0 - \frac{1}{2}(x_0 - 1) = \frac{x_0 + 1}{2}$$

$$\boxed{x_{j+1} = \frac{x_j + 1}{2}}$$

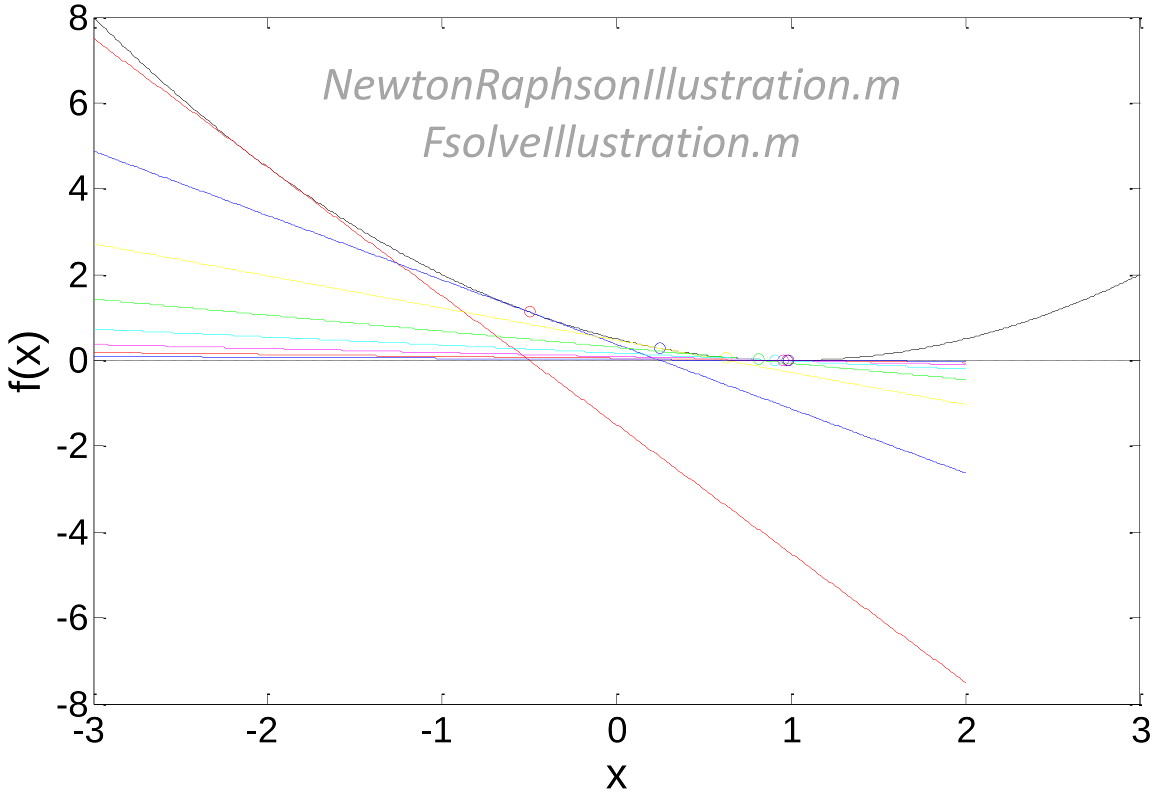
Matlab Code: Homemade or fsolve

```
function NewtonRaphsonIllustration

clear all;close all;clc

plot([-3:0.01:3],0.5*([-3:0.01:3]-1).^2,'k',[-3:3],[0 0 0 0 0 0 0],'k--');hold on;pause

StartingPoint=-2;

while abs(0.5*(StartingPoint-1)^2)>0.0001
    plot([-3:0.01:3],0.5*(StartingPoint-1)^2+(StartingPoint-1)*([-3:0.01:3]-StartingPoint))

    StartingPoint=0.5*(StartingPoint+1),pause
end
SolutionFound=[StartingPoint 0.5*(StartingPoint-1)^2]
```

```
function FsolveIllustration

clear all;close all;clc

StartingPoint=-2;
SolutionFound=fsolve('Quadratic',StartingPoint)
```

```
function y=Quadratic(x)

y=0.5*(x-1)^2
```

Result


NewtonRaphsonIllustration.m
FsolveIllustration.m
8
6
4
2
0
-2
-4
-6
-8
f(x)
-3
-2
-1
0
1
2
3
x

State Space Formulation

$$\mathrm{M}\ddot{x}(t) + Kx(t) + f_{nl}\{x(t), \dot{x}(t)\} = 0$$

$$\dot{z} = g(z, t)$$

State-space form

$$z^T = [x^T \;\; \dot{x}^T]$$

where

$$g(z) = \begin{bmatrix} \dot{x} \\ -M^{-1}[Kx + f_{nl}(x, \dot{x})] \end{bmatrix}$$

Shooting Algorithm

$$\mathbf{H}(\mathbf{z}_{p0}, T) \equiv \mathbf{z}_p(T, \mathbf{z}_{p0}) - \mathbf{z}_{p0} = \mathbf{0}$$

