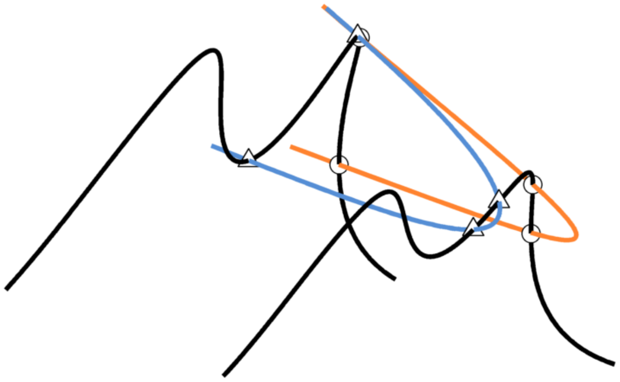
Nonlinear Vibrations of Aerospace Structures

University of Liège, Belgium

L06 Stability & Bifurcations

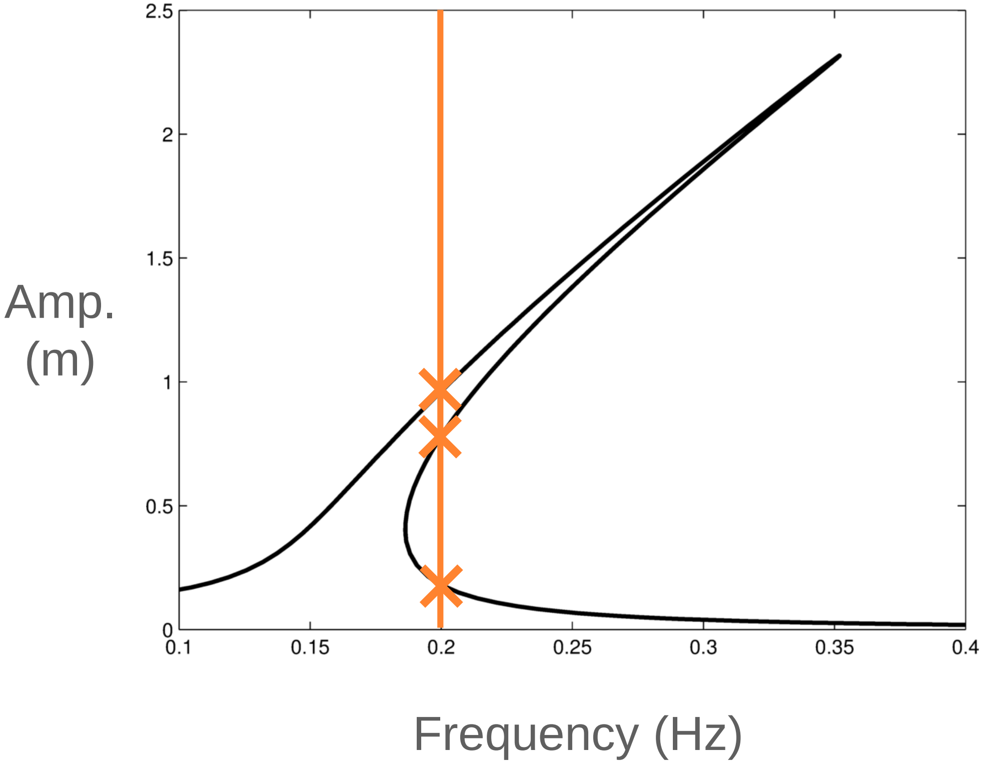
- Floquet theory
- Basins of attraction
- Bifurcation detection & tracking

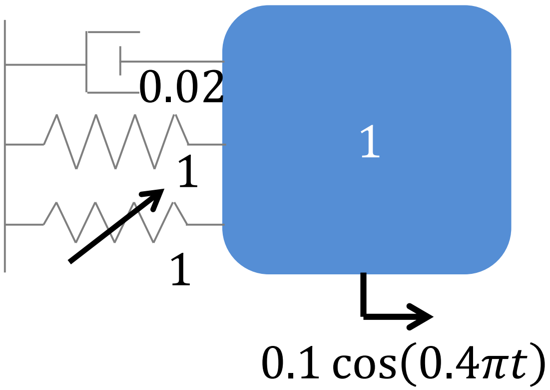


Ghislain Raze | g.raze@uliege.be
Thibaut Detroux

Motivation for This Lecture

For fixed excitation parameters, the solution of a nonlinear system may not be unique.





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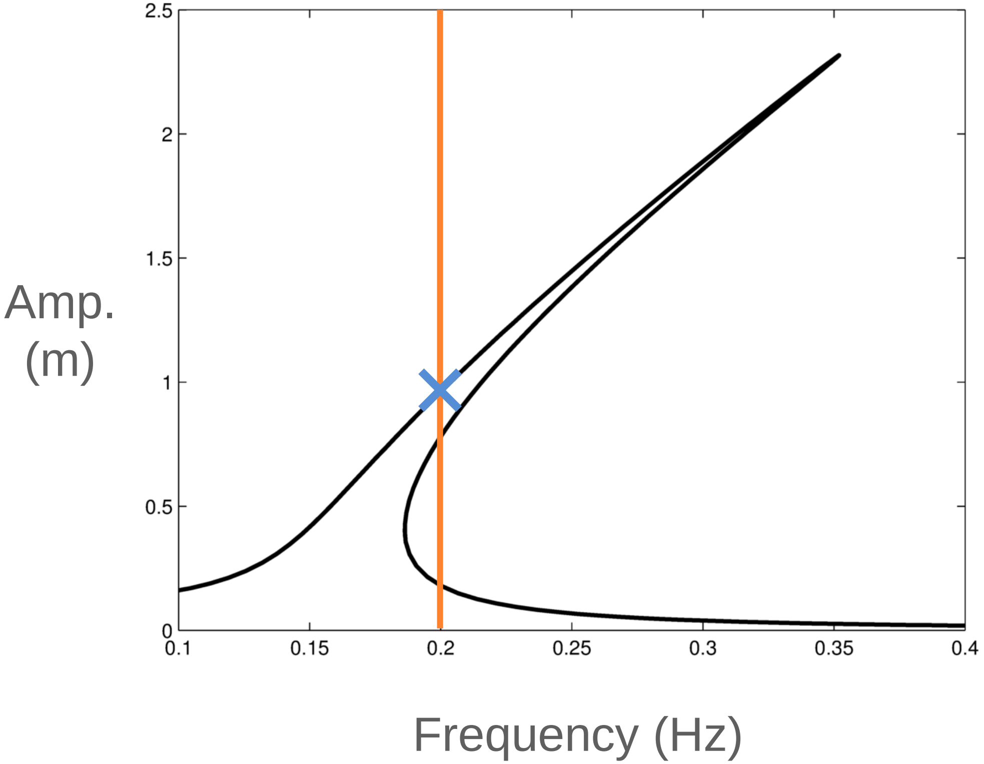

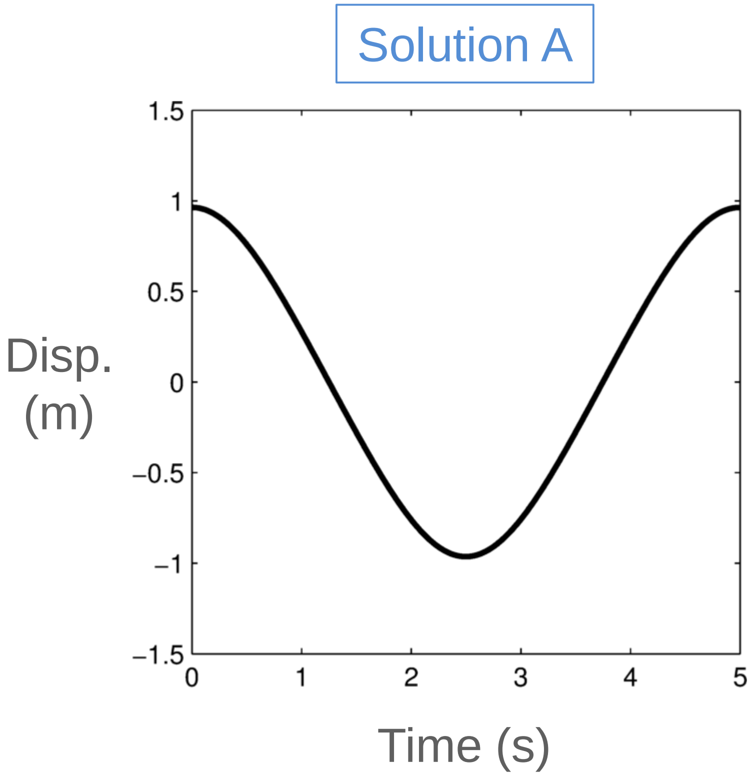

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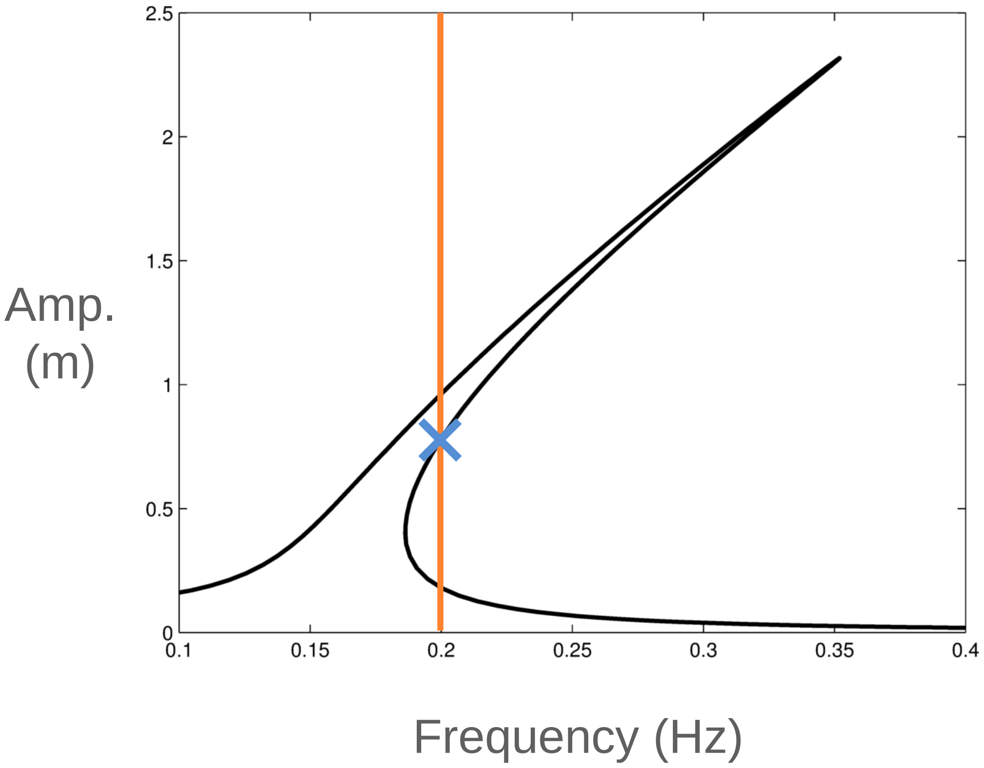

