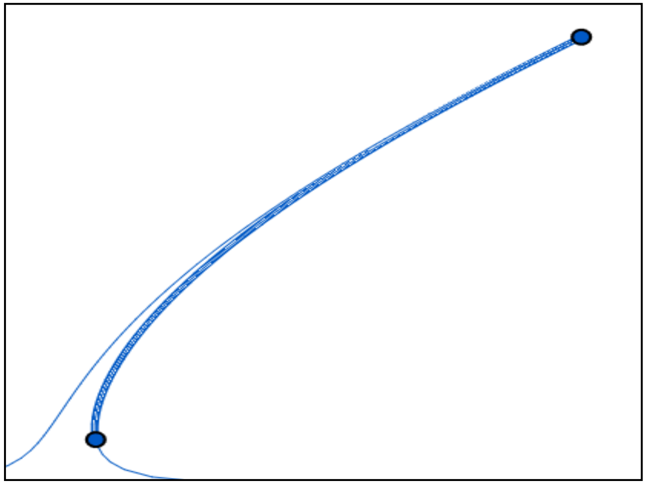
# Nonlinear Vibrations of Aerospace Structures

University of Liège, Belgium

**L02 Impact of NL**

Nonlinear FRFs

4 new concepts

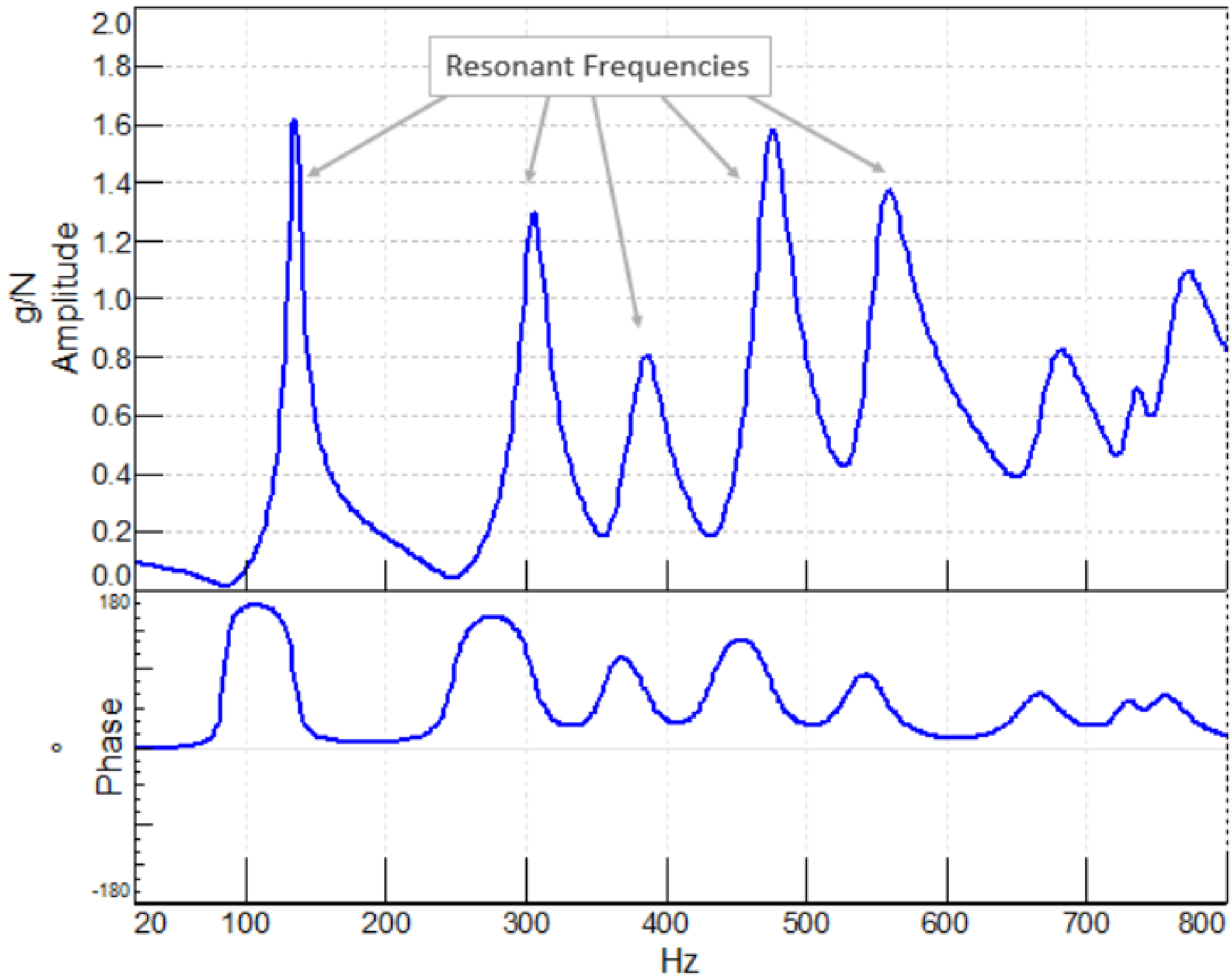


# The Frequency Response Function

$$X(\omega) = H(\omega)F(\omega)$$

Output FRF Input

# Frequency Response Function: Bode Plot




Bode plot of amplitude and phase of a FRF function. Amplitude has peaks corresponding to resonances . Phase has shift at resonant frequency.

# How Do We Calculate the FRF ?

$$m\ddot{x} + kx = F sin\,\omega t$$

$x(t) = X sin\,\omega t$

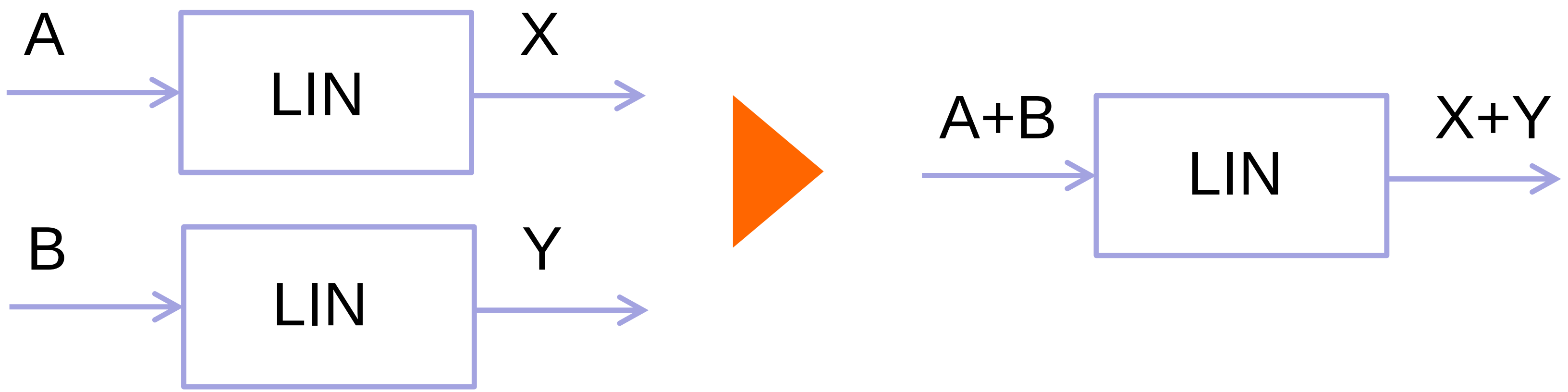
$$-m\omega^2 X + kX = F$$

$$\boxed{X = \frac{F}{k - \omega^2 m}}$$

**Linear system**: output directly proportional to input.

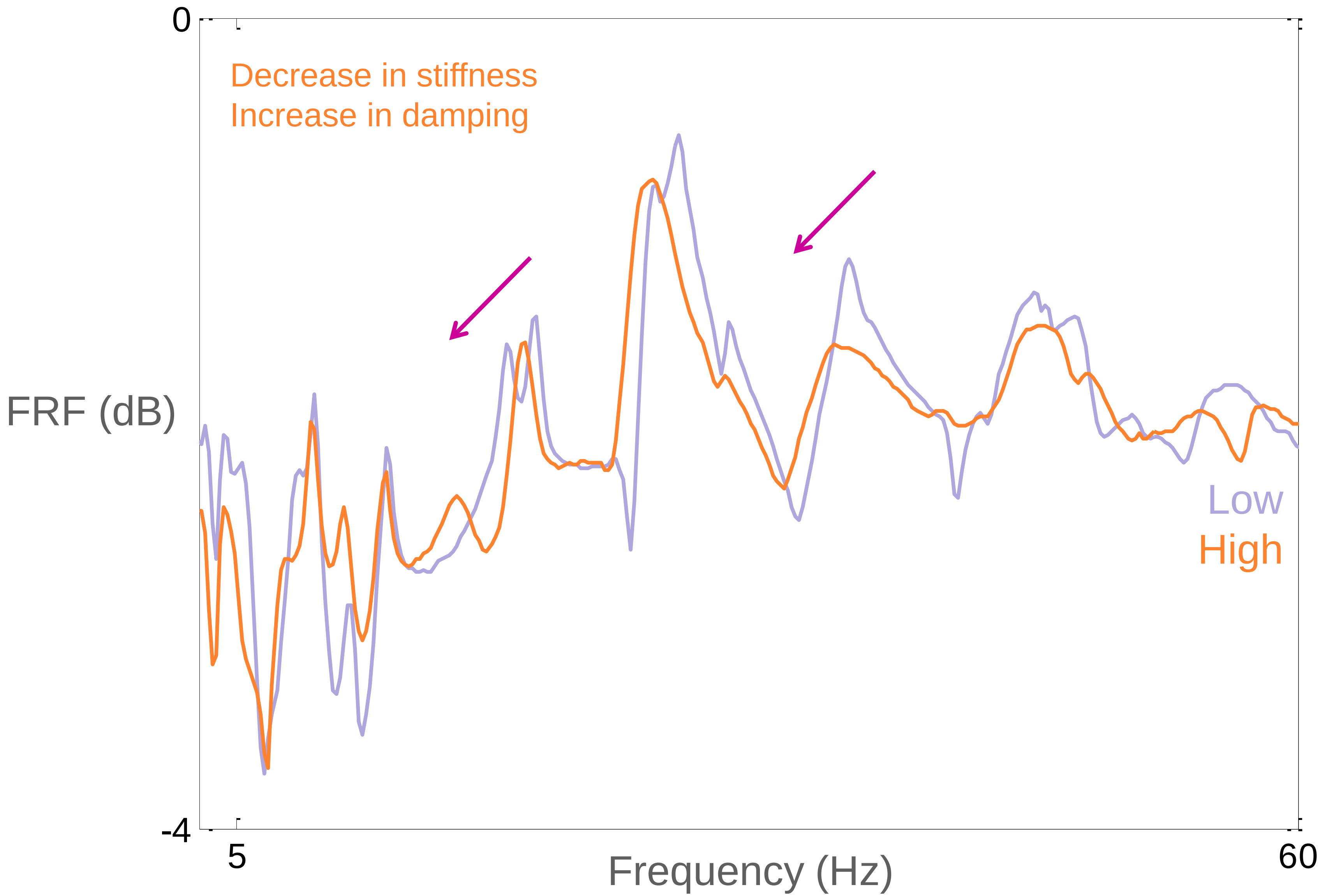
# Superposition Principle

Cornerstone of linear theory:



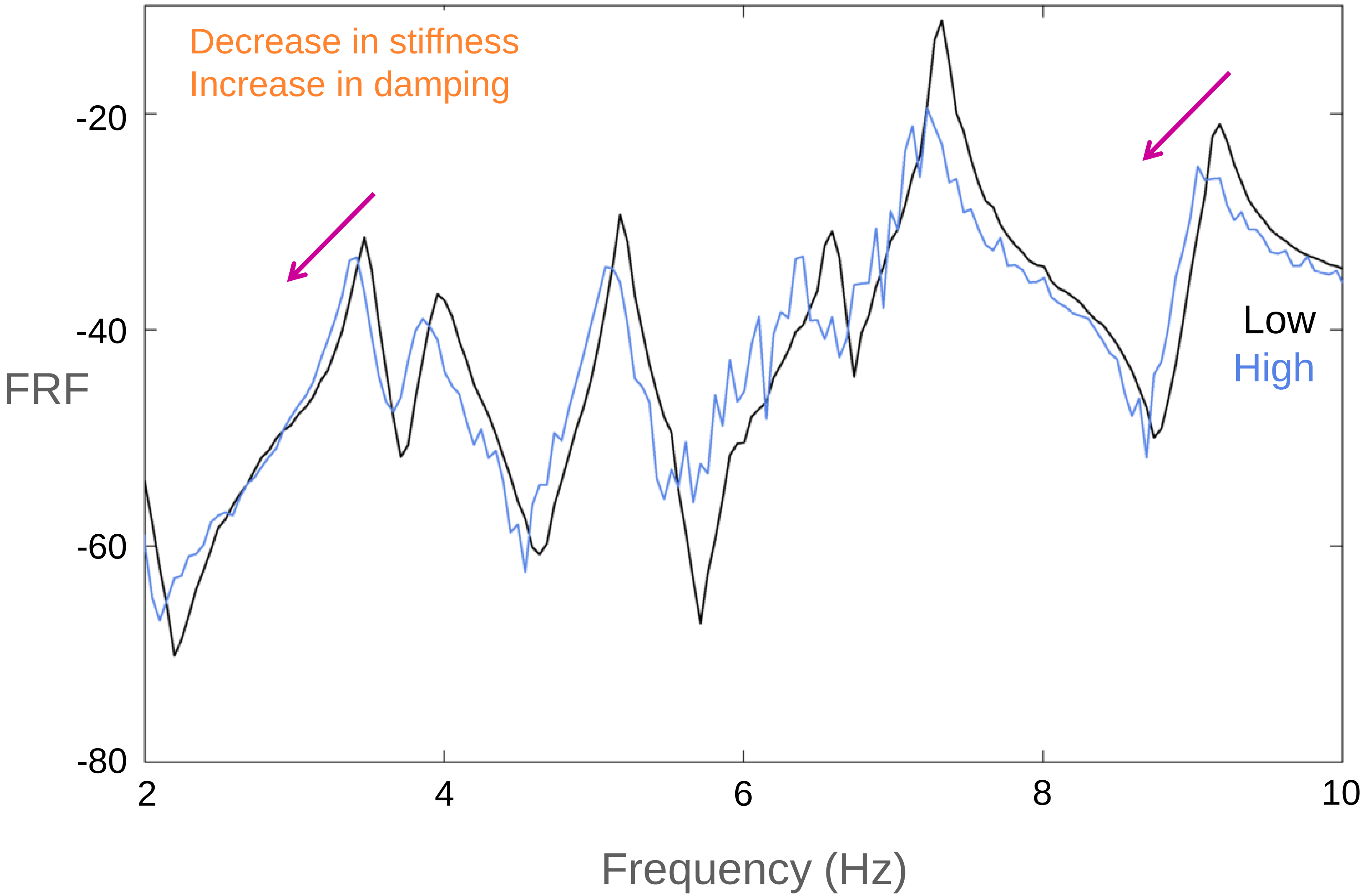
# MS760 Aircraft Has Nonlinear Bolted Joints