Periodicity condition
(2-point BVP)

Numerical solution through iterations:

$$\mathbf{H}\left(\mathbf{z}_{p0}^{(0)}, T^{(0)}\right) + \boxed{\left.\frac{\partial \mathbf{H}}{\partial \mathbf{z}_{p0}}\right|_{\left(\mathbf{z}_{p0}^{(0)}, T^{(0)}\right)}} \Delta \mathbf{z}_{p0}^{(0)} + \boxed{\left.\frac{\partial \mathbf{H}}{\partial T}\right|_{\left(\mathbf{z}_{p0}^{(0)}, T^{(0)}\right)}} \Delta T^{(0)} + \cancel{\mathrm{H.O.T.}} = 0$$

$$\boxed{\frac{\partial \mathbf{H}}{\partial \mathbf{z}_0}(\mathbf{z}_0, T) = \left.\frac{\partial \mathbf{z}(t, \mathbf{z}_0)}{\partial \mathbf{z}_0}\right|_{t=T} - \mathbf{I}}$$

$$\boxed{\begin{aligned} \frac{\partial \mathbf{H}}{\partial T}(\mathbf{z}_0, T) &= \left.\frac{\partial \mathbf{z}(t, \mathbf{z}_0)}{\partial t}\right|_{t=T} \\ &= \mathbf{g}\left(\mathbf{z}\left(T, \mathbf{z}_0\right)\right) \end{aligned}}$$