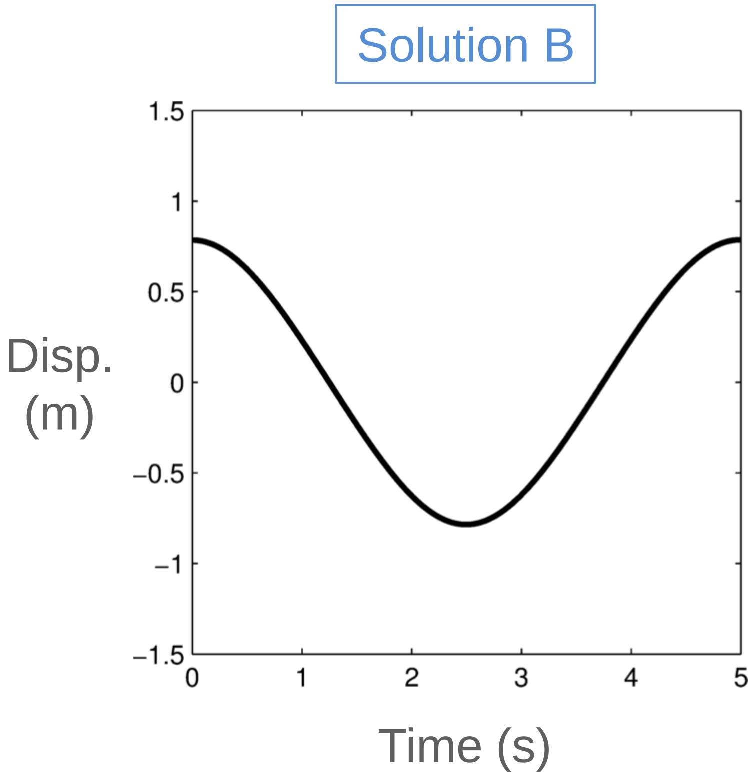

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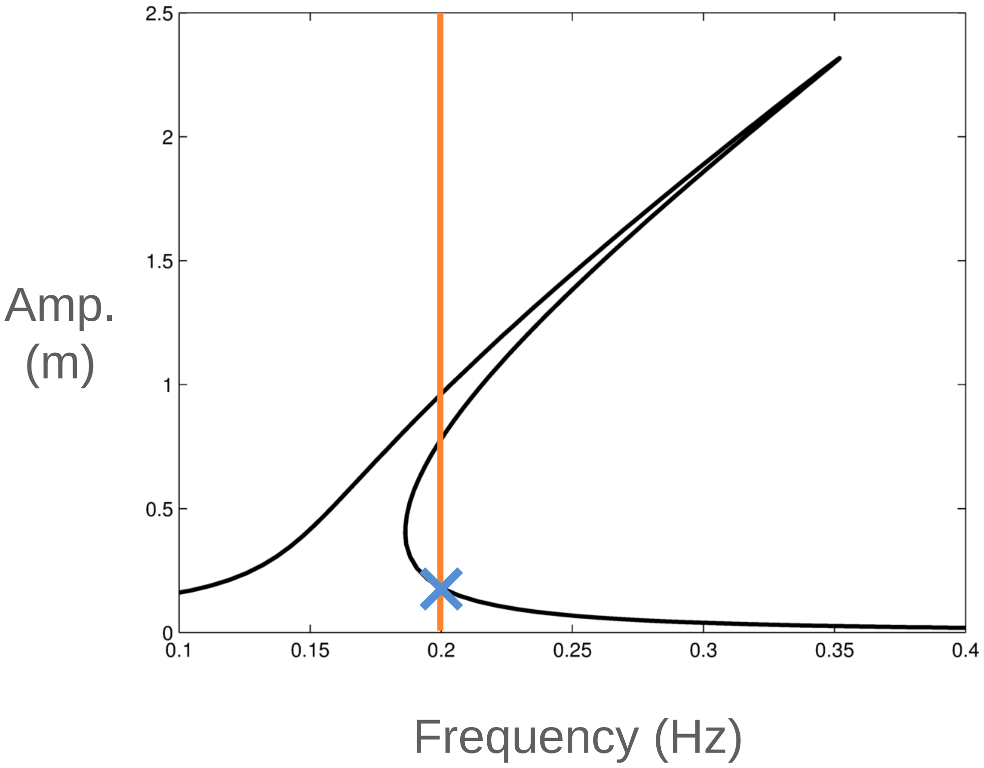
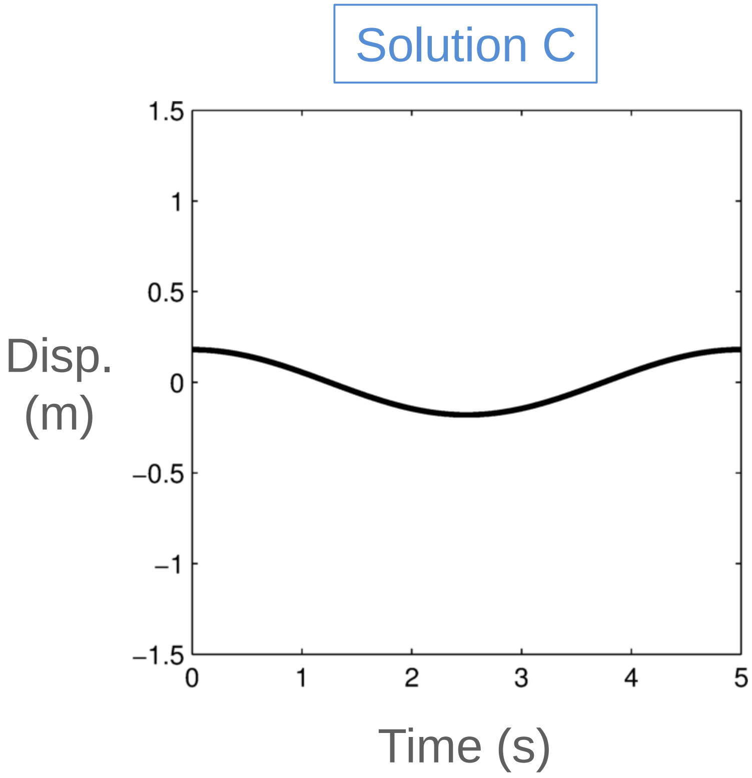
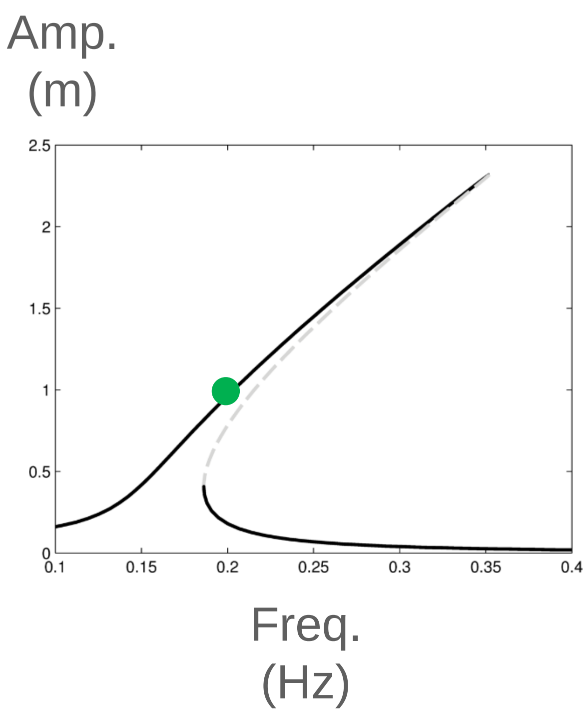
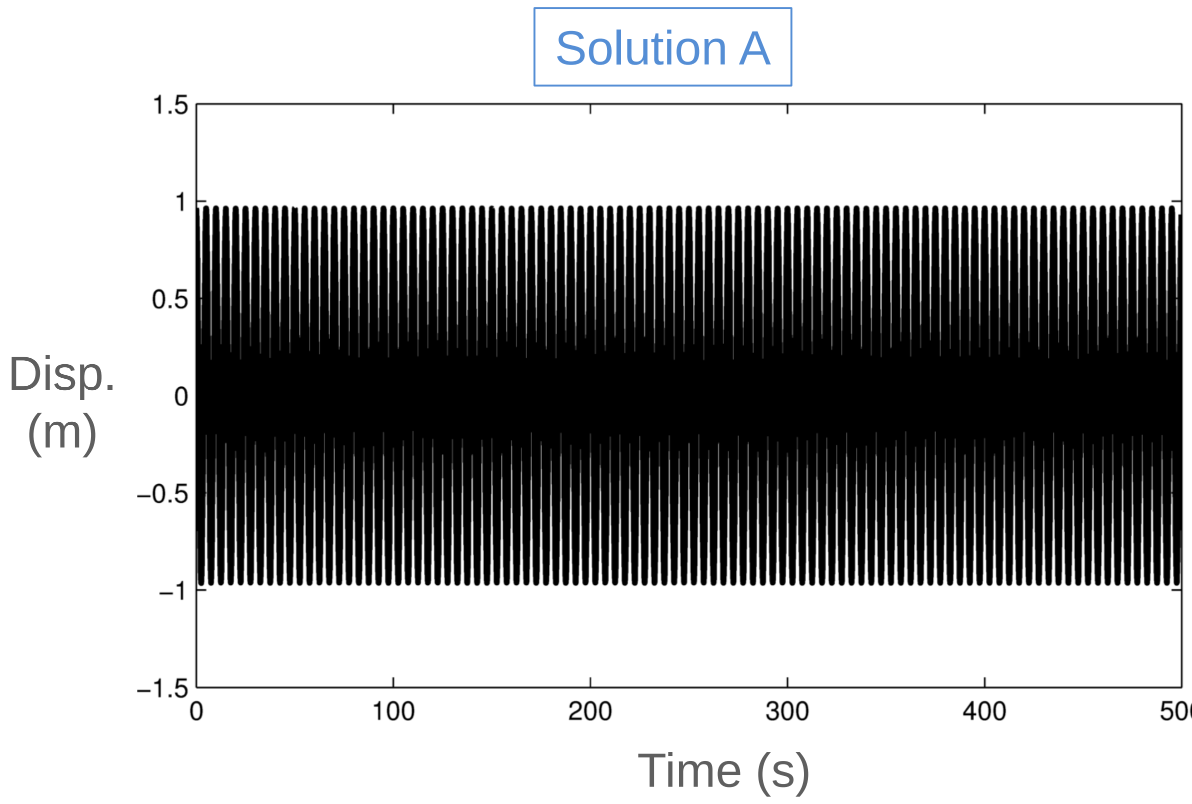




Motivation for This Lecture

As opposed to linear systems, this solution can be either stable or unstable.