# Impact on Measured FRFs

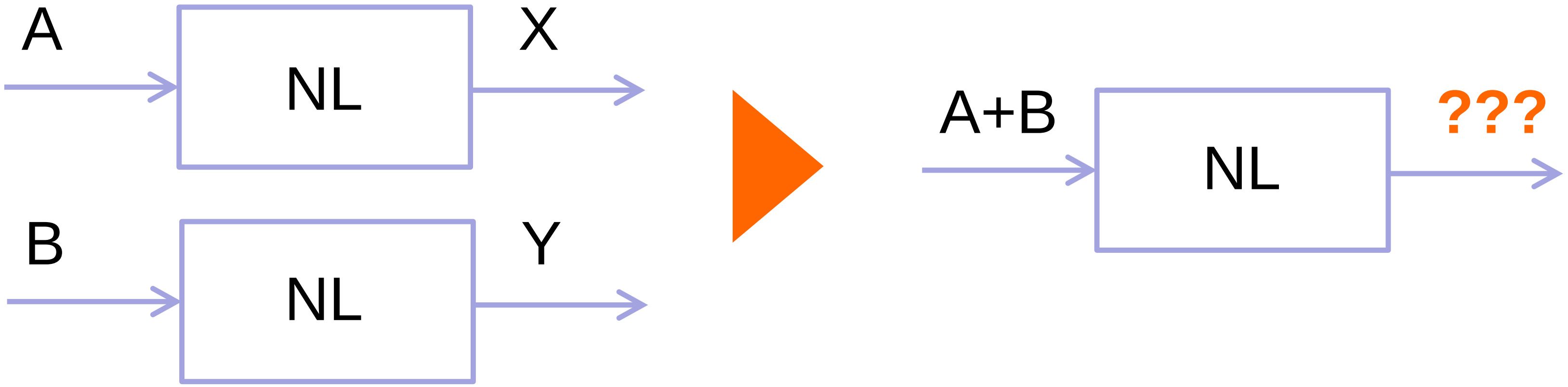


# F-16 Aircraft Has A Nonlinear Sliding Joint

# Impact on Measured FRFs

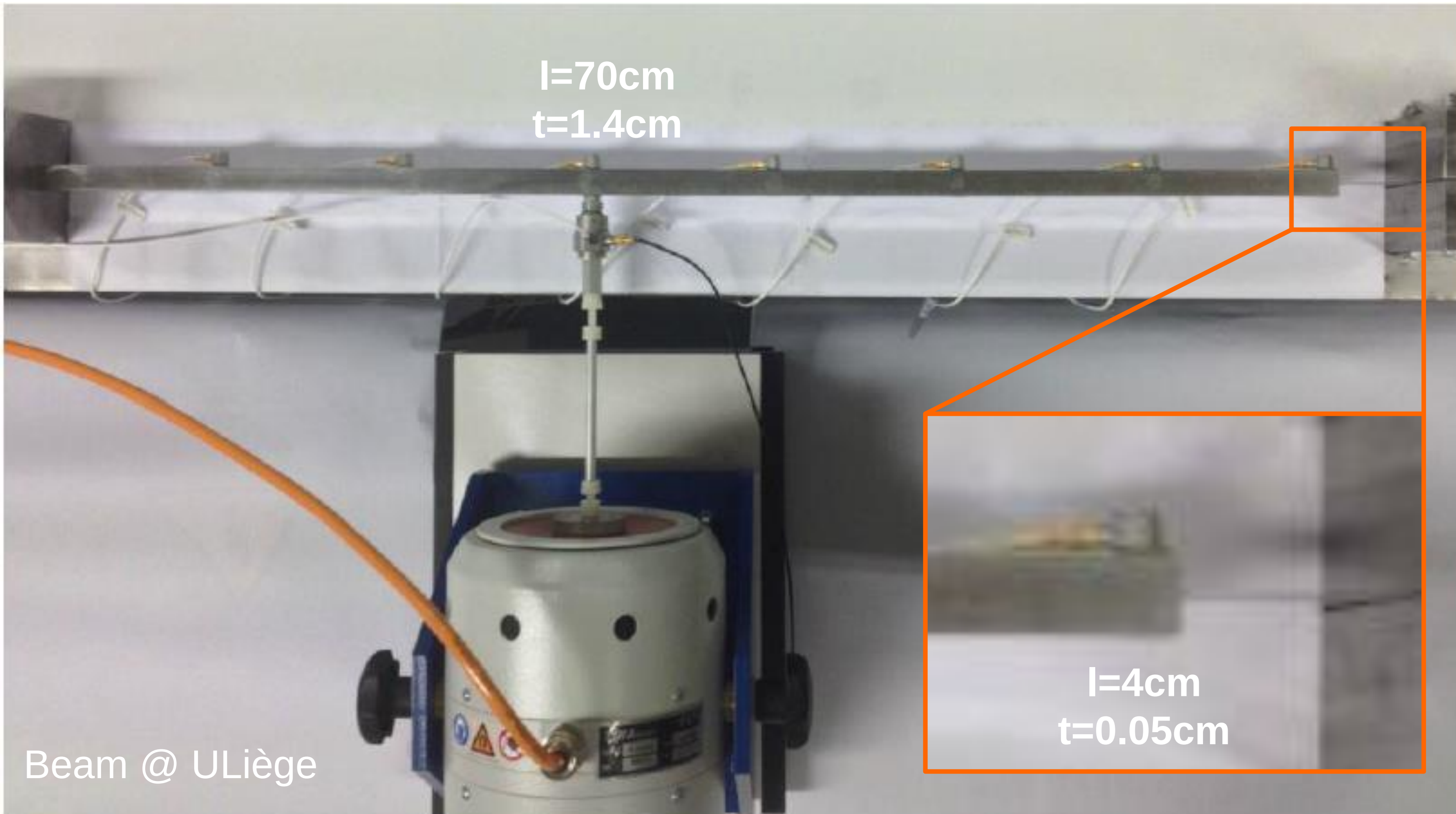


# So, No Superposition Principle for a NL System !



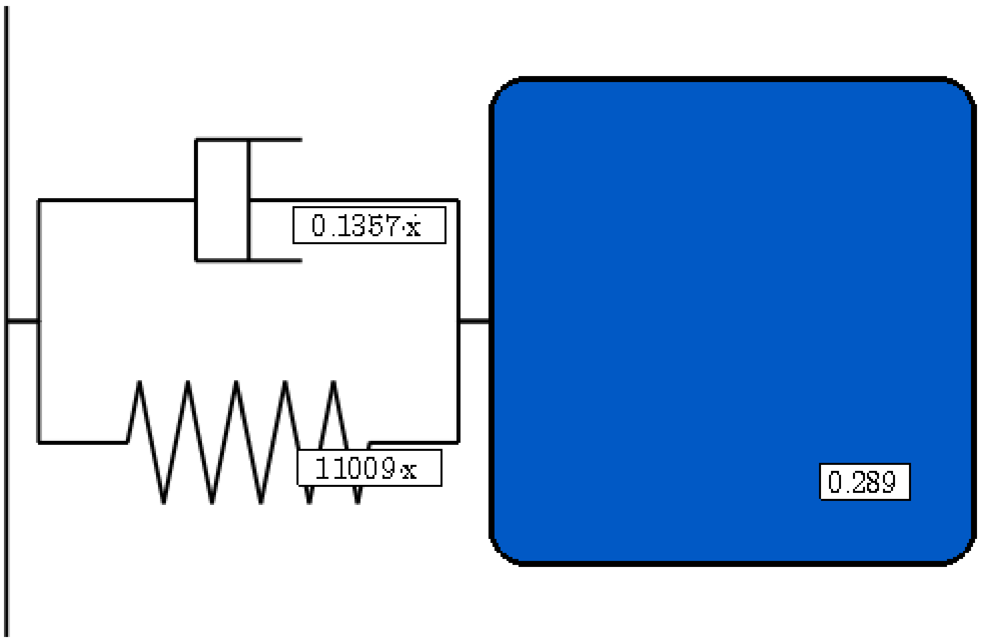
# Let’s Go Back to Basics

Cantilever beam with at very thin beam at its tip:



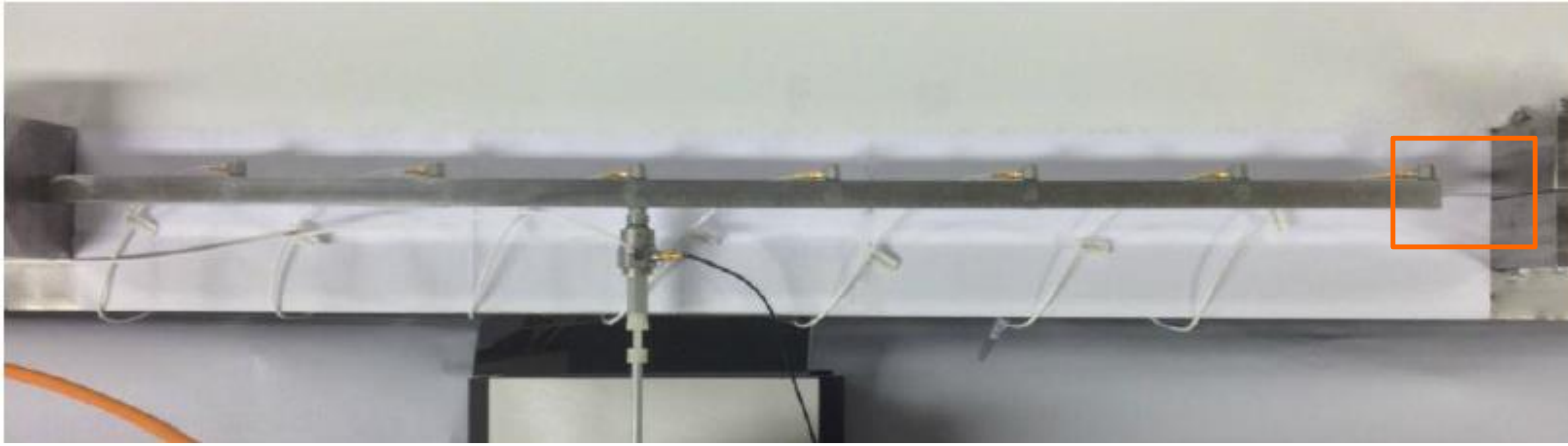
# A 1DOF Model of the First Beam Mode Identified

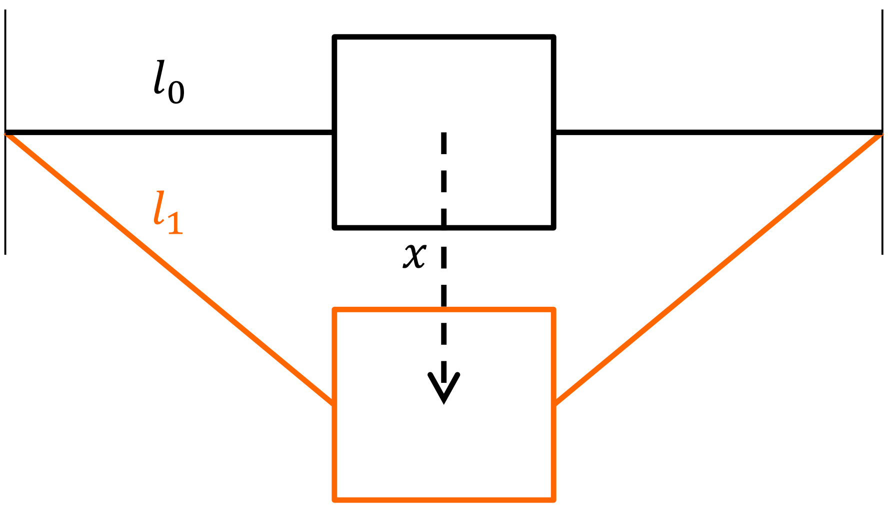
$$0.289\ddot{x} + 0.1357\dot{x} + 11009x = F sin\omega t$$