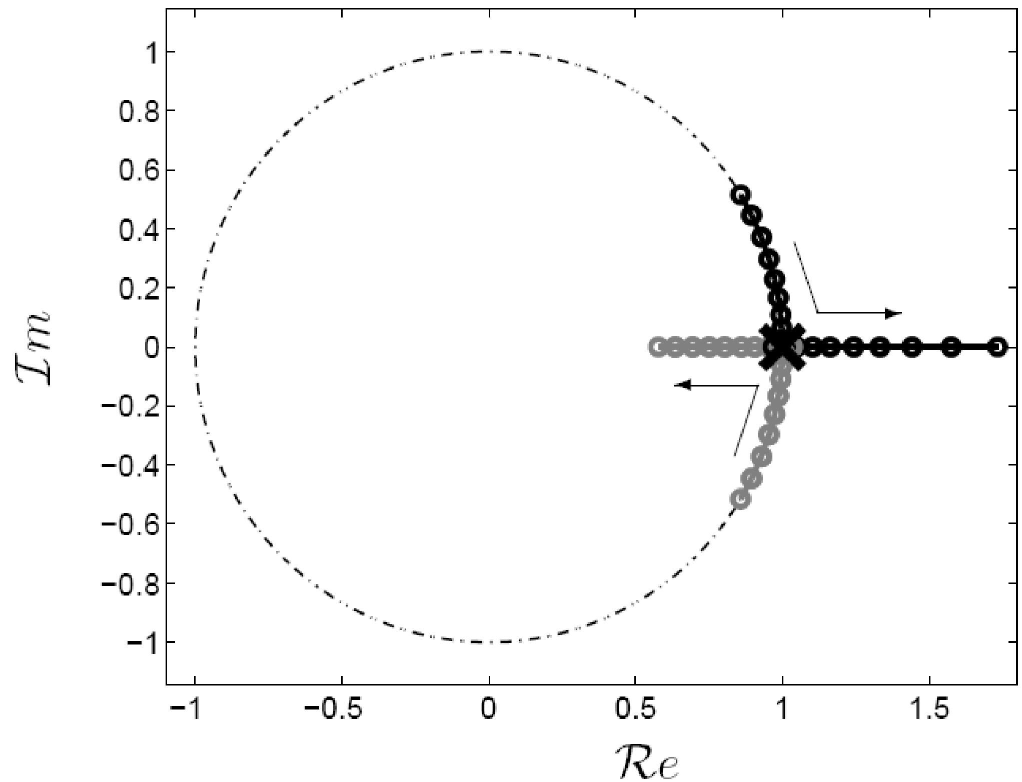
2n x 2n — Monodromy matrix

2n x 1

Jacobian Matrix through Linear ODEs

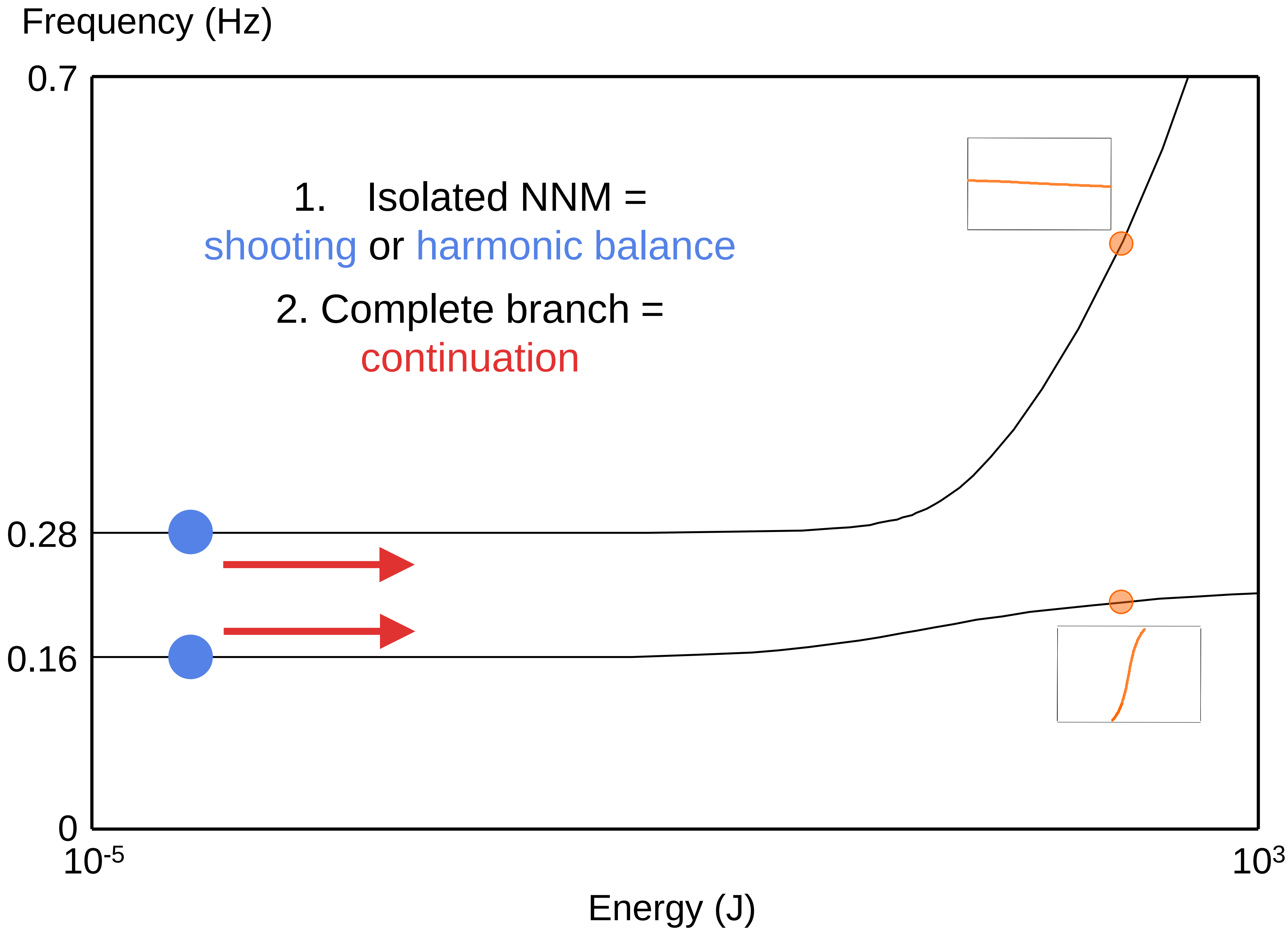
$$\left[\frac{\partial \mathbf{z}(t, \mathbf{z}_0)}{\partial \mathbf{z}_0}\right] \longrightarrow \text{Floquet multipliers} \longrightarrow \text{NNM stability}$$

If a Floquet multiplier has a magnitude larger than one, then the periodic solution is *unstable*; otherwise, it is *stable* in the linear sense.



Floquet multipliers

Combining Shooting with Continuation



More Details

Nonlinear normal modes, Part II: Toward a practical computation using numerical continuation techniques

M. Peeters, R. Viguié, G. Sérandour, G. Kerschen *, J.-C. Golinval

Structural Dynamics Research Group, Department of Aerospace and Mechanical Engineering, University of Liège, 1 Chemin des Chevreuils (B52/3), B-4000 Liège, Belgium