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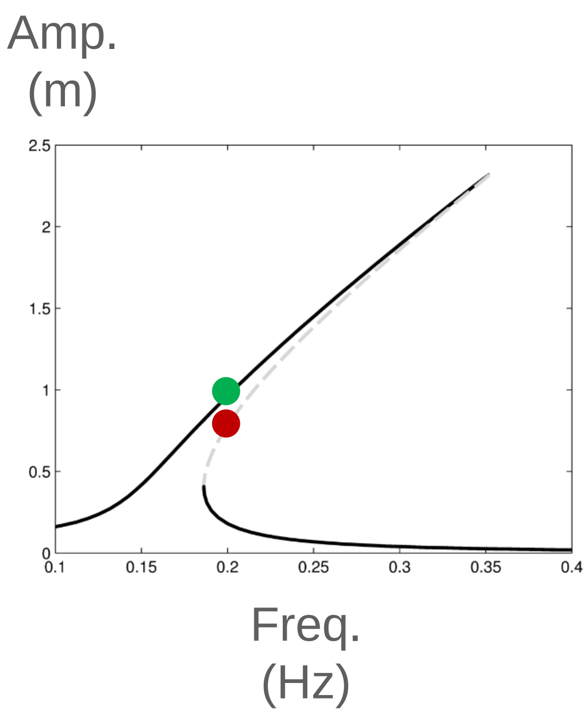

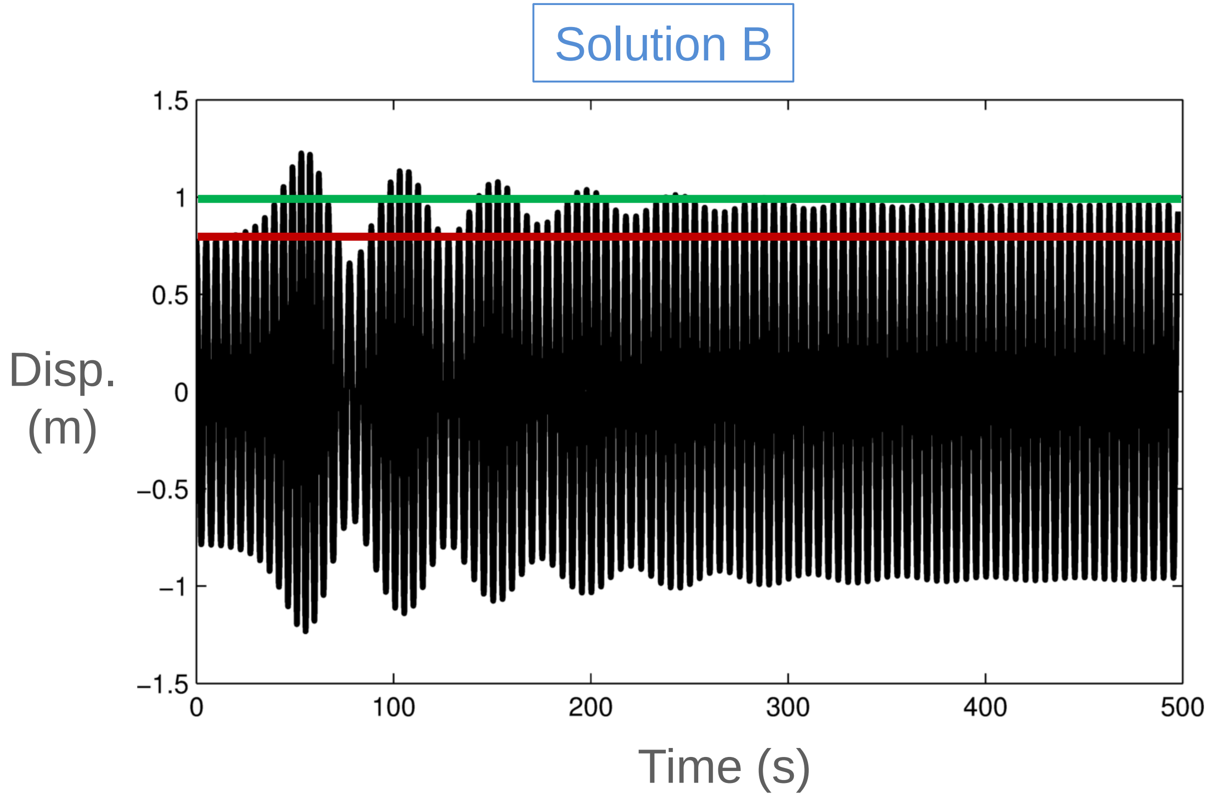

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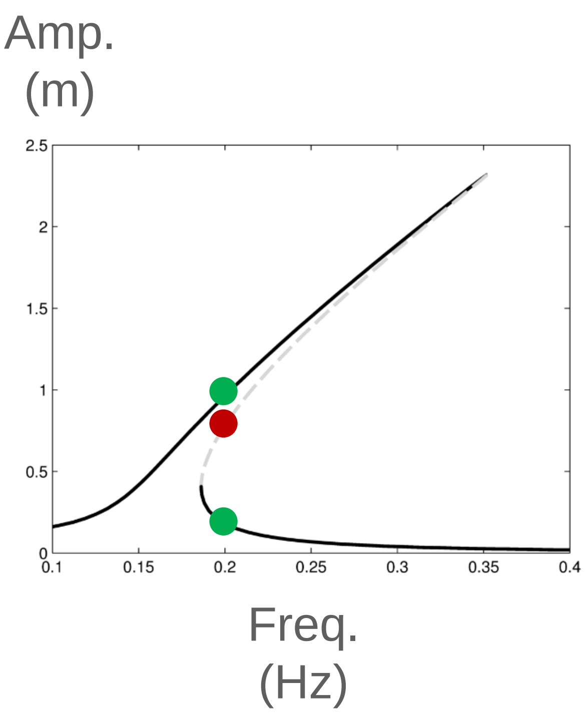
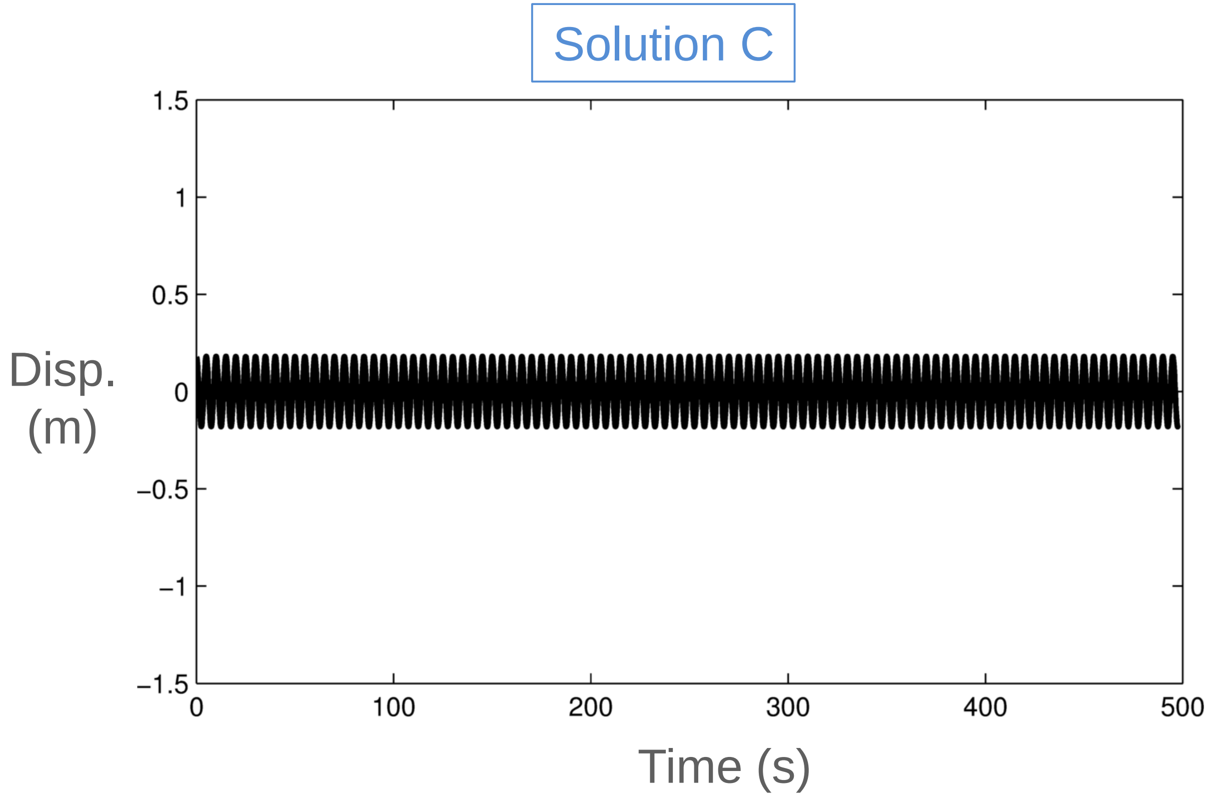