mode 1: 31.0631 Hz / 0.12 %

# However, The Thin Beam Is Geometrically Nonlinear

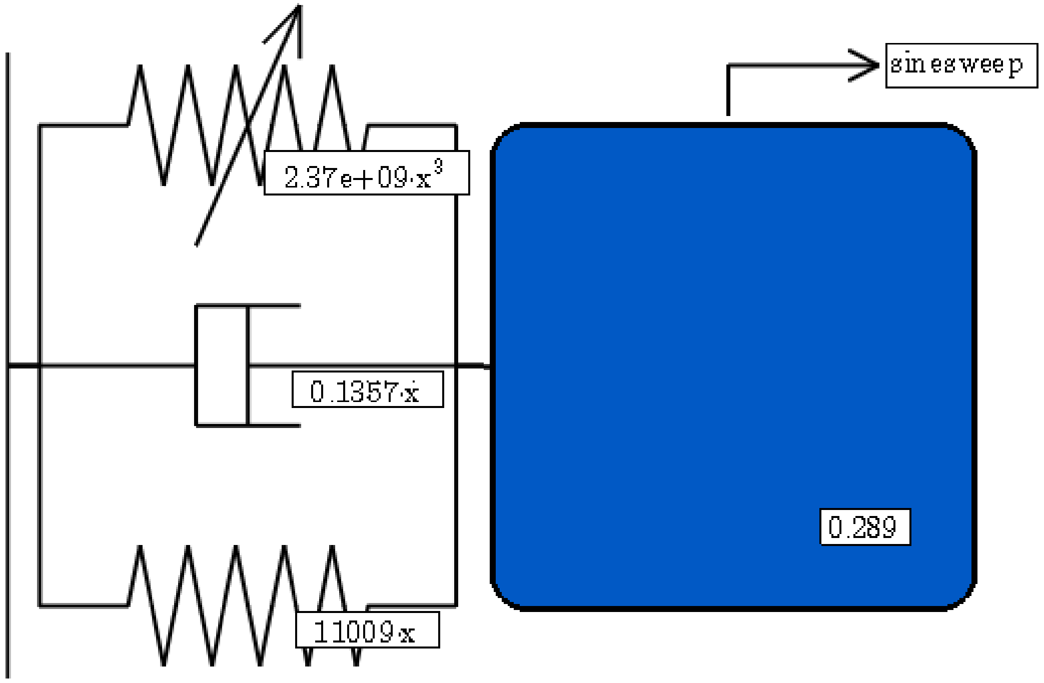





Behavior of a cubic spring

# A 1DOF Nonlinear Model of the First Beam Mode

$$0.289\ddot{x} + 0.1357\dot{x} + 11009x + 2.37.10^9\, x^3 = F sin\omega t$$



# How Do We Calculate the Nonlinear FRF ?

$$m\ddot{x} + kx + k_3x^3 = Fsin\,\omega t$$

$$x = x(F, \omega, t)?$$

# 1. Harmonics

# The Harmonic Balance Method

$$m\ddot{x} + kx + k_3\ x^3 = Fsin\ \omega t$$

$$x(t) = Xsin\ \omega t$$

$$-m\omega^2 Xsin\ \omega t + kXsin\ \omega t + k_3\ X^3 sin^3 \omega t = Fsin\ \omega t$$

$$sin^3 \omega t = (3sin\ \omega t - sin\ 3\omega t)/4$$

Nonlinear relation between *X* and *F*

OPTION 1: EXACT

$$x(t) = Xsin\ \omega t + X_3 sin\ 3\omega t$$

Solution: infinite series of harmonics

OPTION 2: APPROXIMATION

$$-m\omega^2 X + kX + \frac{3}{4}k_3\ X^3 = F$$

Solve a 3rd order polynomial in *X*
(! BIFURCATIONS !)

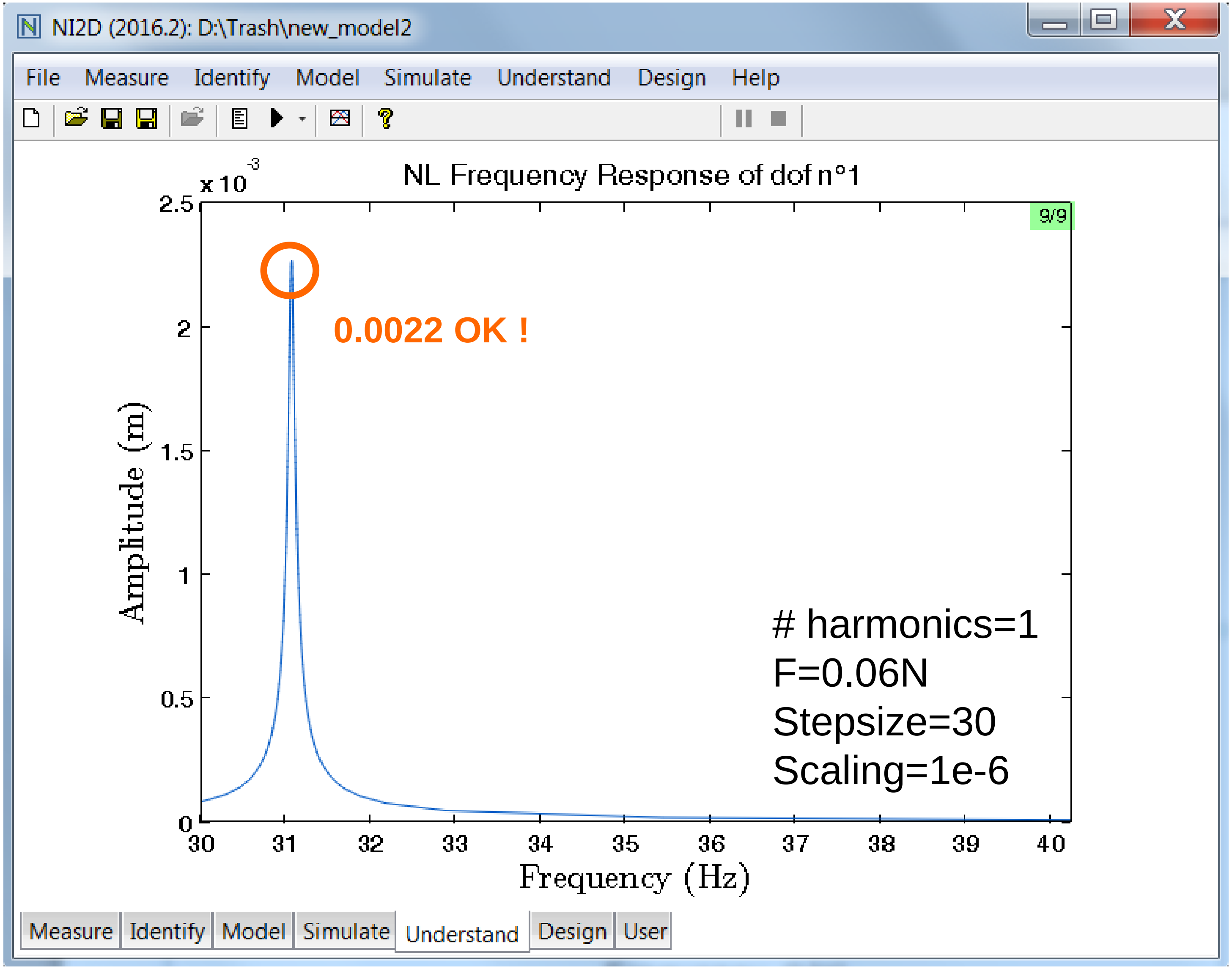
# The Harmonic Balance Method (with Damping)

$$m\ddot{x} + c\dot{x} + kx + k_3x^3 = Fsin\,\omega t$$

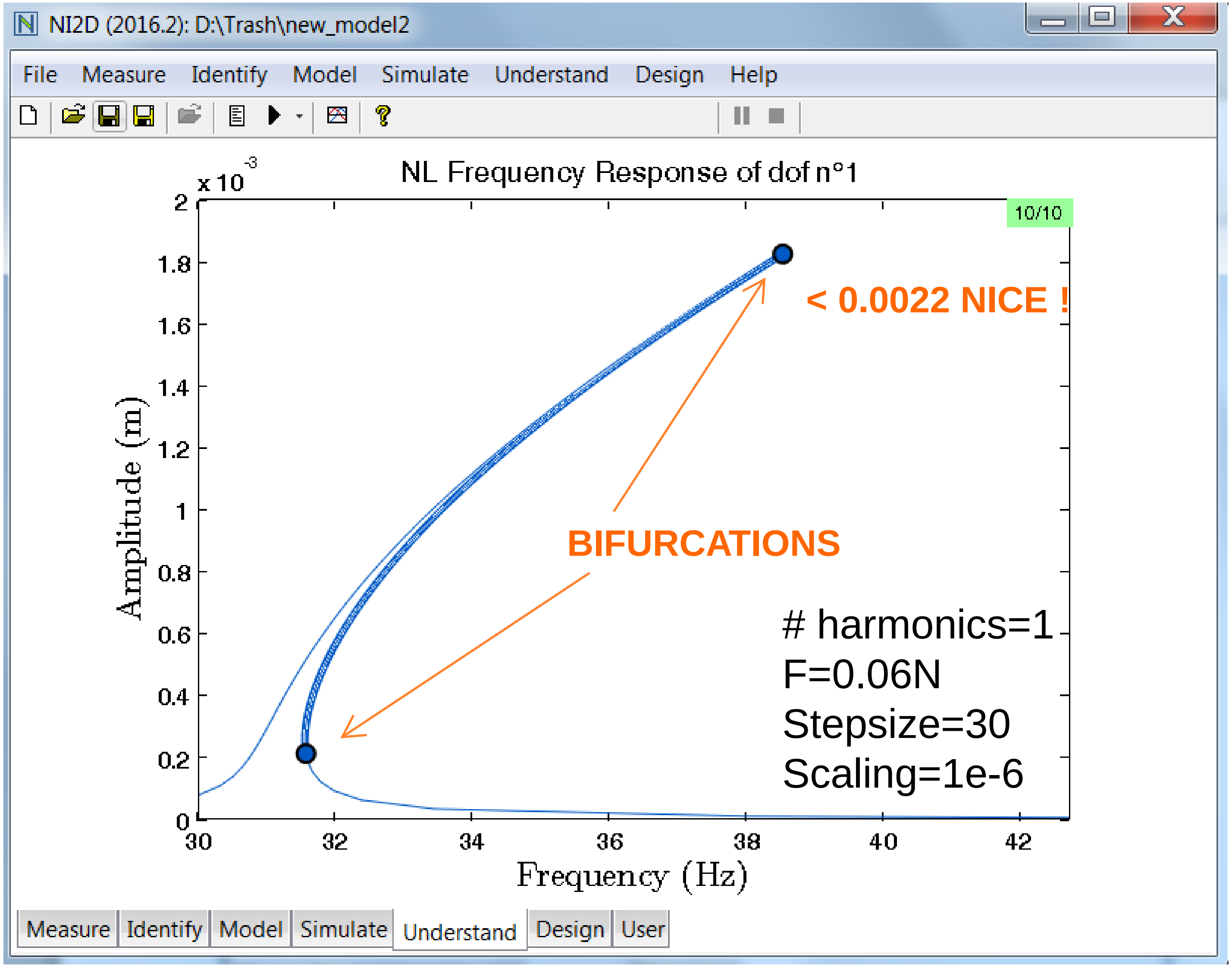
$$x(t) = Xsin\,\omega t + Ycos\,\omega t$$