Mechanical Systems and Signal Processing 23 (2009) 195–216

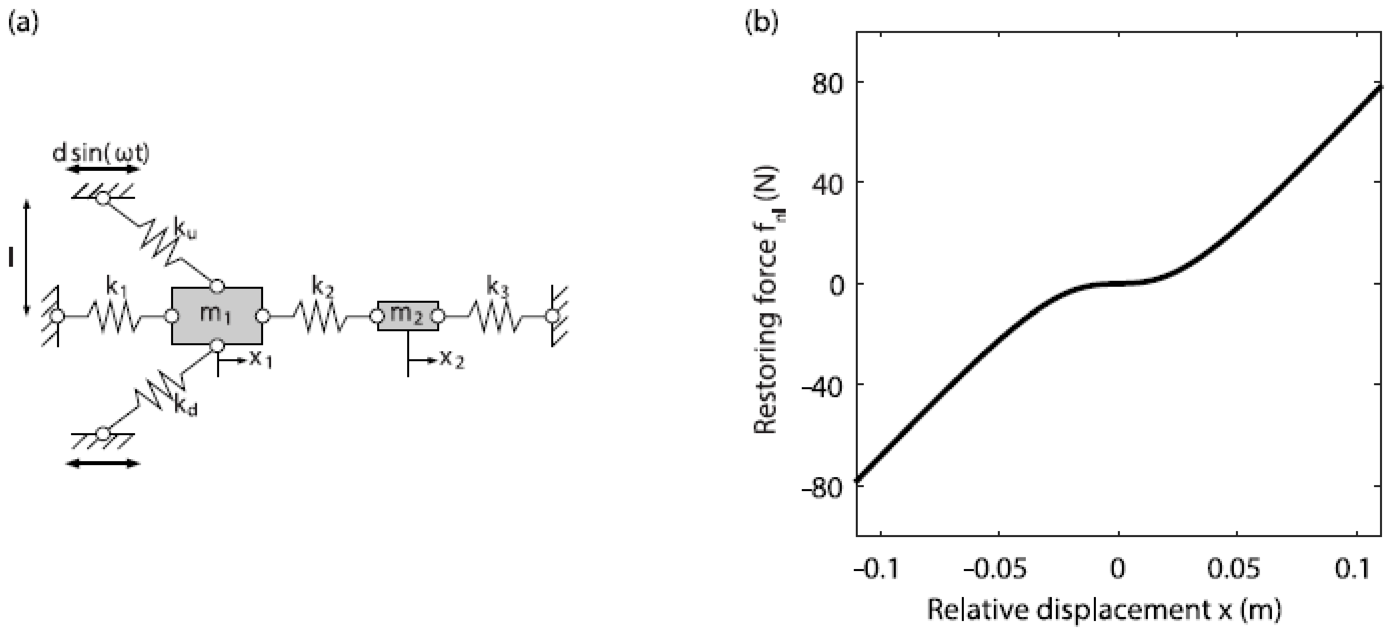
Outline of this Lecture

Discover the properties of nonlinear modes

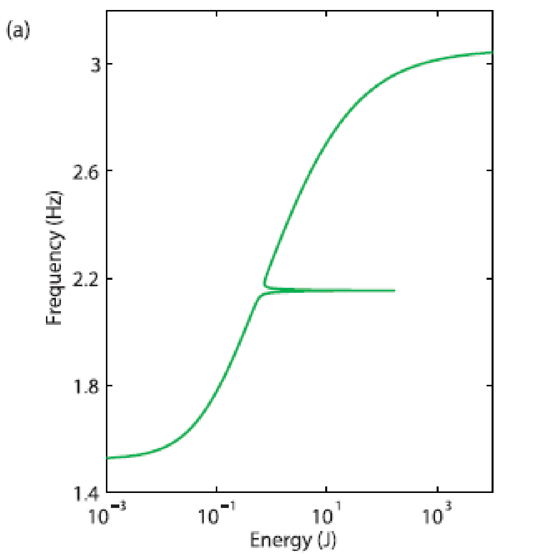
Numerical computation of nonlinear modes

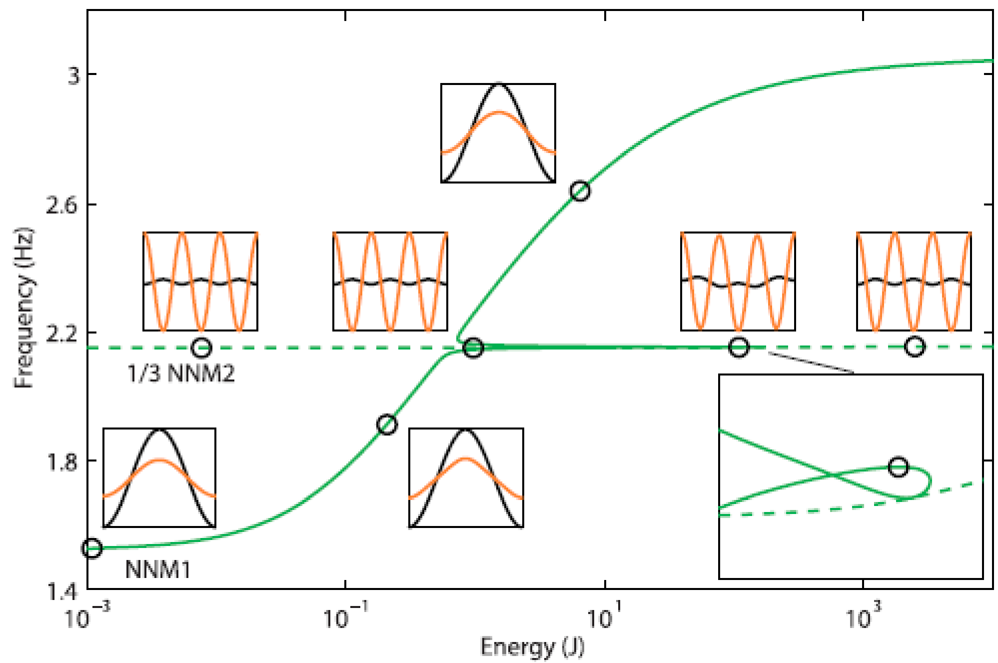
Another puzzling result !