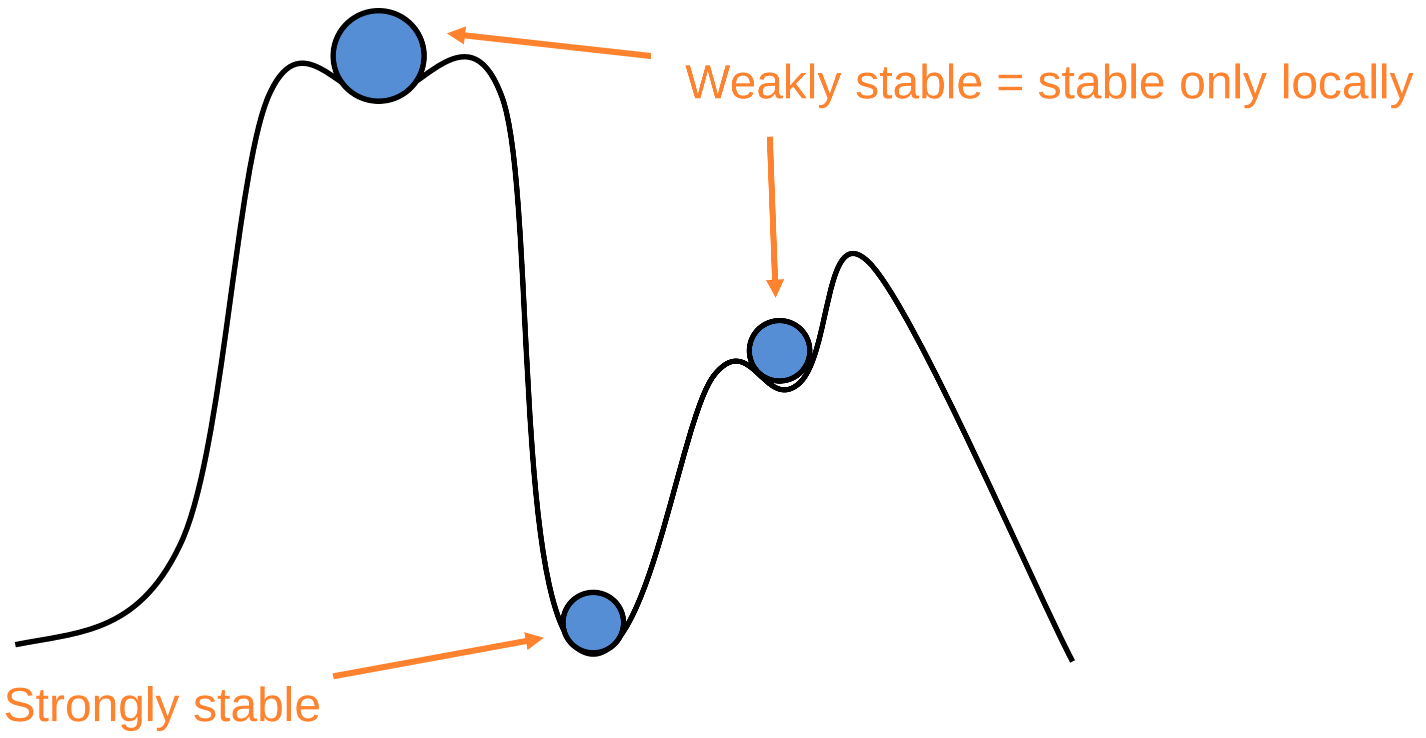


How to analyse stability? What happens at stability changes?

Stability Analysis

Local Stability Analysis

Locally stable means that the periodic solution is stable for small perturbations.



We recast the equations of motion in state-space form:

$$\dot{\mathbf{y}}(t) = \mathbf{L}\mathbf{y}(t) - \mathbf{g}_{nl}(\mathbf{y}) + \mathbf{g}_{ext}(\omega, t)$$

and perturb an equilibrium solution $\mathbf{y}_*(t)$

$$\mathbf{y}(t) = \mathbf{y}_*(t) + \mathbf{y}_p(t), \qquad \left|\mathbf{y}_p(t)\right| \ll |\mathbf{y}_*(t)|,$$

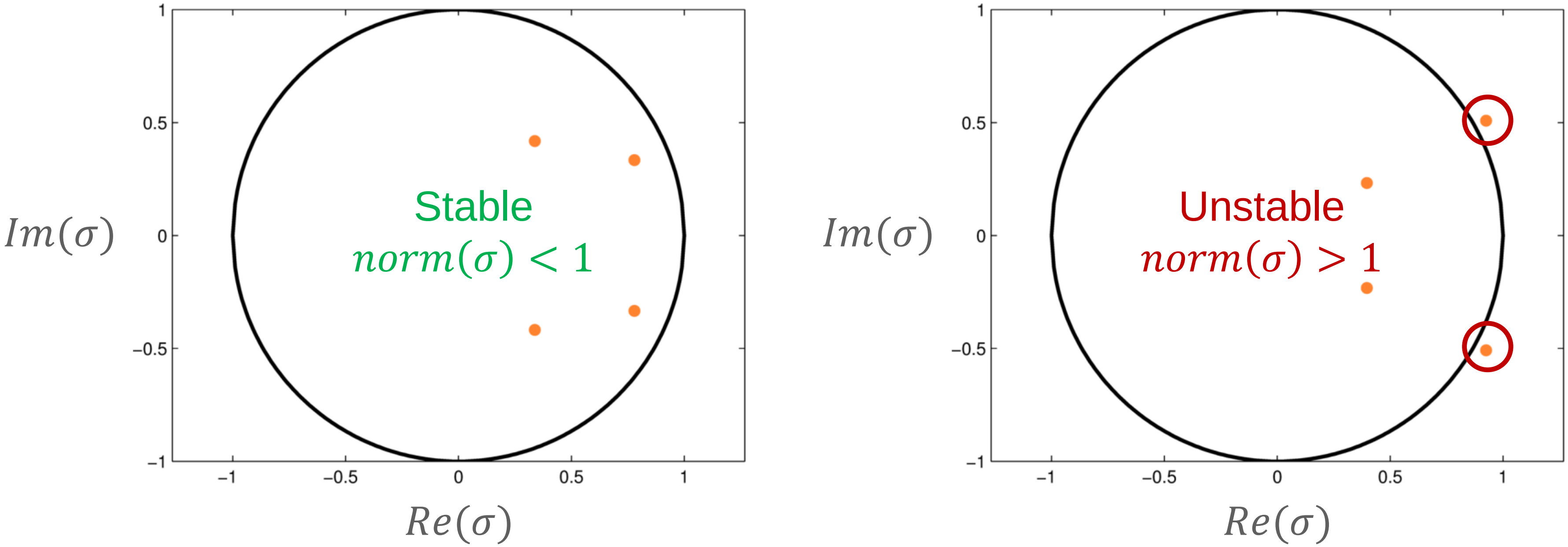
which yields

$$\dot{\mathbf{y}}_\mathrm{p}(t) \approx \left(\mathbf{L} - \left.\frac{\partial \mathbf{g}_{nl}}{\partial \mathbf{y}}\right|_{\mathbf{y}=\mathbf{y}_*(t)} \right) \mathbf{y}_p(t) = \mathbf{J}(t)\mathbf{y}_p(t).$$

The stability of this time-varying system can be assessed with Floquet theory.

Local Stability Analysis with Floquet Theory

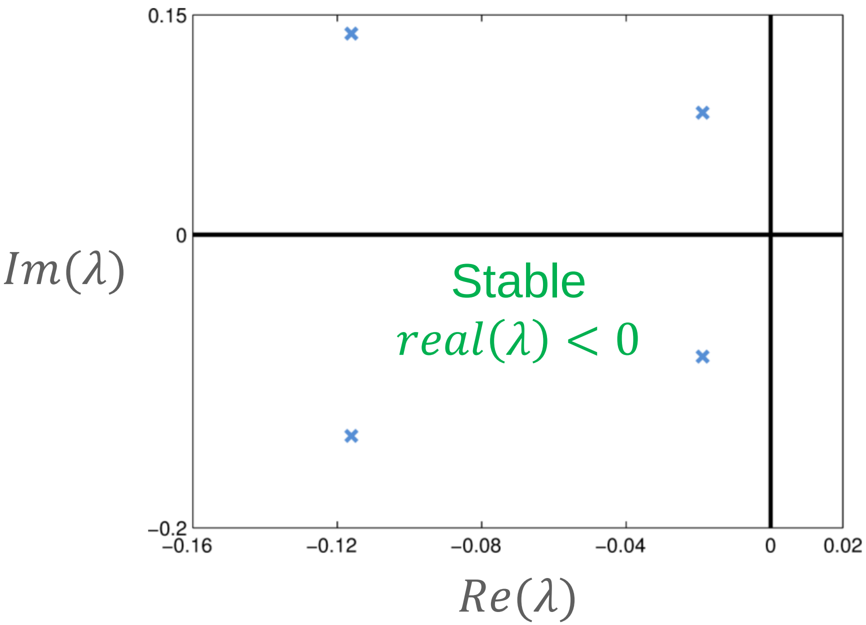
For each periodic solution, Floquet theory provides multipliers σ_i.

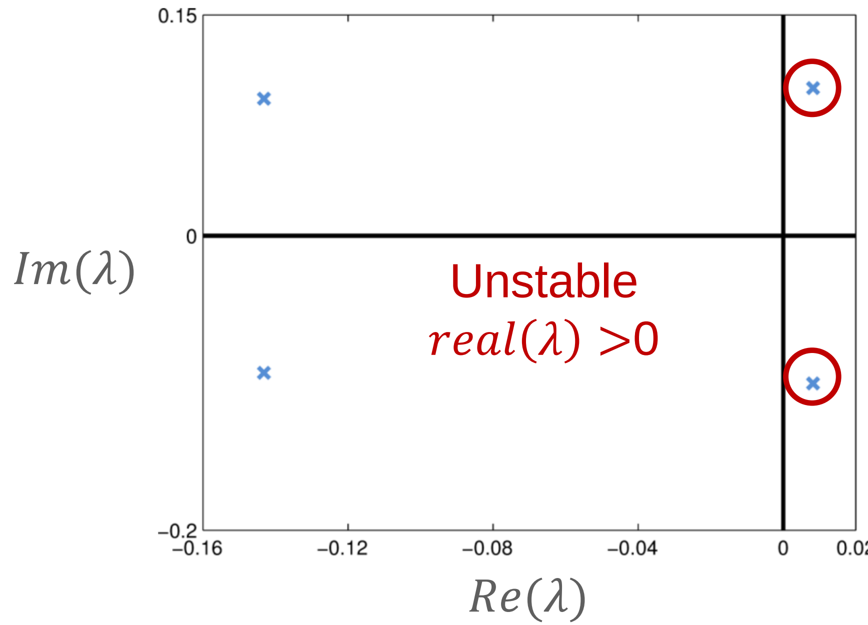


If at least one Floquet multiplier has a magnitude greater than 1, then the solution is unstable, otherwise it is stable.