Same machinery

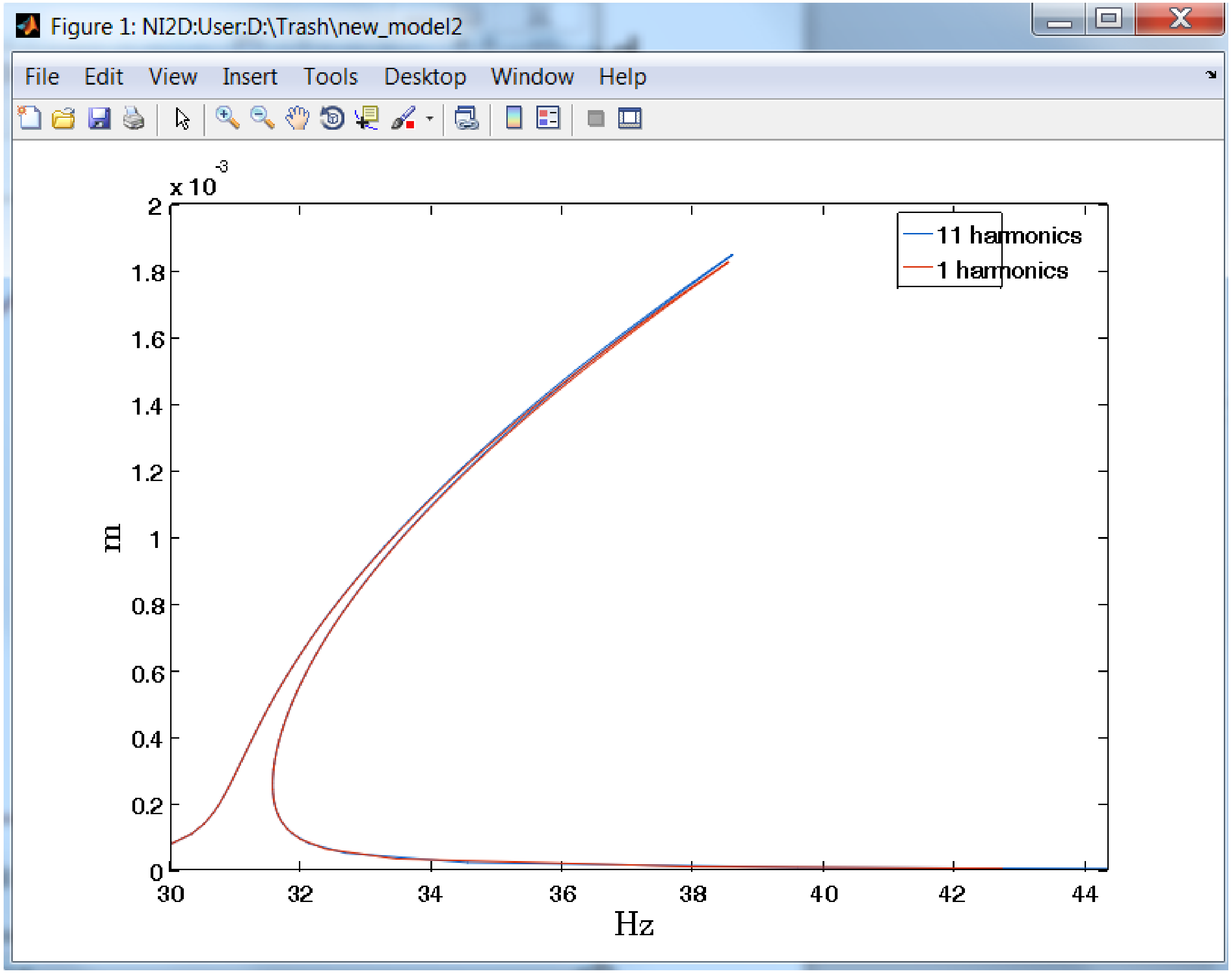
# HB for the Linear System in NI2D



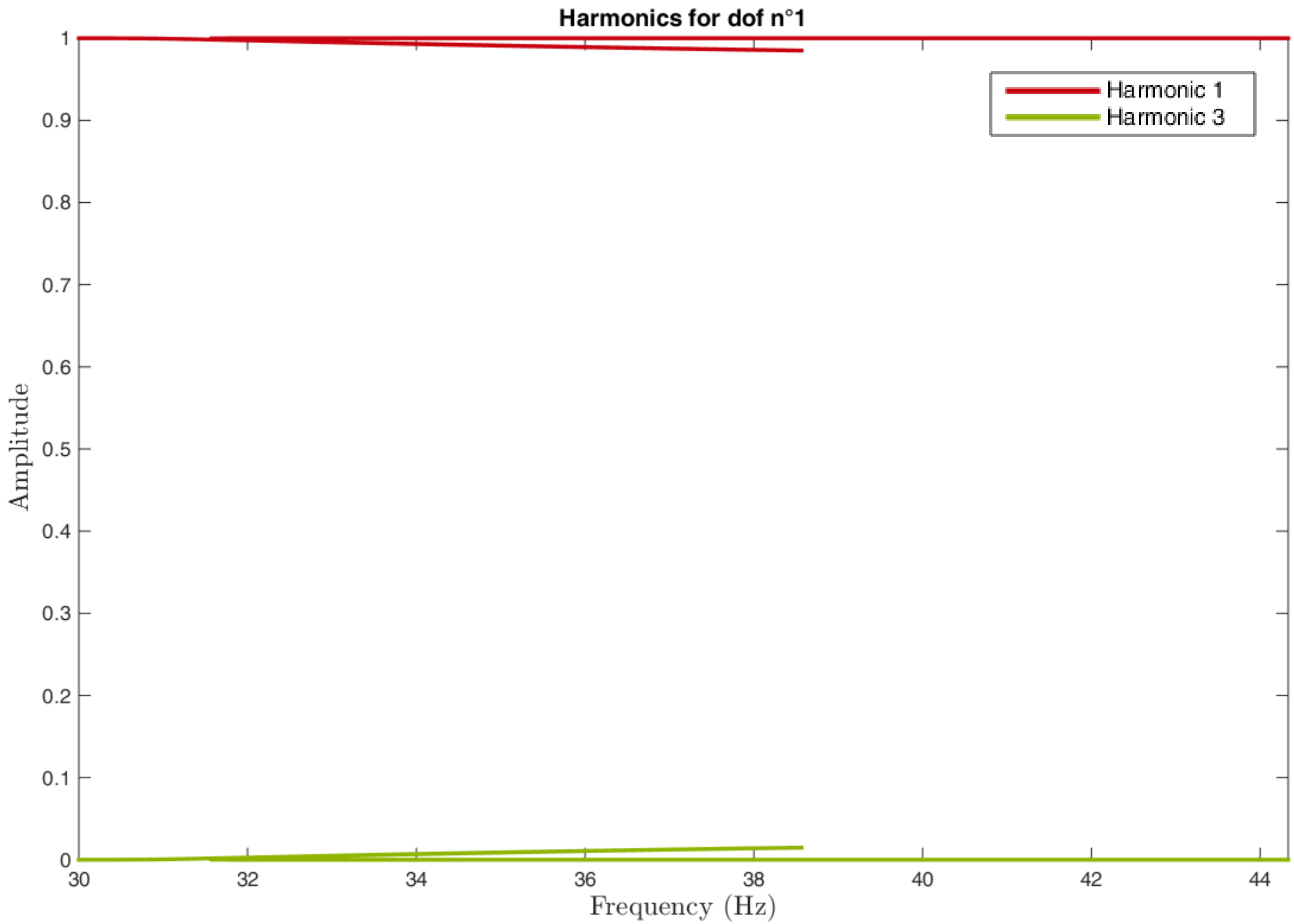
# HB for the Nonlinear System in NI2D



# Very Fast Convergence of HB in this Case

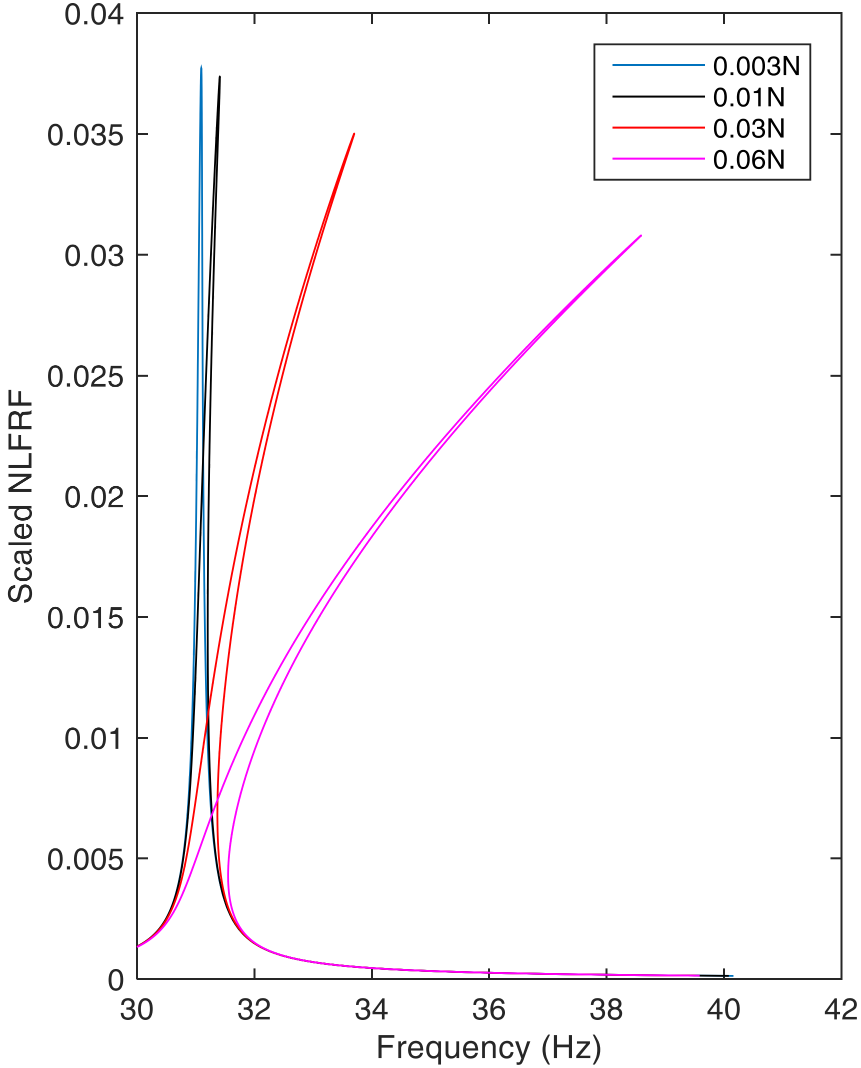


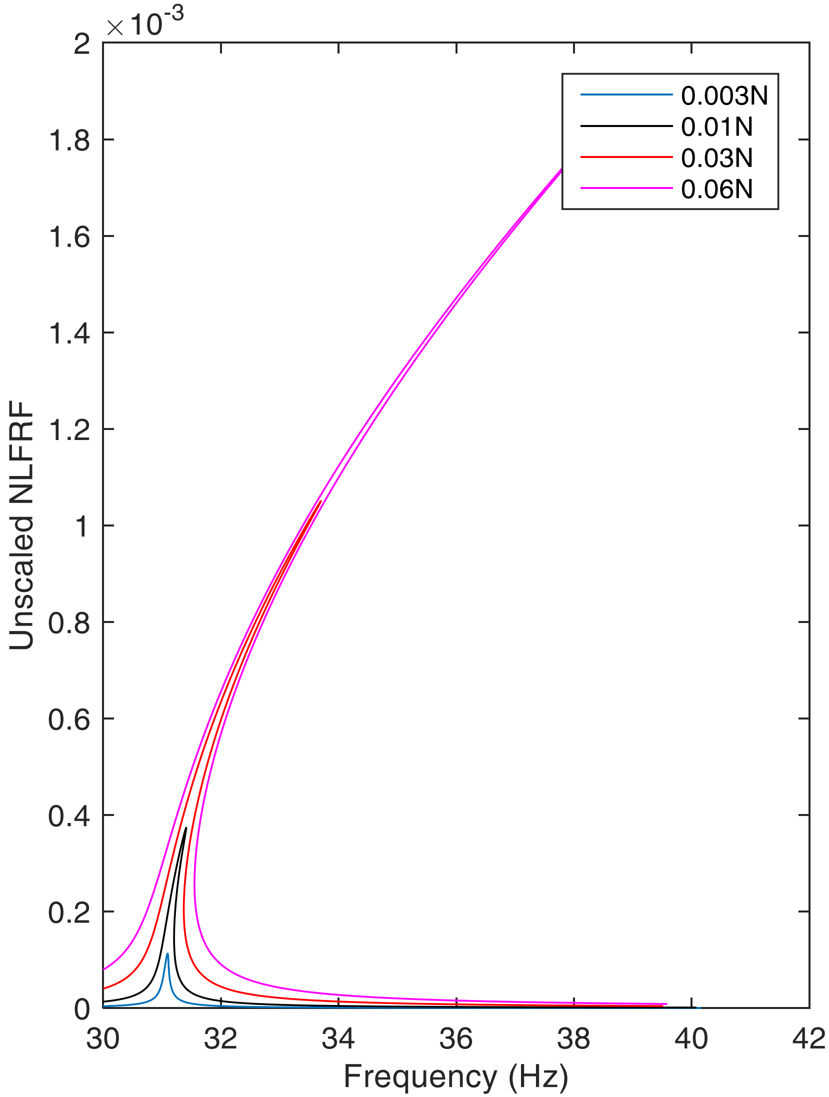
# Importance of Harmonics



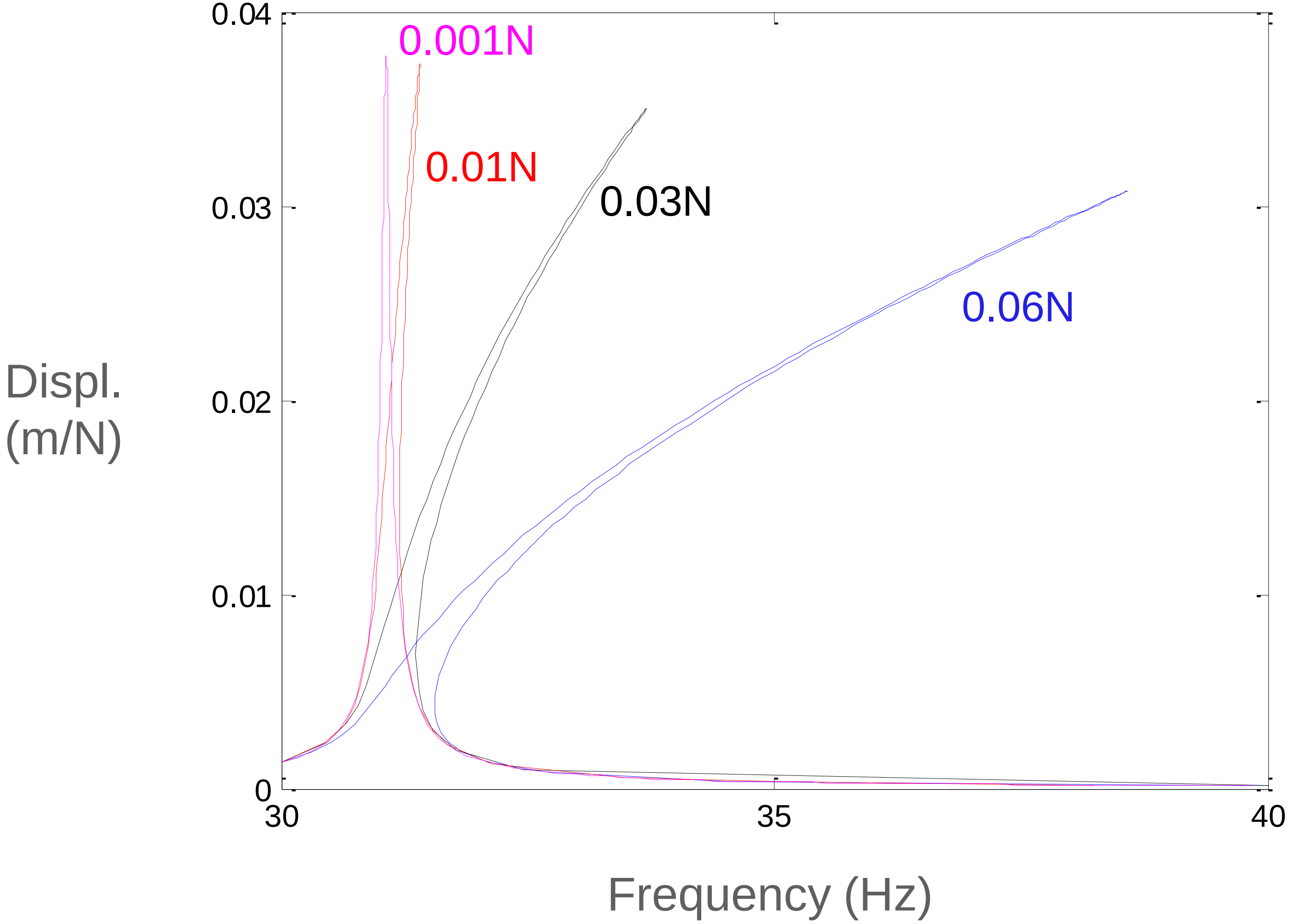