2DOF System Excited from the Base

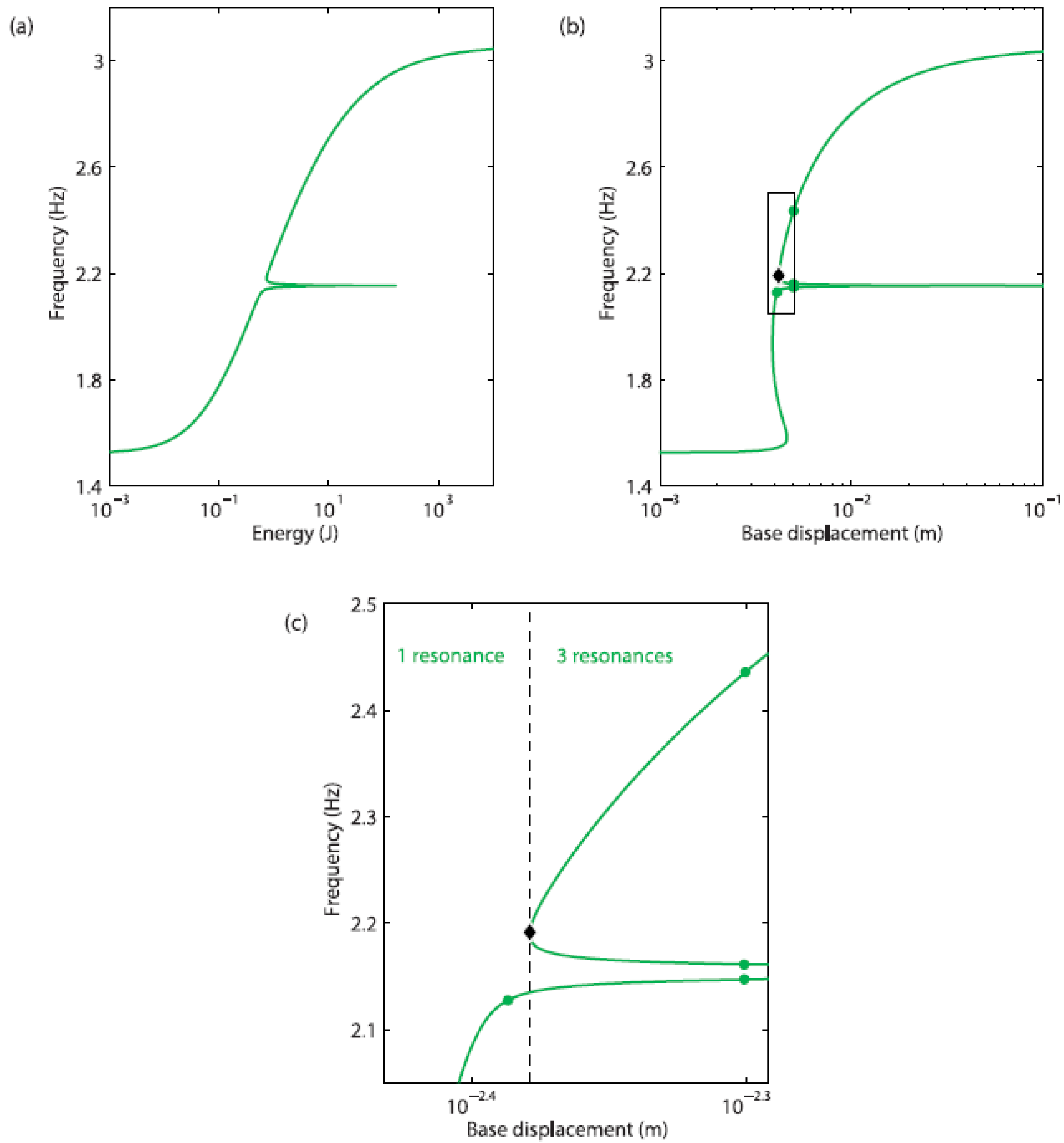


Mode 1 Features a Modal Interaction