Local Stability Analysis with Floquet Theory

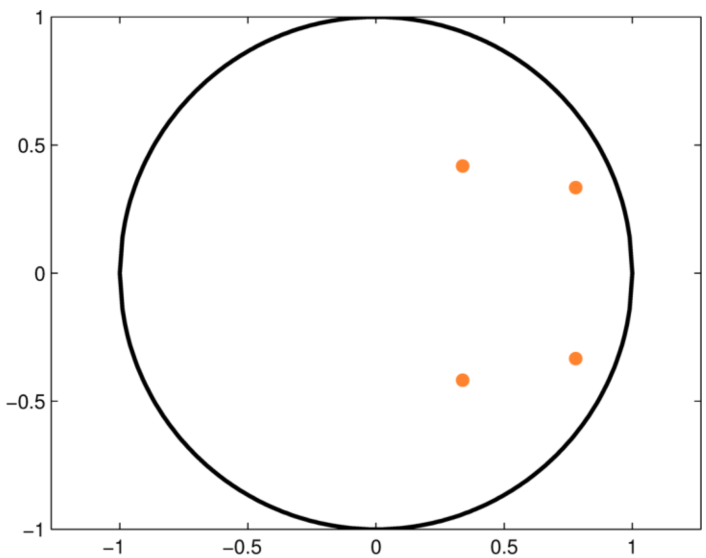
For each periodic solution, Floquet theory provides exponents λ_i.





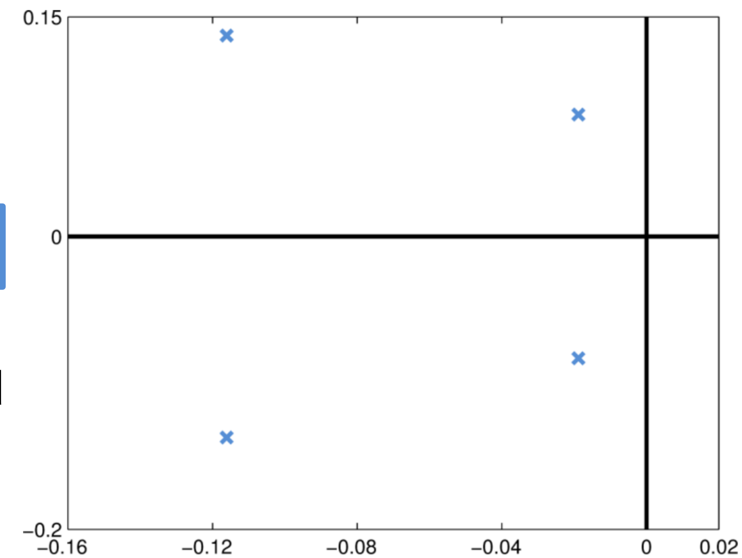
If at least one Floquet exponent has a real part greater than 0, then the solution is unstable, otherwise it is stable.

Multipliers and Exponents Are Related via a Simple Mapping



$$\sigma = e^{\lambda T}$$

Period



Periodic solutions of a system with n DOFs possesses $2n$ Floquet exponents/multipliers.

How to Perform Stability Analysis using Floquet Theory?

- Time-domain methods

  Monodromy matrix computation (not in this lecture).

- Frequency-domain methods

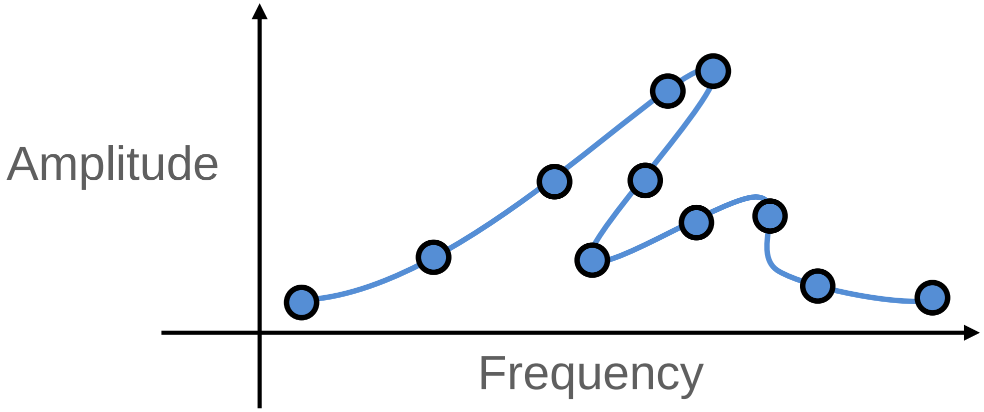
  Hill’s matrix computation.

Reminder: Harmonic Balance Method

Harmonic balance equation for periodic solutions:

$$\mathbf{h}(\mathbf{z}, \omega) \equiv \mathbf{A}(\omega)\mathbf{z} - \mathbf{b}(\mathbf{z}) = \mathbf{0}$$

Given $\mathbf{h}(\mathbf{z}, \omega)$, and the Jacobian matrices $\mathbf{h}_{\mathbf{z}}$ and $\mathbf{h}_{\omega}$, a continuation procedure can compute branches of periodic solutions.



In the Frequency Domain: Computation of the Hill's Matrix

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{f}(\mathbf{x}, \dot{\mathbf{x}}, \omega, t)$$
$$= \mathbf{f}_{ext}(\omega, t) - \mathbf{f}_{nl}(\mathbf{x}, \dot{\mathbf{x}})$$

A periodic solution $\mathbf{x}^*(t)$ satisfying the EOMs is perturbed with a periodic solution modulated by an exponential decay:

$$\mathbf{x}(t) = \mathbf{x}^*(t) + e^{\lambda t}\mathbf{s}(t)$$

Introducing the perturbation in the EOMs leads to:

$$\mathbf{M}\ddot{\mathbf{x}}^* + \mathbf{C}\dot{\mathbf{x}}^* + \mathbf{K}\mathbf{x}^*$$
$$+[\lambda^2\mathbf{M}\mathbf{s} + \lambda(2\mathbf{M}\dot{\mathbf{s}} + \mathbf{C}\mathbf{s}) + \mathbf{M}\ddot{\mathbf{s}} + \mathbf{C}\dot{\mathbf{s}} + \mathbf{K}\mathbf{s}]e^{\lambda t}$$
$$= \mathbf{f}(\mathbf{x}, \dot{\mathbf{x}}, \omega, t)$$

In the Frequency Domain: Computation of the Hill's Matrix

$$\mathbf{M}\ddot{\mathbf{x}}^* + \mathbf{C}\dot{\mathbf{x}}^* + \mathbf{K}\mathbf{x}^*$$
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$$= \mathbf{f}(\mathbf{x}, \dot{\mathbf{x}}, \omega, t)$$

Fourier series approximation:

$$\mathbf{x}^*(t) = (\mathbf{Q}(t) \otimes \mathbf{I}_n)\mathbf{z}^*$$

$$\mathbf{s}(t) = (\mathbf{Q}(t) \otimes \mathbf{I}_n)\mathbf{u}$$

\+ Galerkin procedure + Linearisation

$$(\boldsymbol{\Delta}_2\lambda^2 + \boldsymbol{\Delta}_1\lambda + \mathbf{h}_\mathbf{z})e^{\lambda t}\mathbf{u} = \mathbf{0}$$

In the Frequency Domain: Computation of the Hill's Matrix

$$(\mathbf{\Delta}_2\lambda^2 + \mathbf{\Delta}_1\lambda + \mathbf{h_z})e^{\lambda t}\mathbf{u} = \mathbf{0}$$

where

$$\mathbf{\Delta}_1 = \mathbf{\nabla} \otimes \mathbf{2M} + \mathbf{I}_{2N_H+1} \otimes \mathbf{C}$$

$$= \begin{bmatrix} \mathbf{C} & & & & & \\ & \mathbf{C} & -2\omega\mathbf{M} & & & \\ & 2\omega\mathbf{M} & \mathbf{C} & & & \\ & & & \ddots & & \\ & & & & \mathbf{C} & -2N_H\omega\mathbf{M} \\ & & & & 2N_H\omega\mathbf{M} & \mathbf{C} \end{bmatrix}$$

and

$$\mathbf{\Delta}_2 = \mathbf{I}_{2N_H+1} \otimes \mathbf{M}$$

In the Frequency Domain: Computation of the Hill's Matrix

$$(\boldsymbol{\Delta}_2\lambda^2 + \boldsymbol{\Delta}_1\lambda + \mathbf{h_z})e^{\lambda t}\mathbf{u} = \mathbf{0}$$

The quadratic eigenvalue problem of size $n(2N_H + 1)$

$$(\boldsymbol{\Delta}_2\lambda^2 + \boldsymbol{\Delta}_1\lambda + \mathbf{h_z})\mathbf{v} = 0$$

can be rewritten as a linear eigenvalue problem of doubled size

$$\left(\mathbf{B} - \lambda\mathbf{I}_{2n(2N_H+1)}\right)\mathbf{w} = \mathbf{0}$$

with

$$\mathbf{B} = \begin{bmatrix} -\boldsymbol{\Delta}_2^{-1}\boldsymbol{\Delta}_1 & -\boldsymbol{\Delta}_2^{-1}\mathbf{h_z} \\ \mathbf{I}_{n(2N_H+1)} & \mathbf{0} \end{bmatrix}$$

Computation of the Floquet Exponents from Hill's Matrix

$$\mathbf{B} = \begin{bmatrix} -\boldsymbol{\Delta}_2^{-1}\boldsymbol{\Delta}_1 & -\boldsymbol{\Delta}_2^{-1}\mathbf{h_z} \\ \mathbf{I}_{n(2N_H+1)} & \mathbf{0} \end{bmatrix}$$