# 2. Frequency-amplitude dependence

# Variation of Nonlinear FRFs

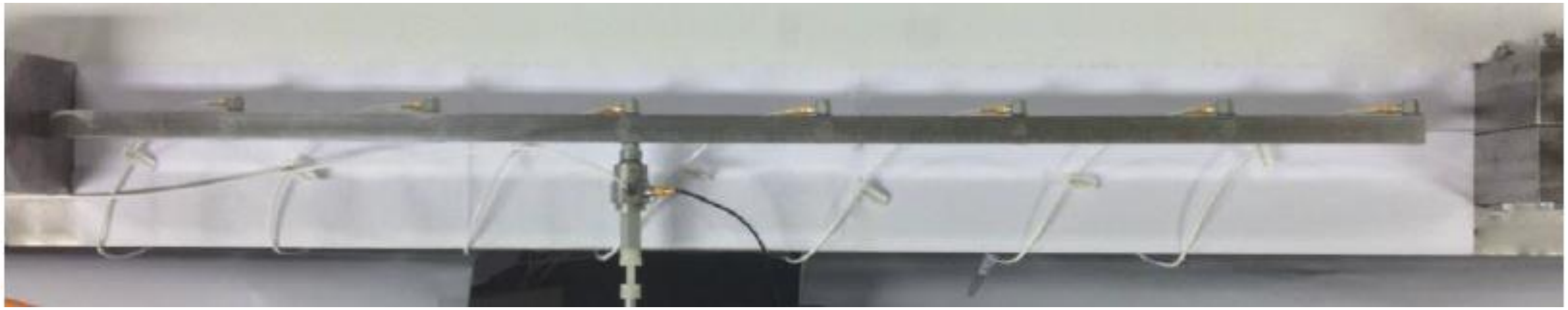


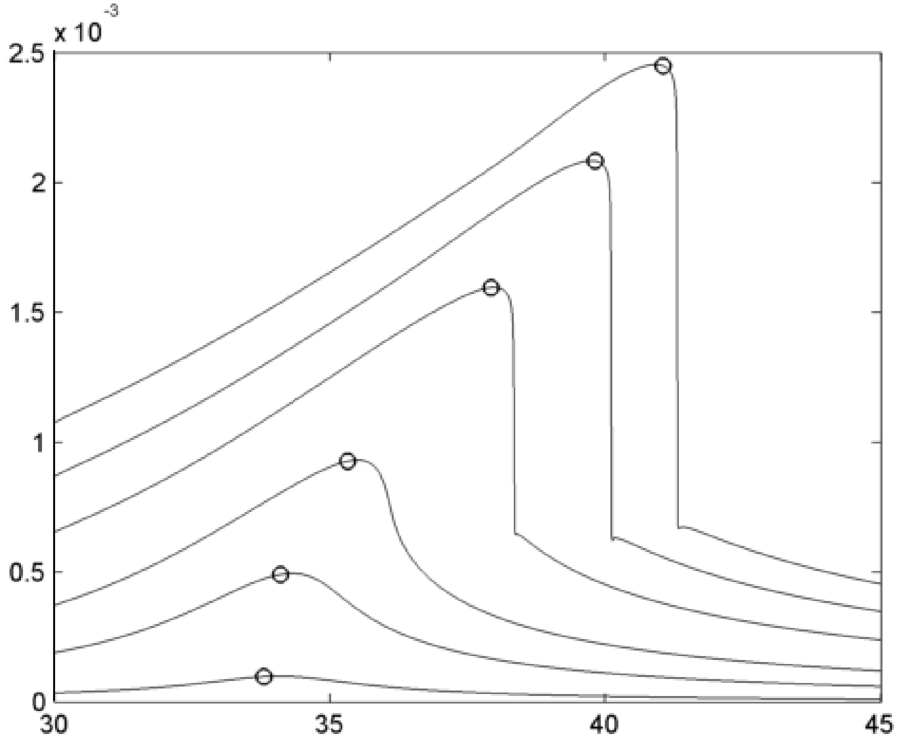



# Variation of Nonlinear FRFs



# Measured (Unscaled) FRFs



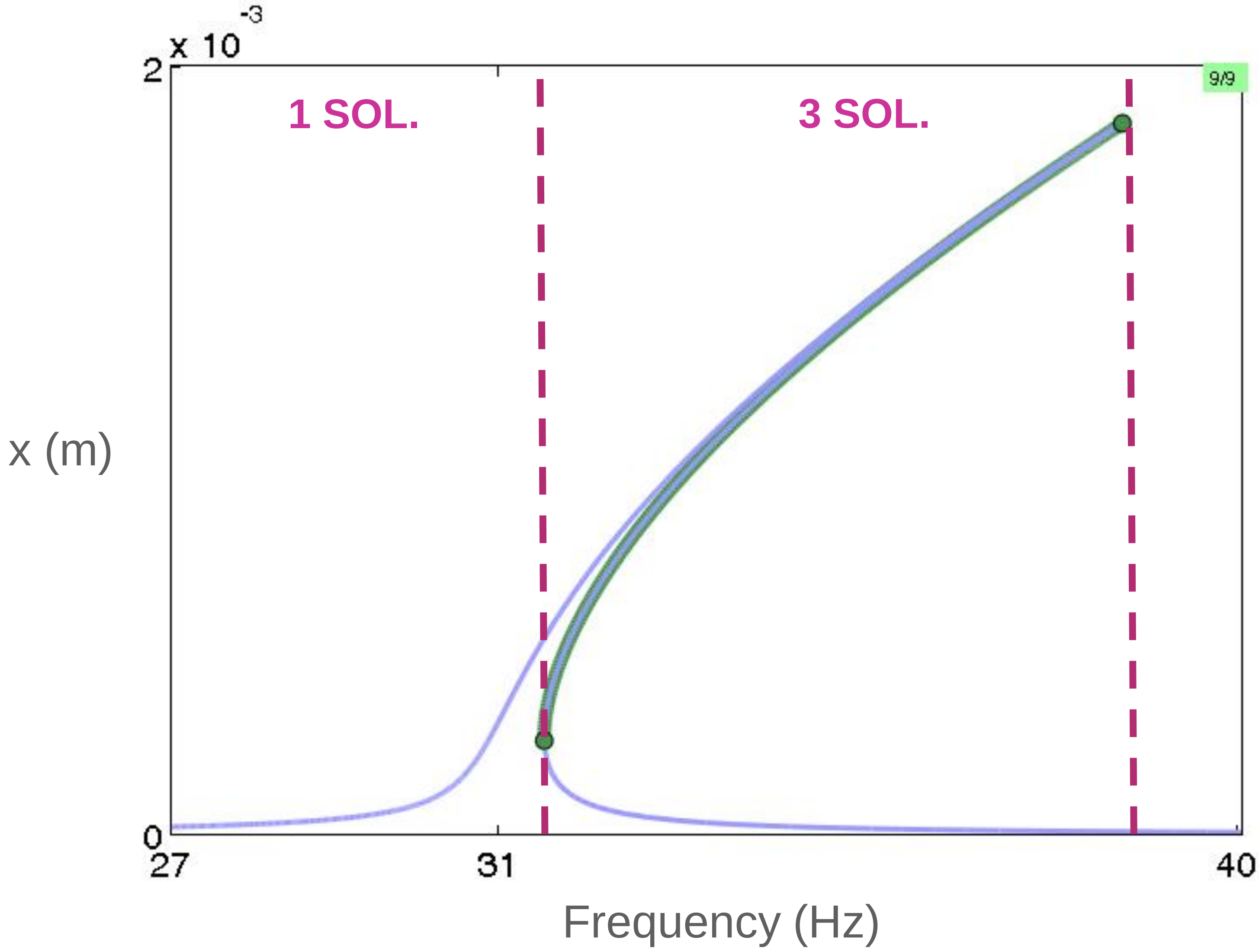



Displ. (m)

Frequency (Hz)

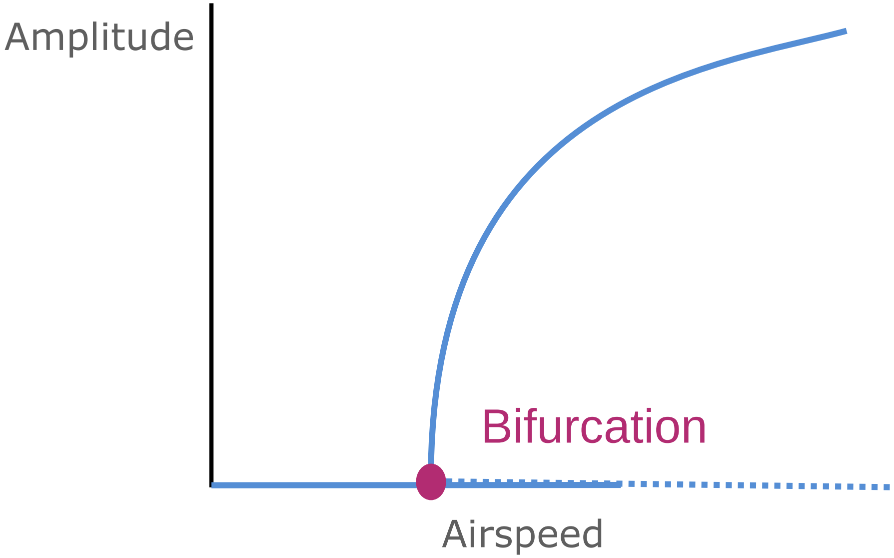
# 3. Bifurcations

# Bifurcations Generate Multi-Valued Response



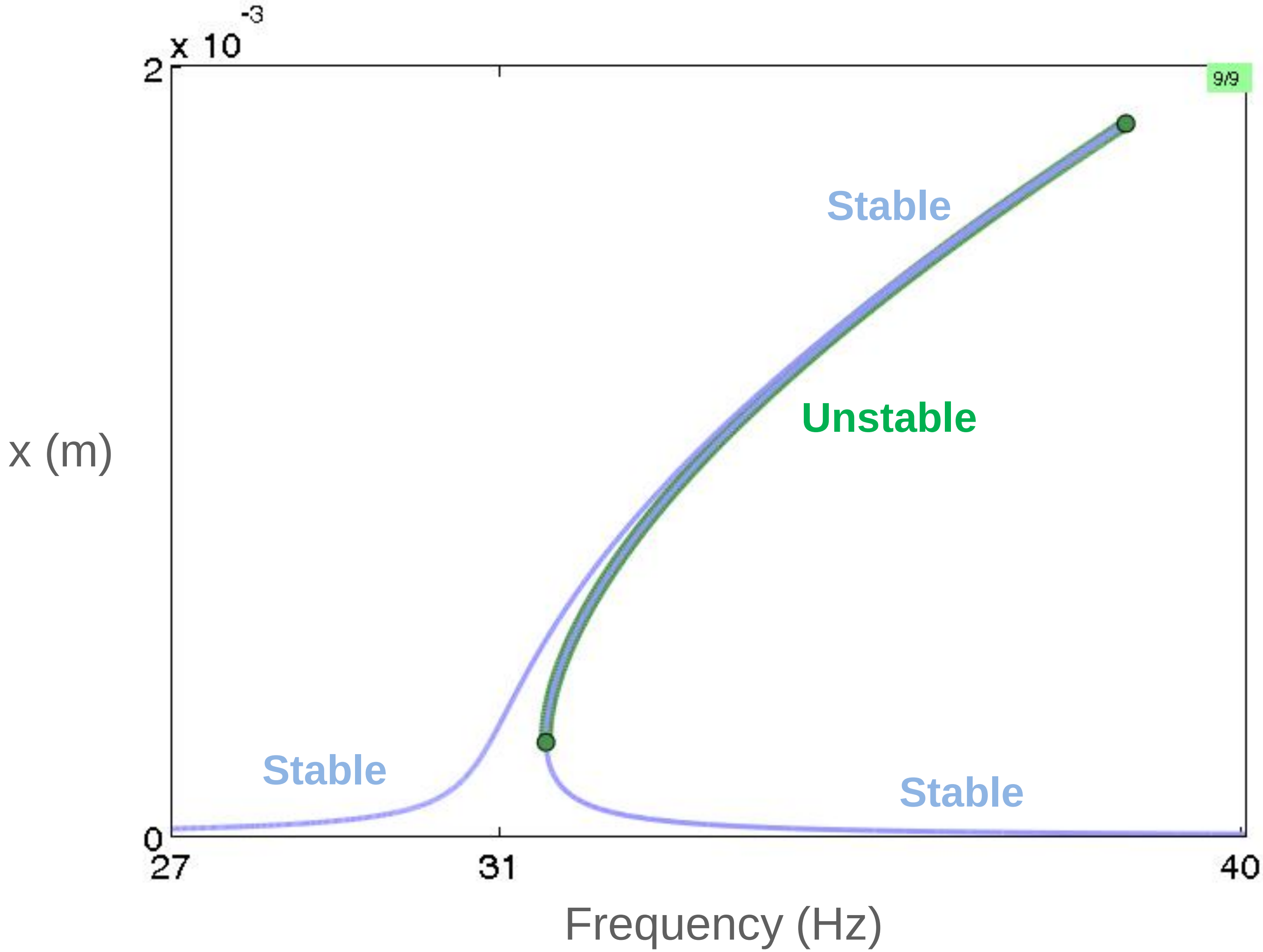
# These Complex Phenomena Exist in Practice