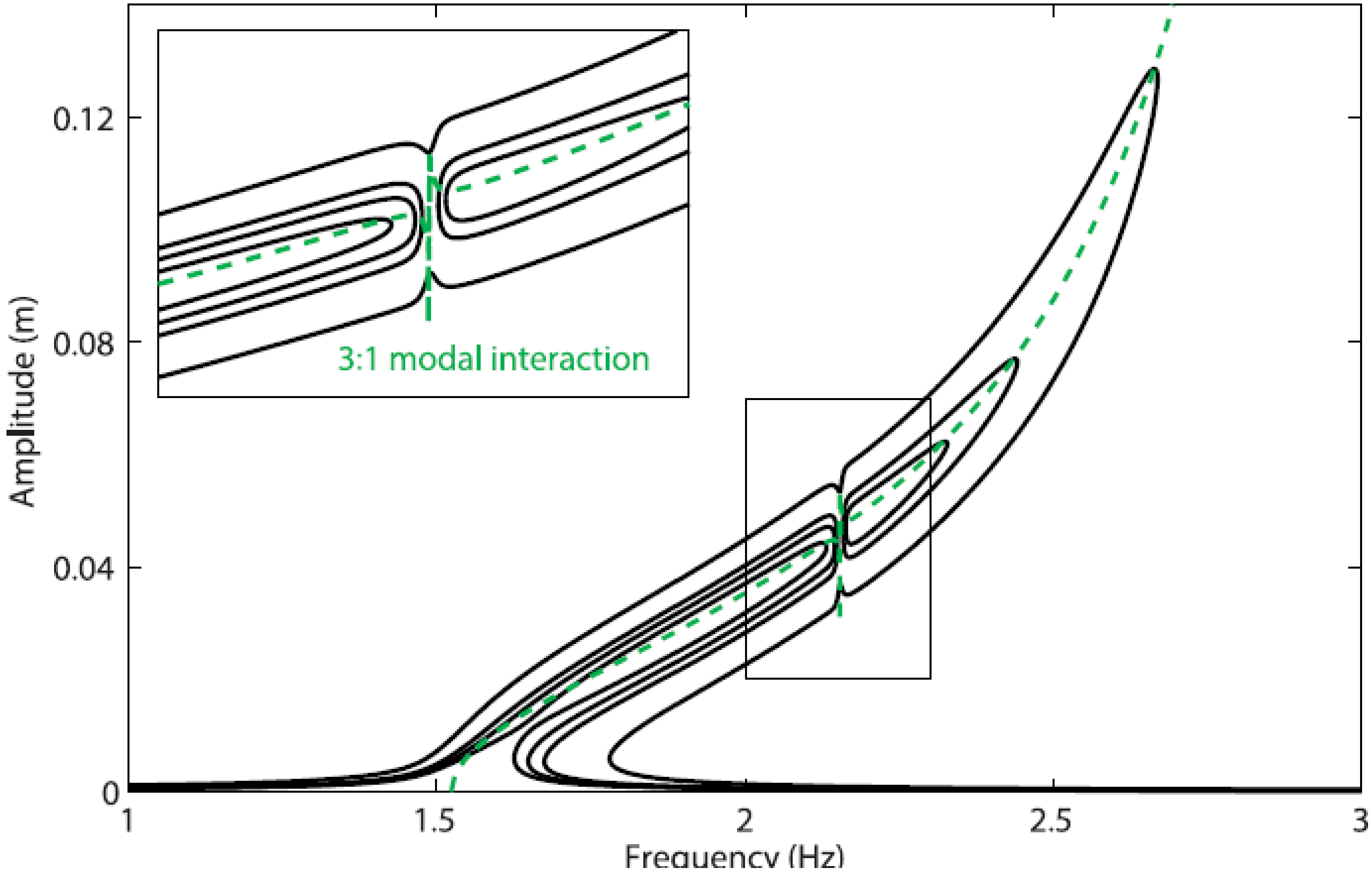




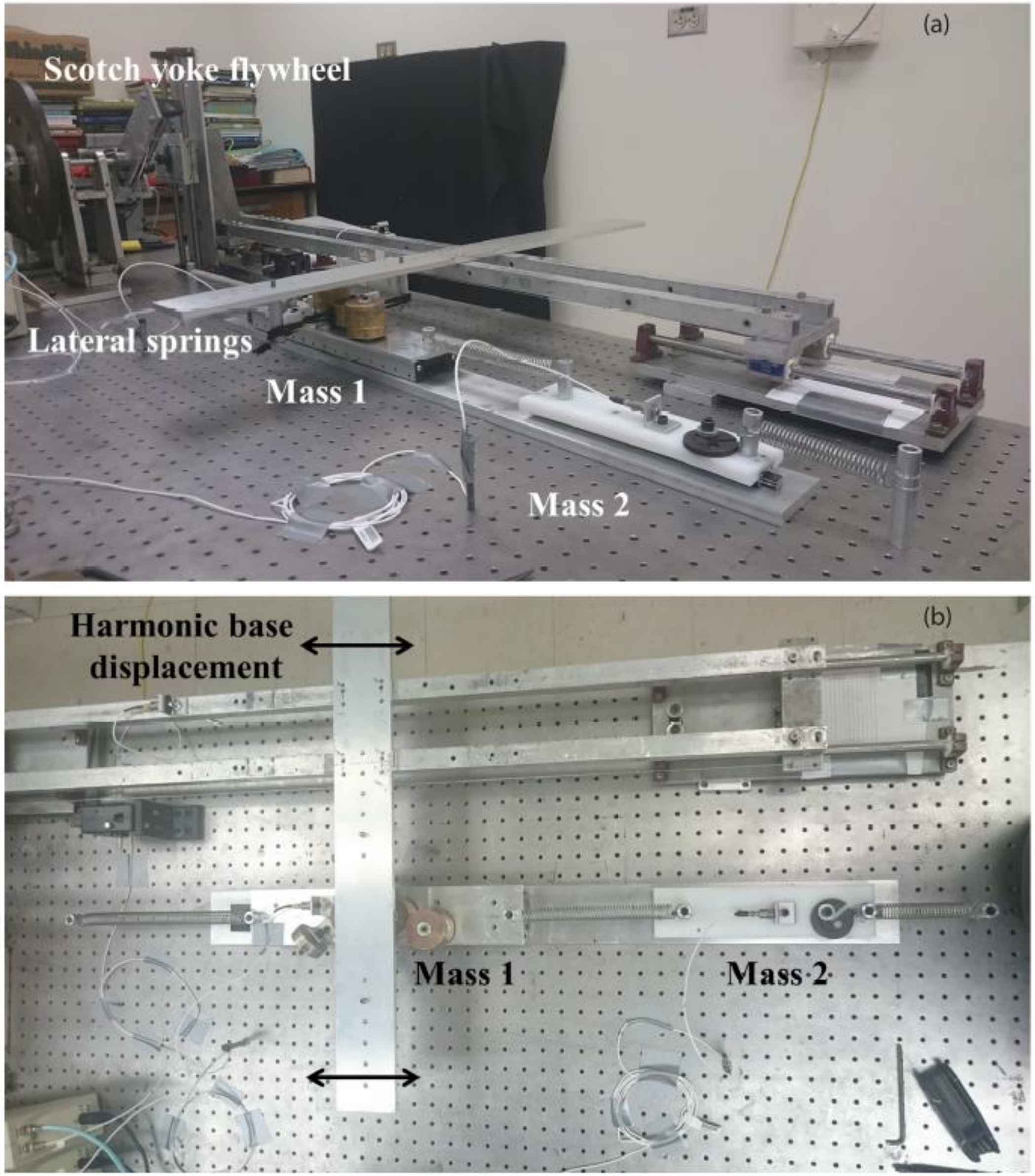
NNM as a Function of