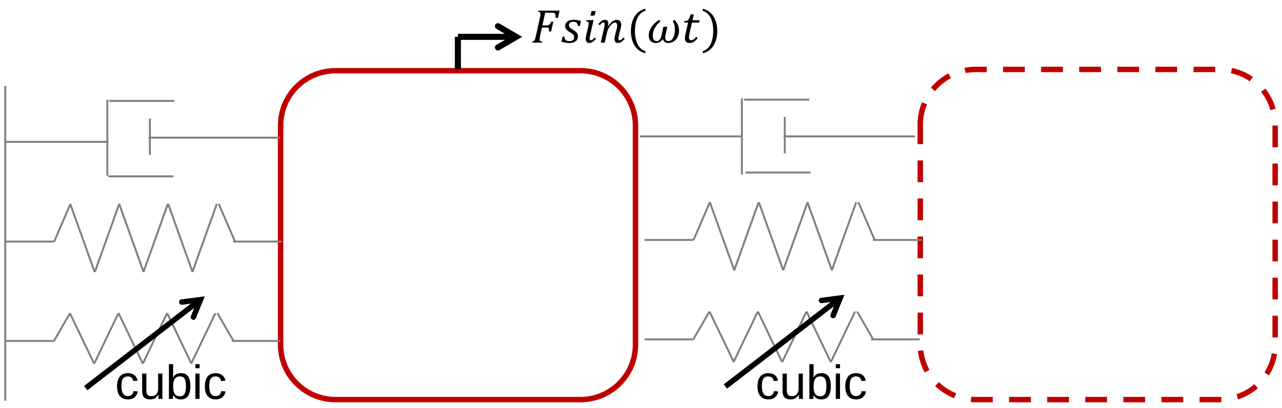
$\mathbf{B}$ is the Hill's matrix and its eigenvalues $\boldsymbol{\lambda}$ are the Hill's coefficients (real or complex conjugates since $\mathbf{B}$ is real).

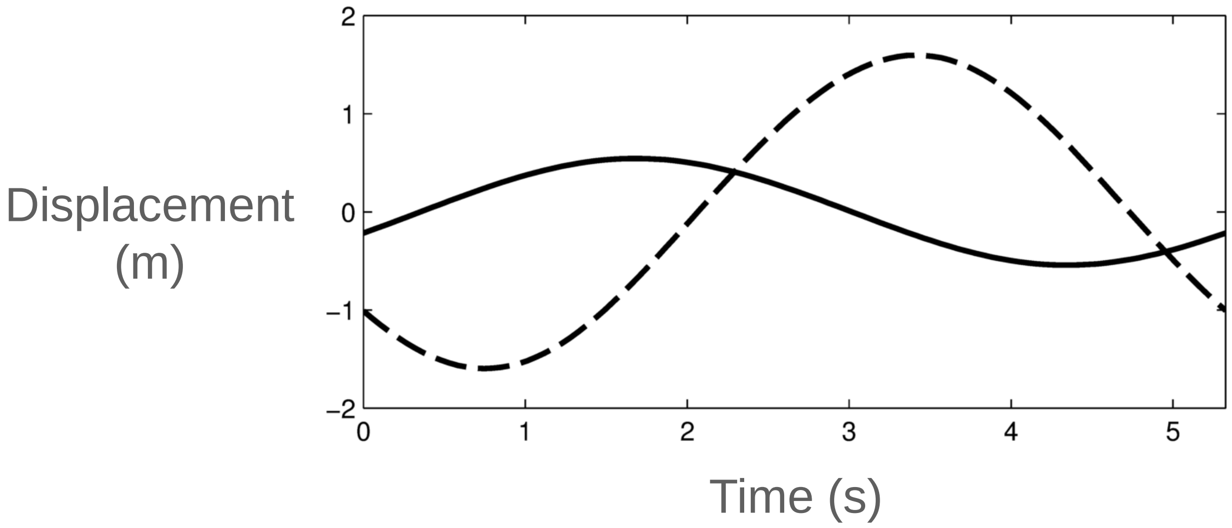
Among these $2n(2N_H + 1)$ eigenvalues, one can find $2n$ Hill's coefficients $\tilde{\boldsymbol{\lambda}}$ that approximate the Floquet exponents of the periodic solution $\mathbf{x}^*$.

The best approximation of the Floquet exponents are the $2n$ Hill's coefficients with the smallest imaginary parts in modulus.

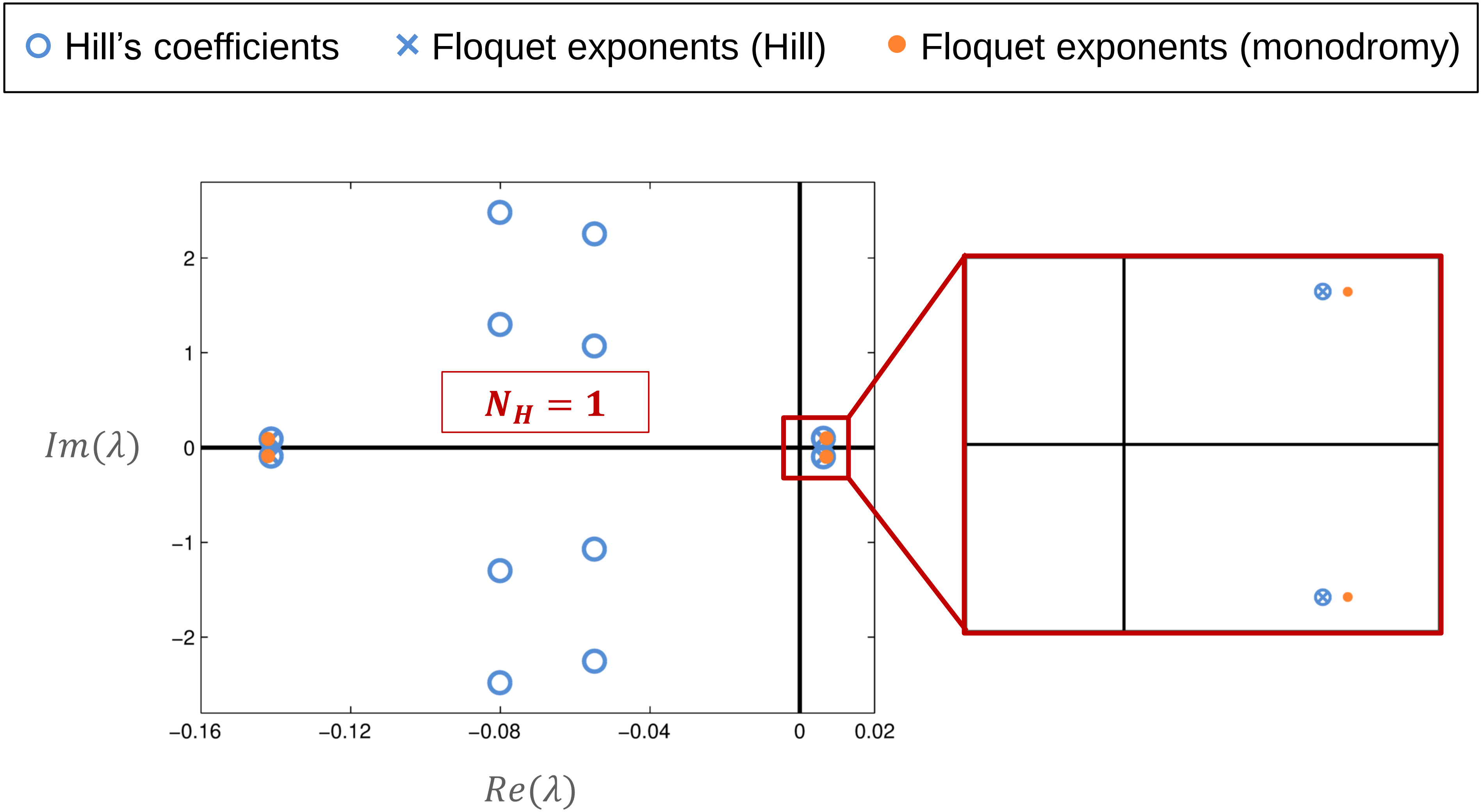
Comparing Hill's Coefficients with Floquet Exponents

Illustration of the sorting criterion on a 2-DOF example:



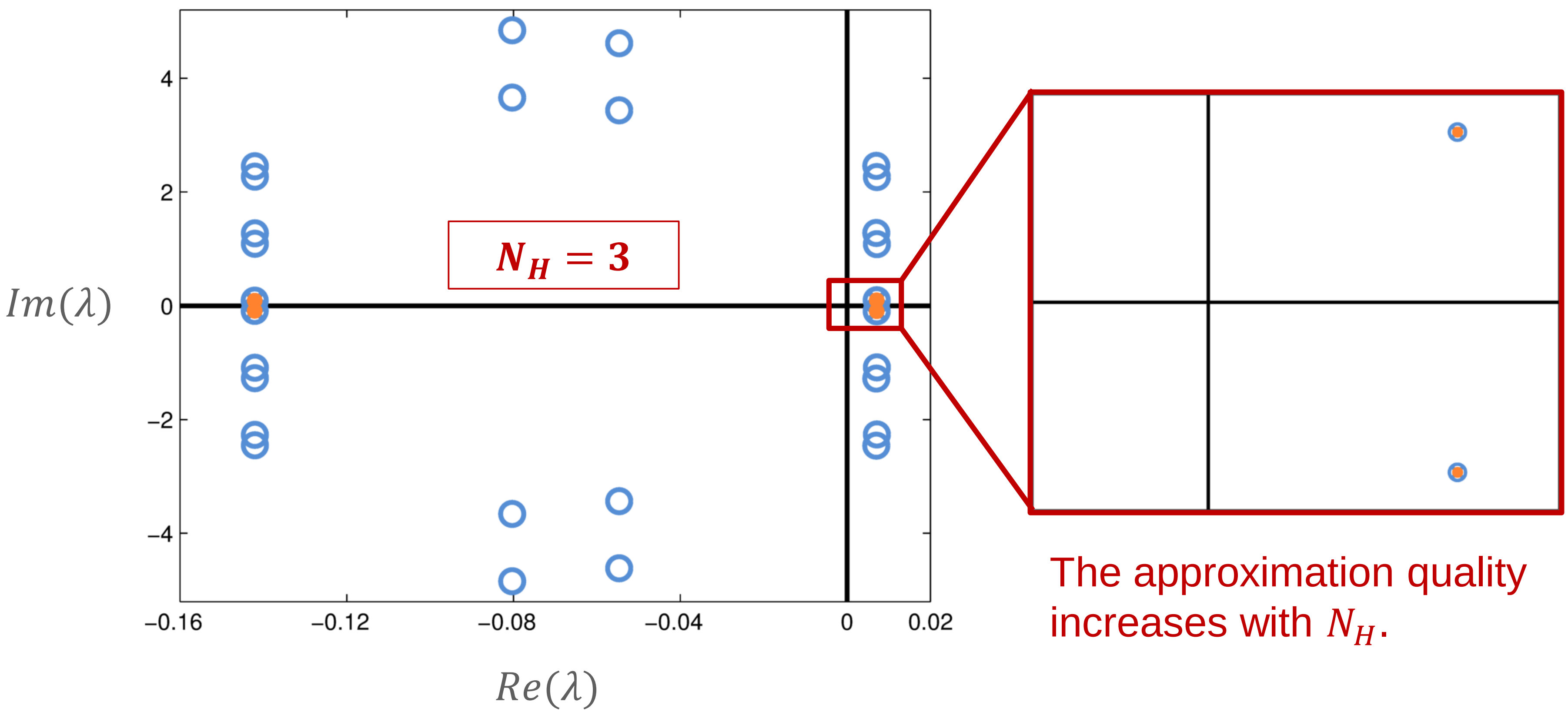


Comparing Hill's Coefficients with Floquet Exponents



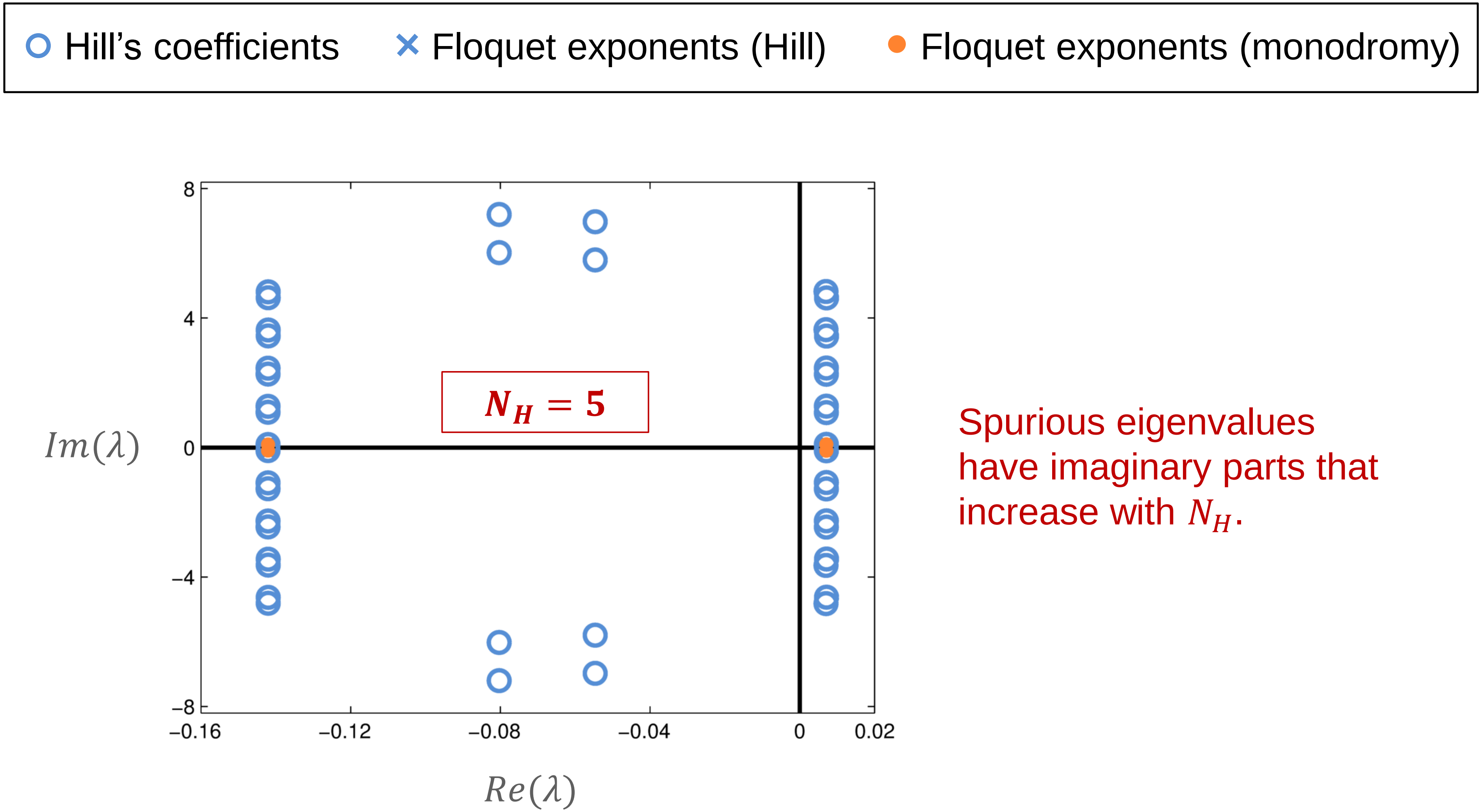
Comparing Hill's Coefficients with Floquet Exponents

○ Hill's coefficients	× Floquet exponents (Hill)	● Floquet exponents (monodromy)



The approximation quality increases with N_H.

Comparing Hill's Coefficients with Floquet Exponents



Spurious eigenvalues have imaginary parts that increase with N_H.

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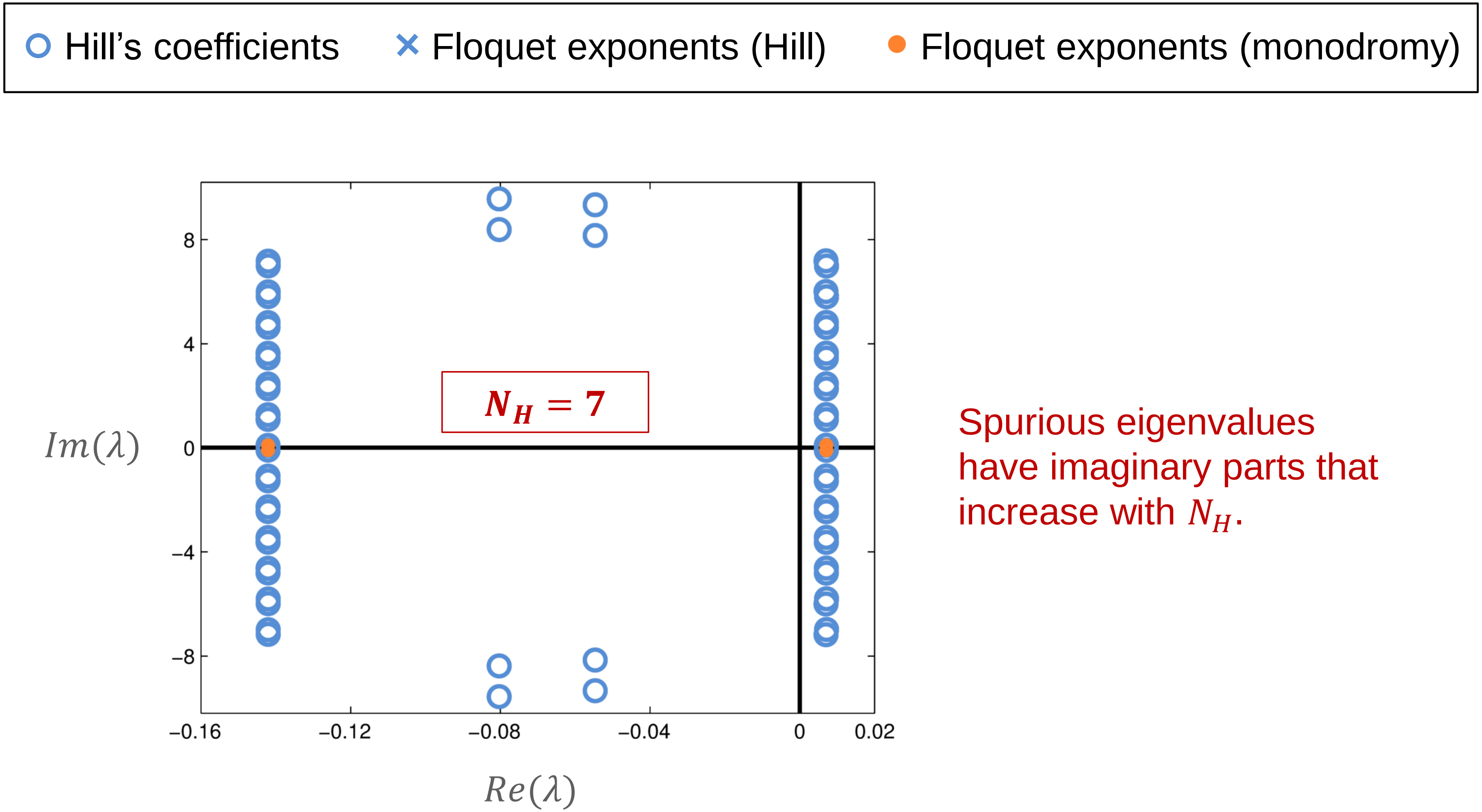