E.g., flutter in aircraft structures

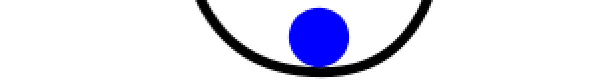


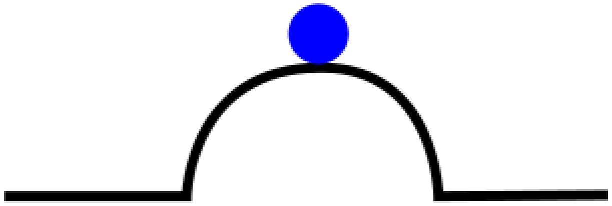

# 4. Stability

# Bifurcations Change Stability

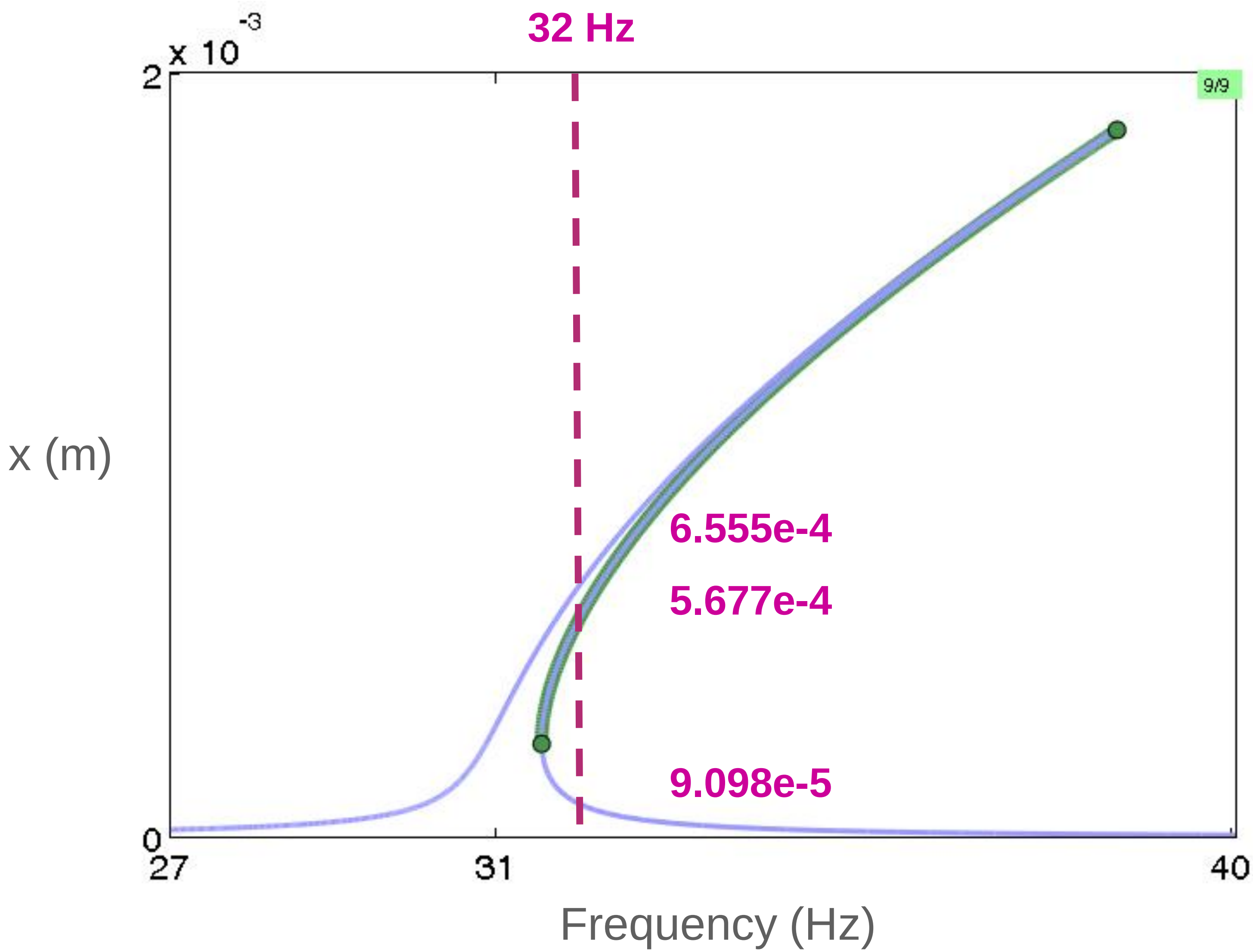


# What Is Stability/Instability ?

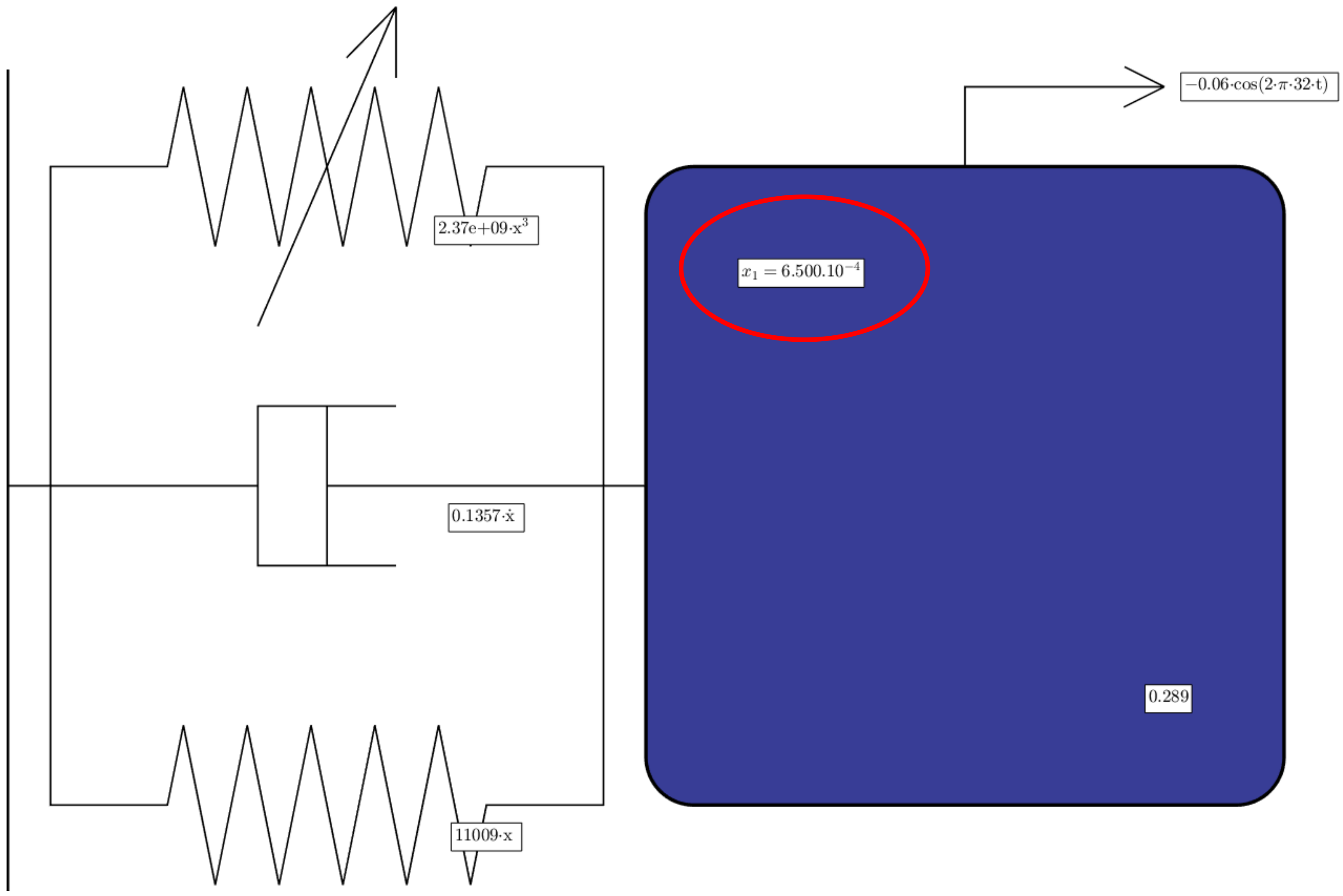




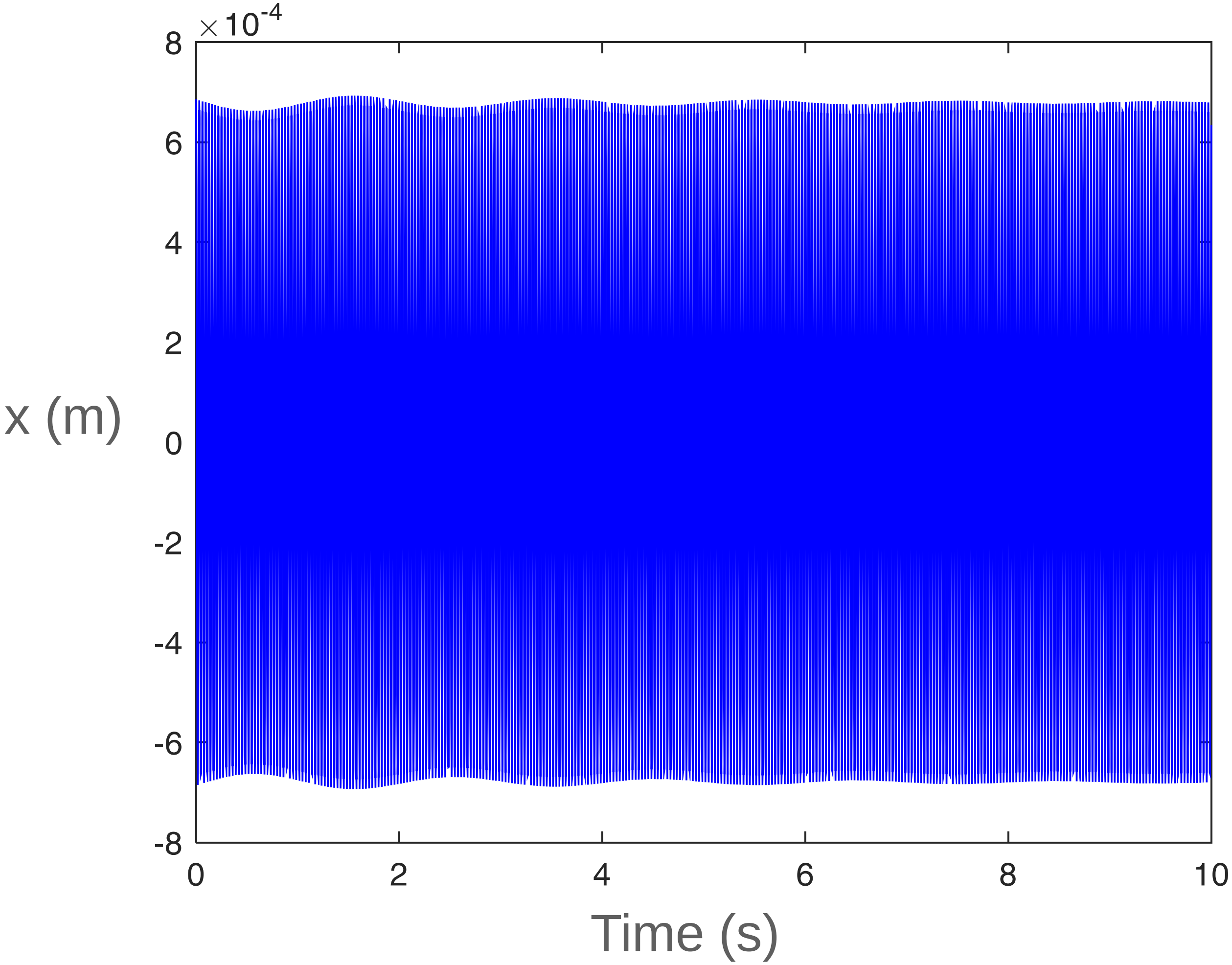
# Starting from a Stable/Unstable Solutions