the Base Displacement



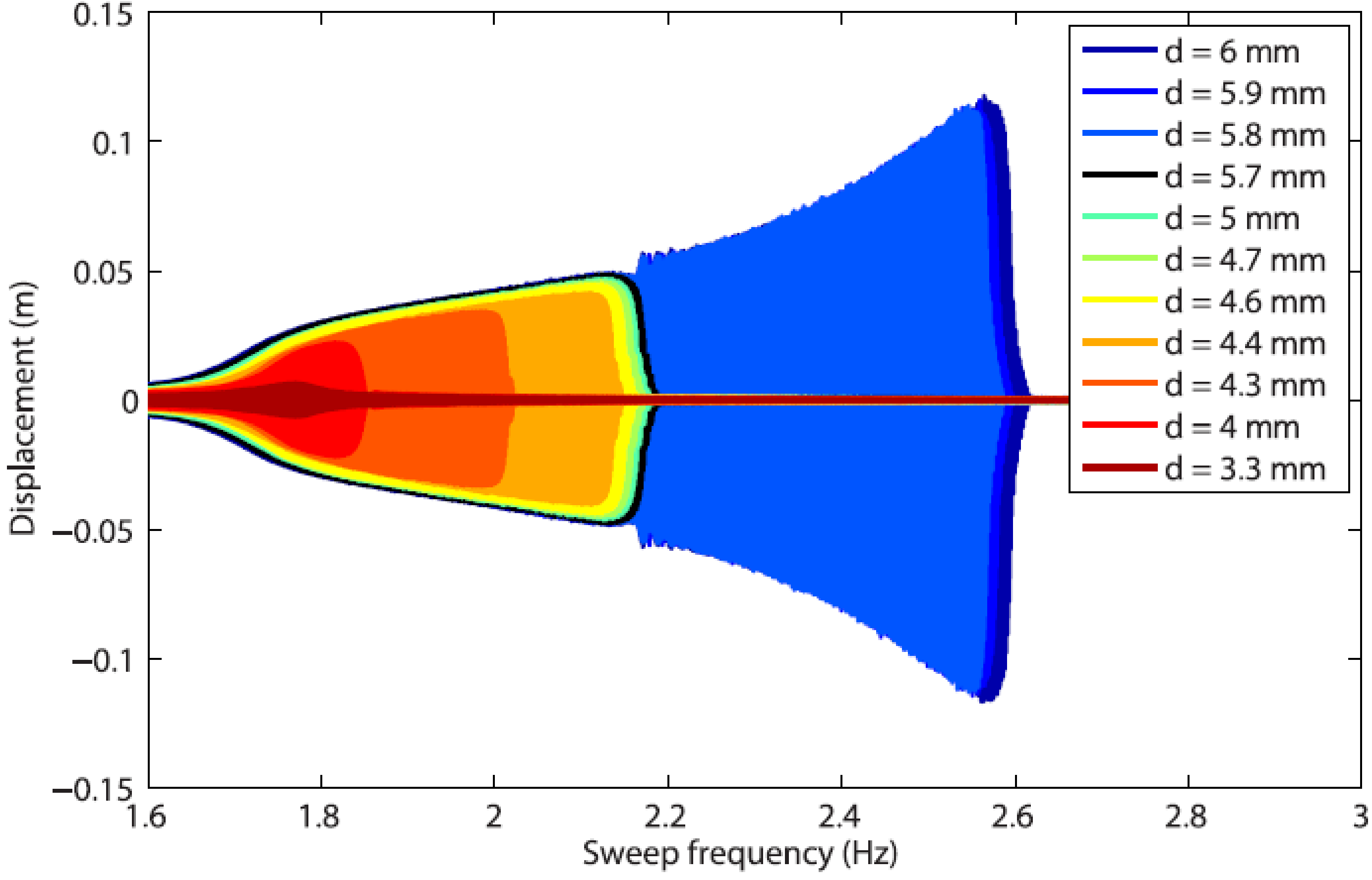
The Forced Damped Response: An Island !