# Starting from a Stable/Unstable Solutions

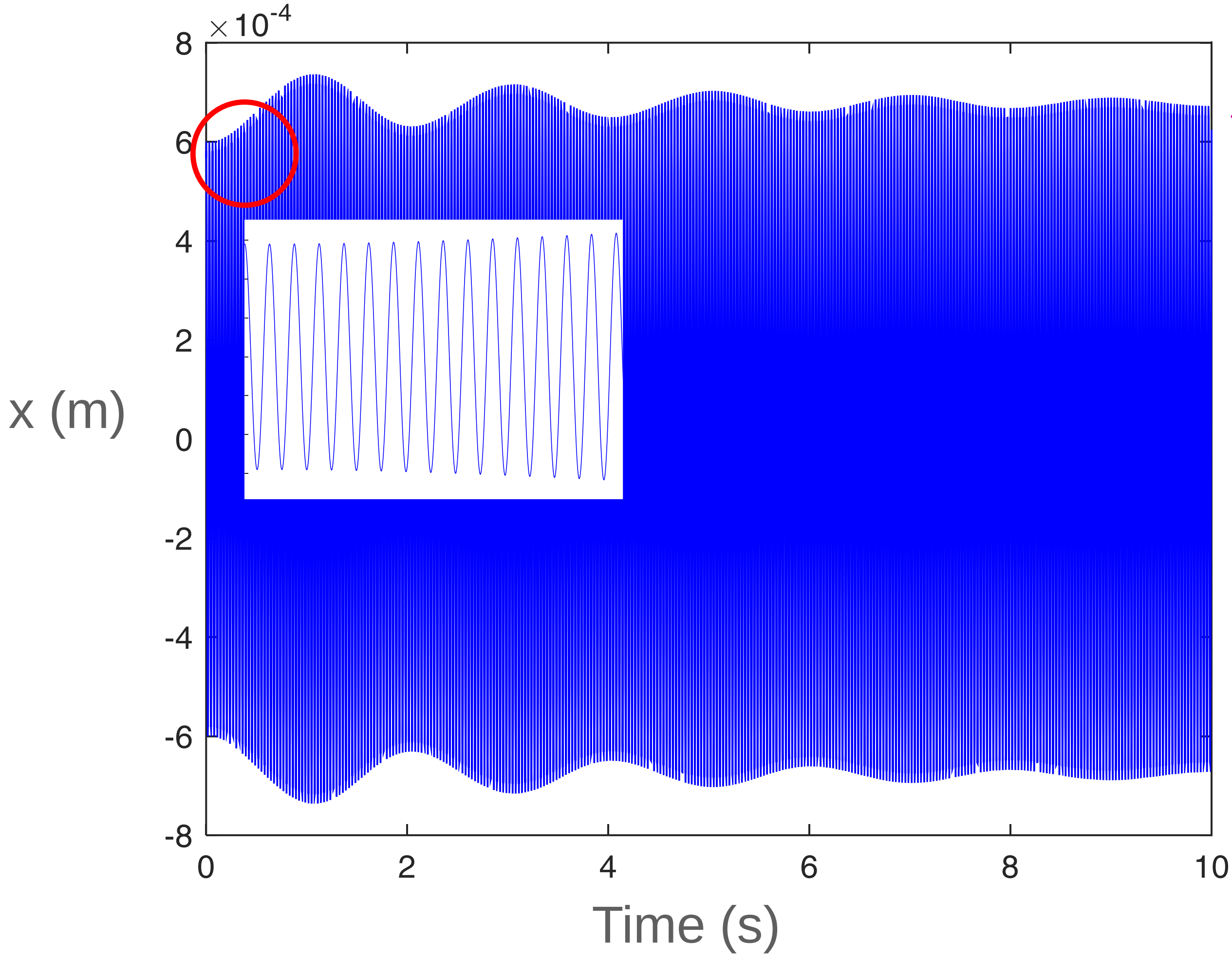


# Starting from a Perturbed Stable Solution



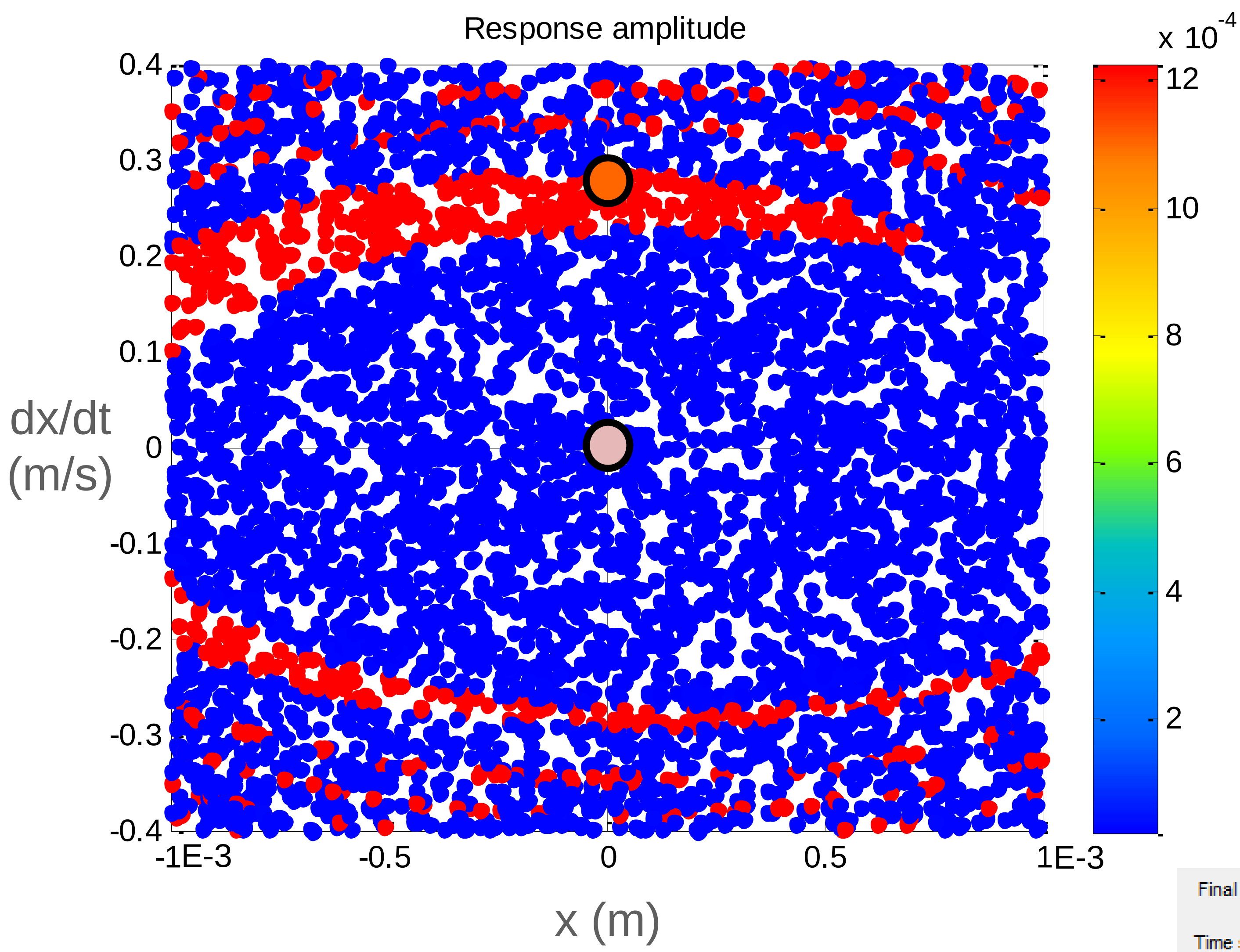

**6.6e-4 (stable sol.)**

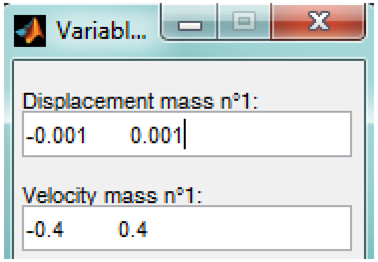
# Starting from a Perturbed Unstable Solution

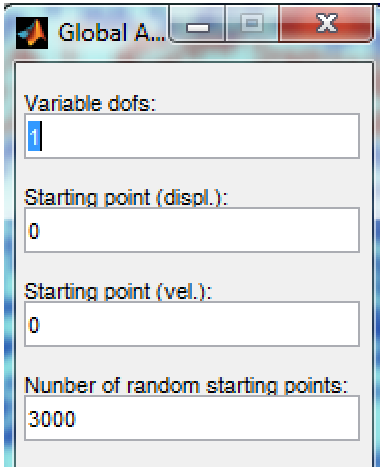


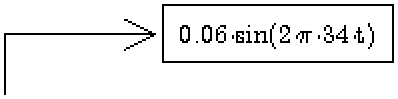

6.6e-4
(stable sol.)

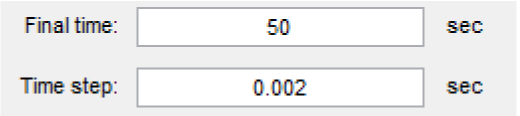
# The Competition Between the Two Stable Solutions



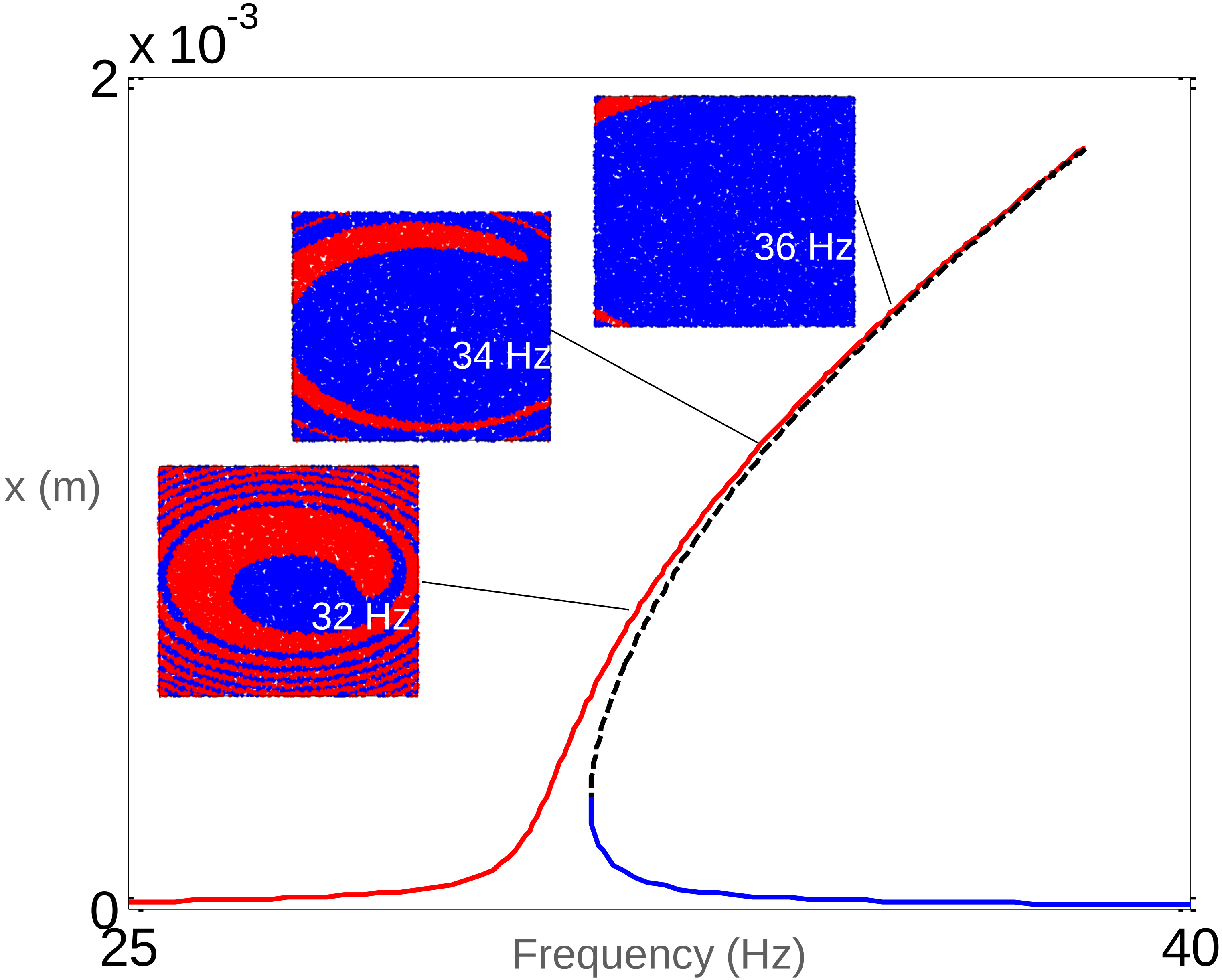



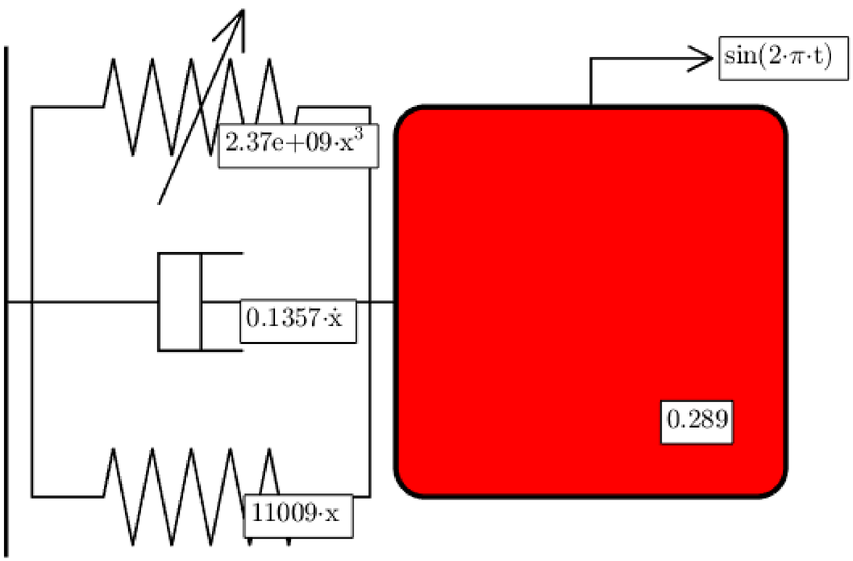



# Evolution of the Basins of Attraction

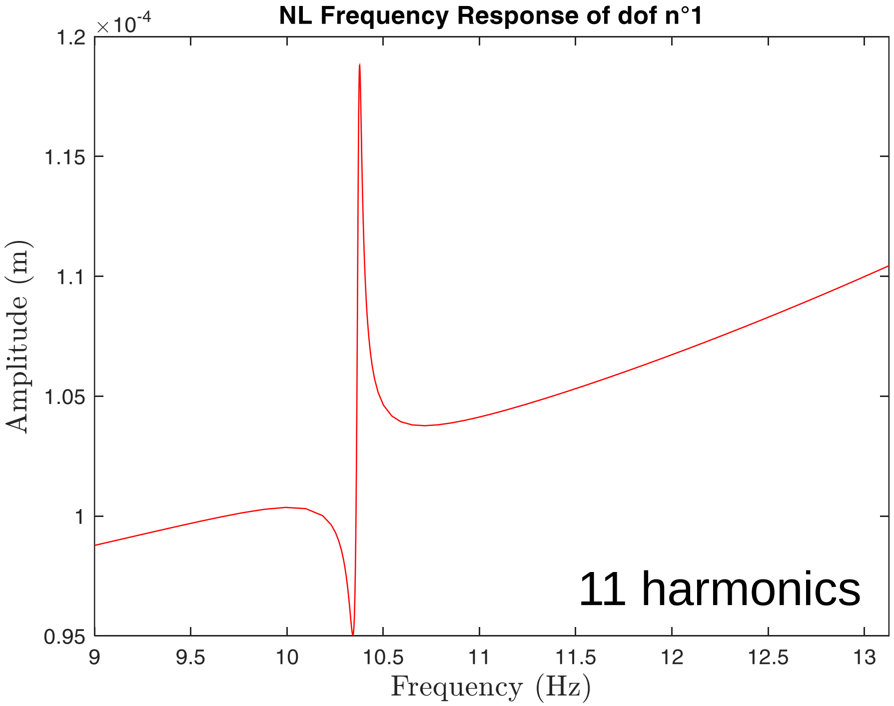


Is that all ?

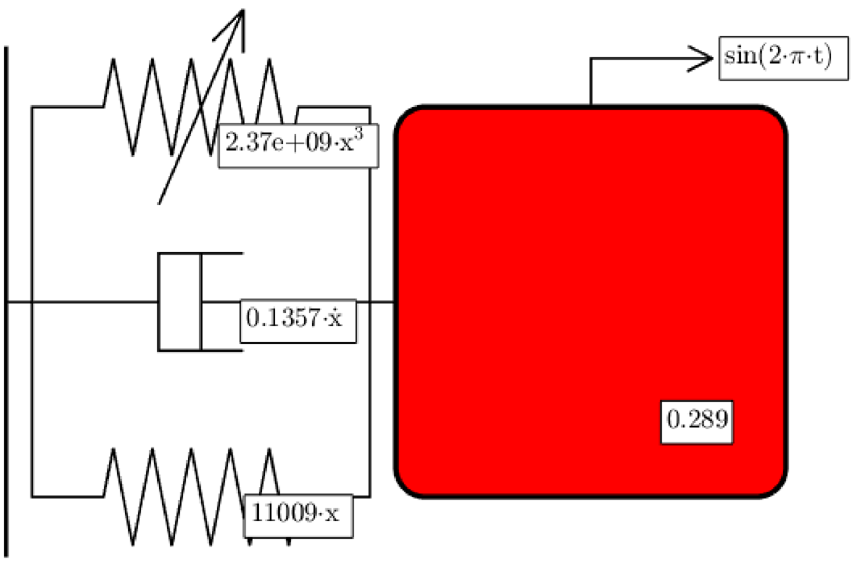
# What’s Going On @ 1N (Far From Resonance) ?



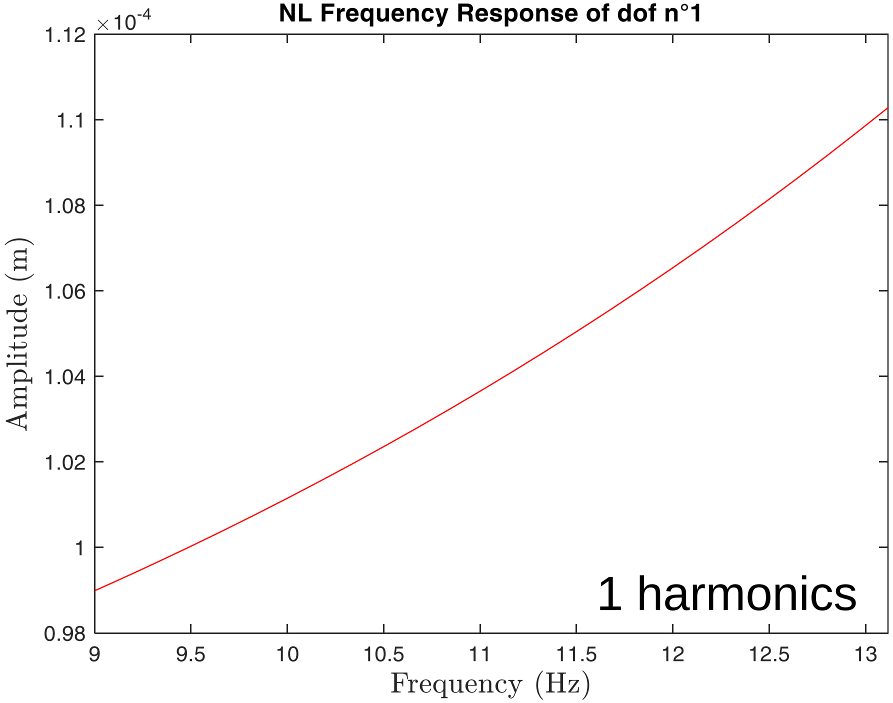

Adaptative stepsize



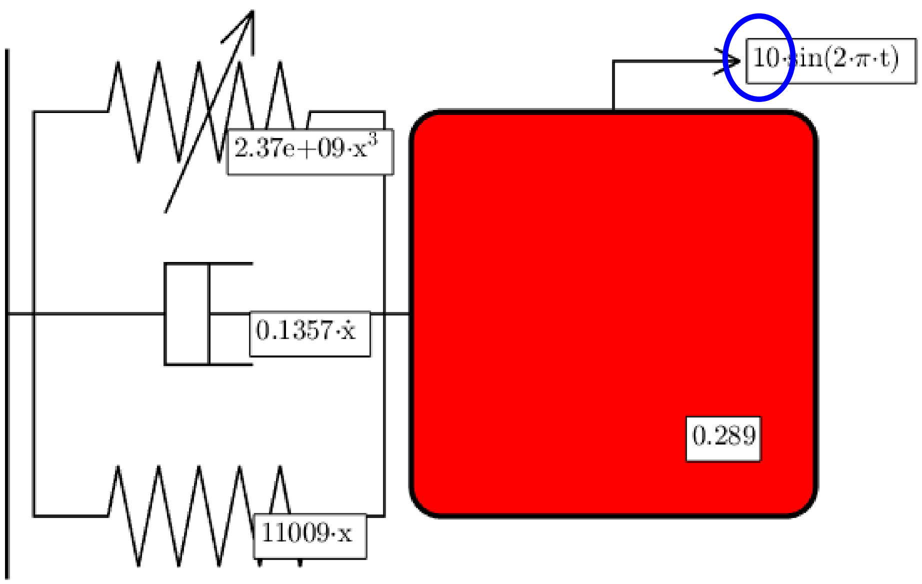
# The New Resonance Has Disappeared !



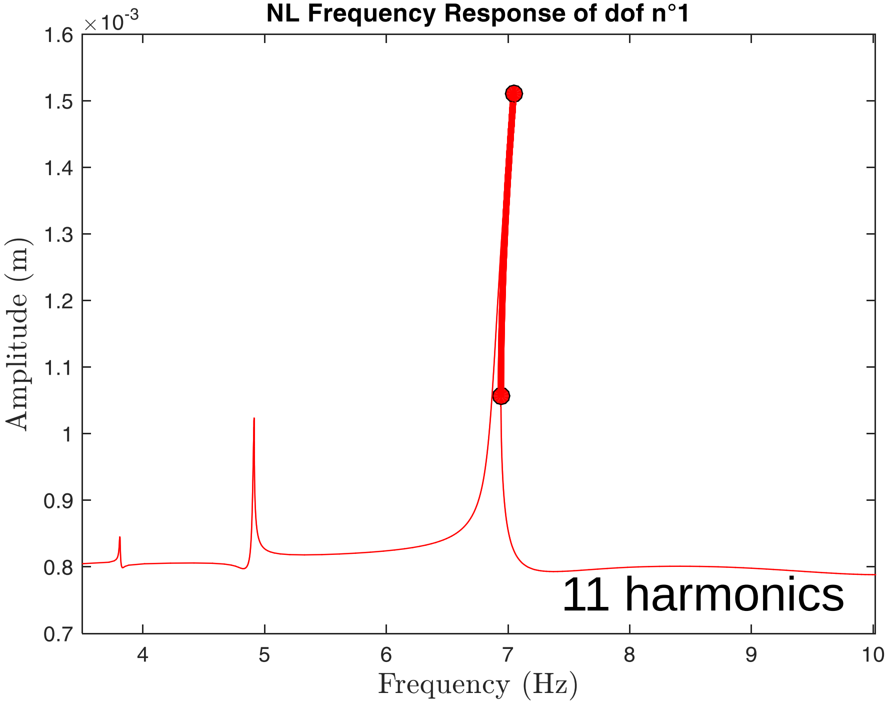

Adaptative stepsize



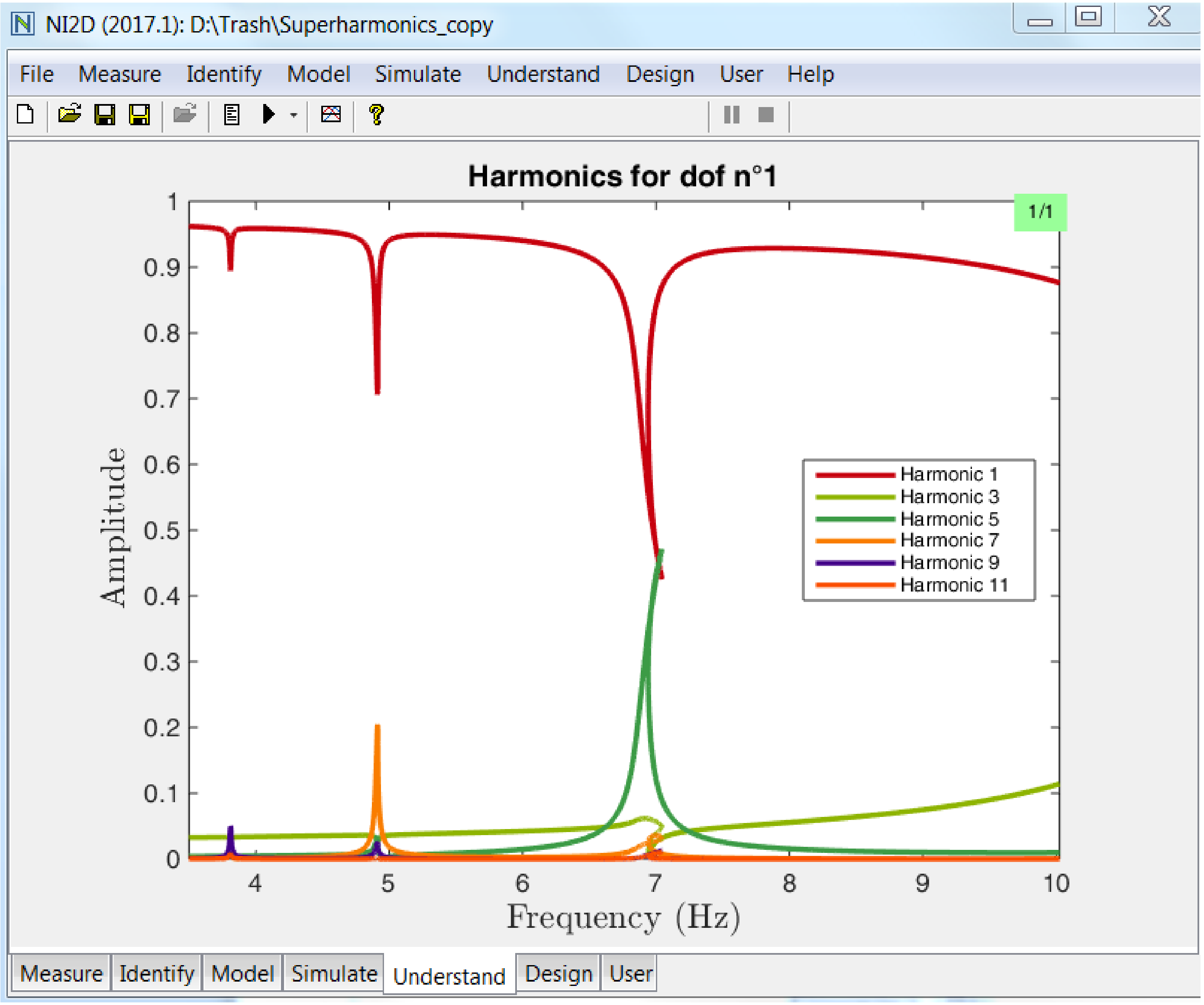
# More Resonances @ 10N

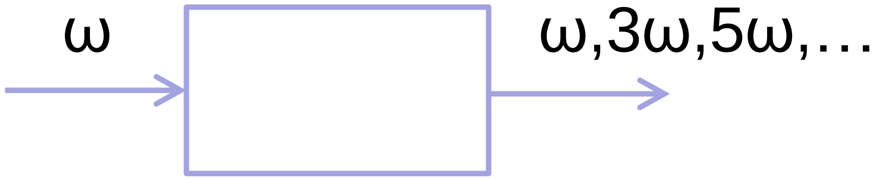



Stepsize=2



# Superharmonic Resonances



| Linear | Nonlinear |
|---|---|
| Superposition principle | **NO !** By definition … |
|  | |
| Simple frequency content | **NO !** Harmonics |
|  |  |
| Uniqueness of the solutions | **NO !** Bifurcations |
| Invariance of FRFs | **NO !** Freq.-energy dependence |

# 4 New Key Concepts

- Harmonics
- Frequency-amplitude dependence
- Bifurcations
- Stability

Rich frequency content

New resonances

Don’t extrapolate !

Nonuniqueness.

New resonances

Stable/unstable solutions

# Usefulness of Nonlinear FRFs