Experimental Set-Up



Experimental Evidence



More Details

RESEARCH ARTICLE

Experimental study of isolas in nonlinear systems featuring modal interactions

Thibaut Detroux[1] *, **Jean-Philippe Noël**[1], **Lawrence N. Virgin**[2], **Gaëtan Kerschen**[1]

1 Space Structures and Systems Laboratory, Aerospace and Mechanical Engineering Department, University of Liège, Liège, Belgium, **2** Nonlinear Dynamics Group, Pratt School of Engineering, Duke University, Durham, North Carolina, United States of America

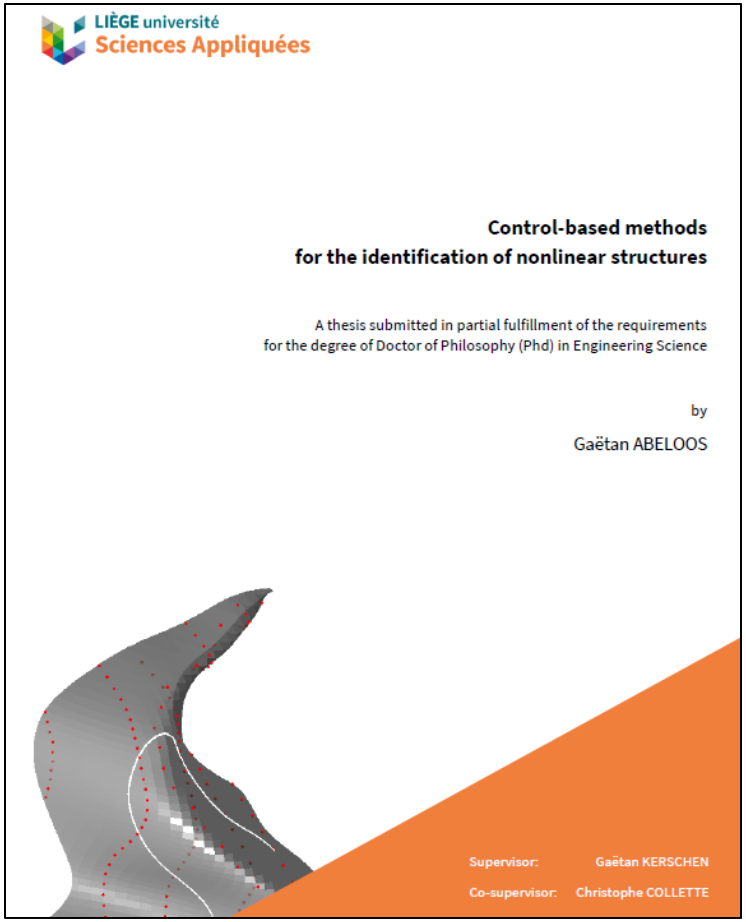
In Summary

Nonlinear normal modes, a rigorous extension of the concept of modes to nonlinear systems.

Similar concept (resonance !) but fundamental differences:

- Frequency-energy dependence
- Bifurcations
- Stability
- Modal interactions (harmonics)

Current Research @ S3L (G. Abeloos' Thesis)



We can now calculate

1. backbones directly from the experiment (without a model)
2. stable and unstable frequency responses including isolated branches

PhD thesis defense on November 24 at 15.00 (02 math)
Possibility to do a new PhD on the topic…

Current Research @ S3L (G. Abeloos' Thesis)