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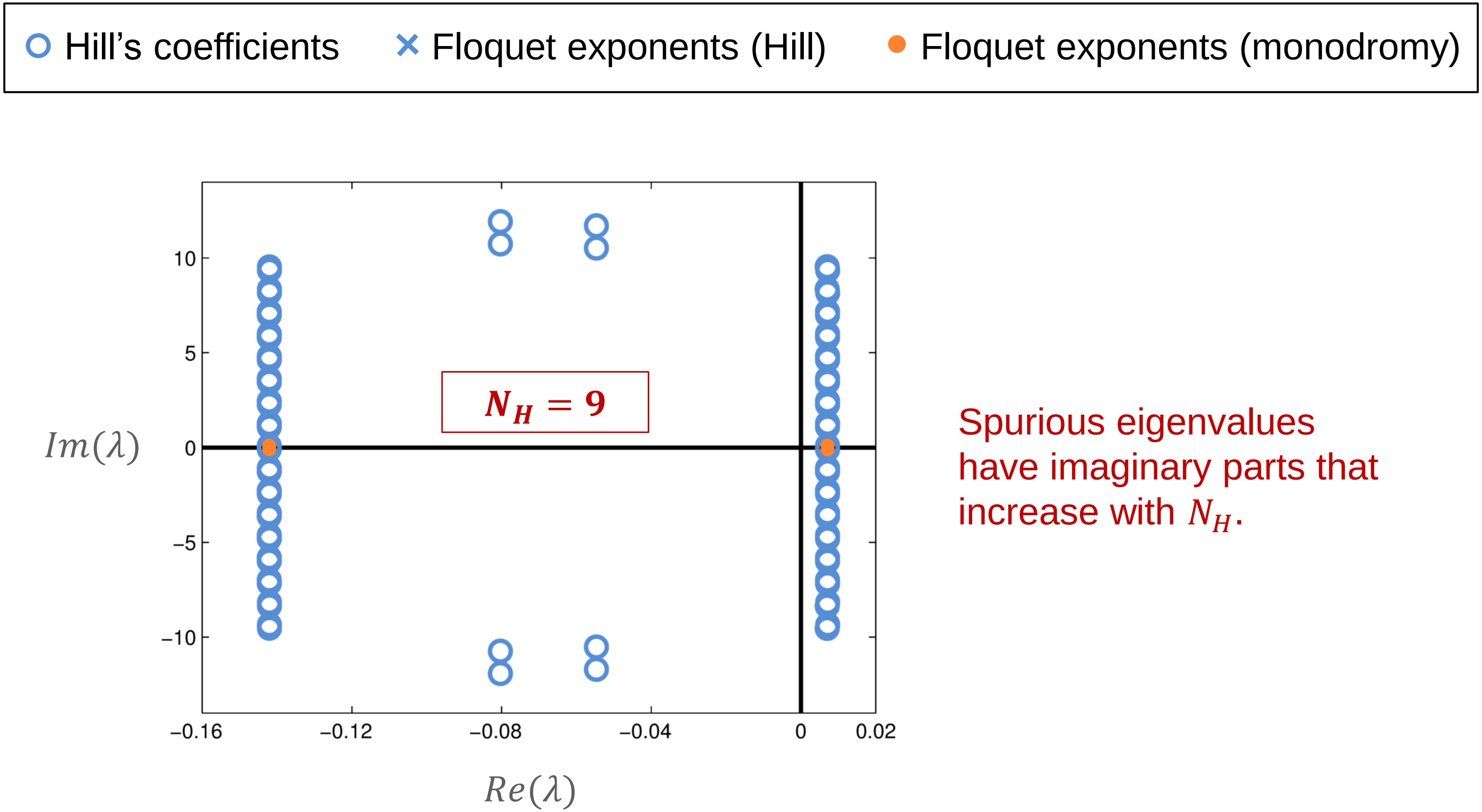


Spurious eigenvalues have imaginary parts that increase with N_H.

Computation of Floquet Exponents with Hill's Method

$$\mathbf{B} = \begin{bmatrix} -\boldsymbol{\Delta}_2^{-1}\boldsymbol{\Delta}_1 & -\boldsymbol{\Delta}_2^{-1}\mathbf{h_z} \\ \mathbf{I}_{n(2N_H+1)} & \mathbf{0} \end{bmatrix}$$

Eigenvalues of $\mathbf{B}$ = approximation of Floquet exponents.

Step 1:
Convergence of the eigenvalues w.r.t. the number of harmonics.

Step 2:
Selection of the eigenvalues $\tilde{\lambda}_i$ with smallest imaginary parts in modulus.

$$\tilde{\mathbf{B}} = \begin{bmatrix} \tilde{\lambda}_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\lambda}_{2n} \end{bmatrix}$$

Hill's Method in Summary

$$\widetilde{\mathbf{B}} = \begin{bmatrix} \tilde{\lambda}_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\lambda}_{2n} \end{bmatrix}$$

Reasonably accurate if

the number of harmonics retained N_H is large enough ;

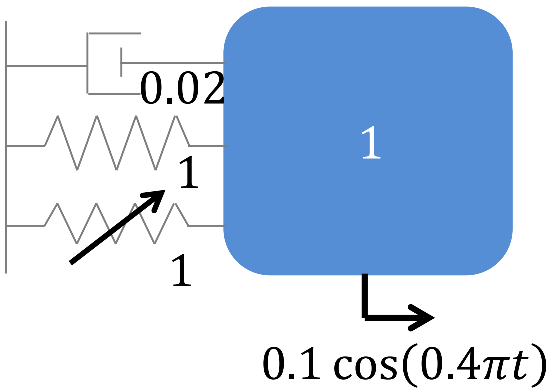
the eigenvalues are correctly sorted.

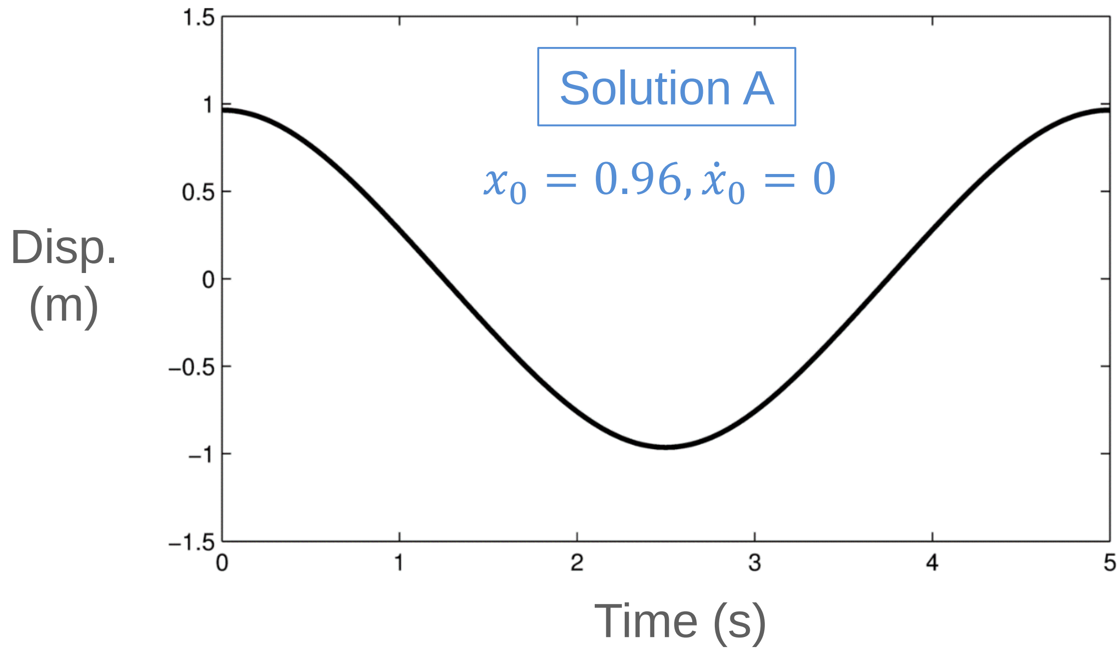
It does not require time integration, but the eigenvalue problem to solve is computationally intensive for large systems.

The only term that needs to be evaluated when $\mathbf{z}$ varies is $\mathbf{h}_{\mathbf{z}}$, which can by obtained as a by-product of the harmonic balance method.

Basins of Attraction of Periodic Solutions

When two stable solutions coexist for the same system and excitation parameters, initial conditions dictate which solution will attract the dynamics and eventually stabilise.





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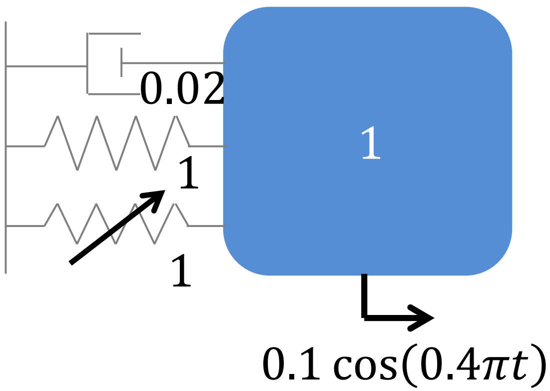
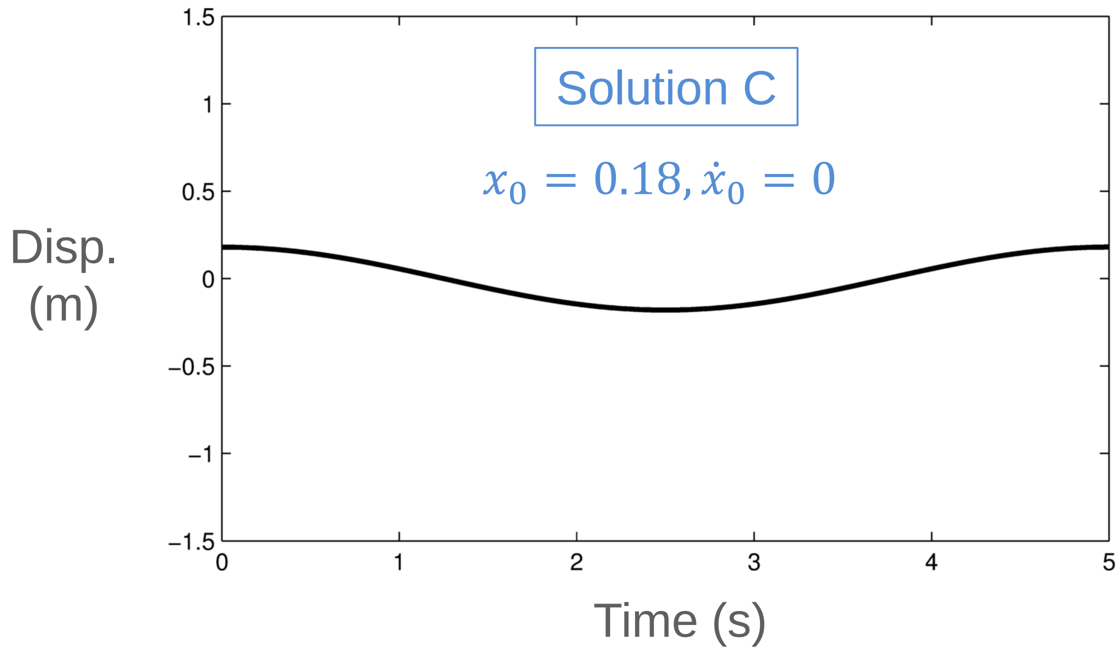
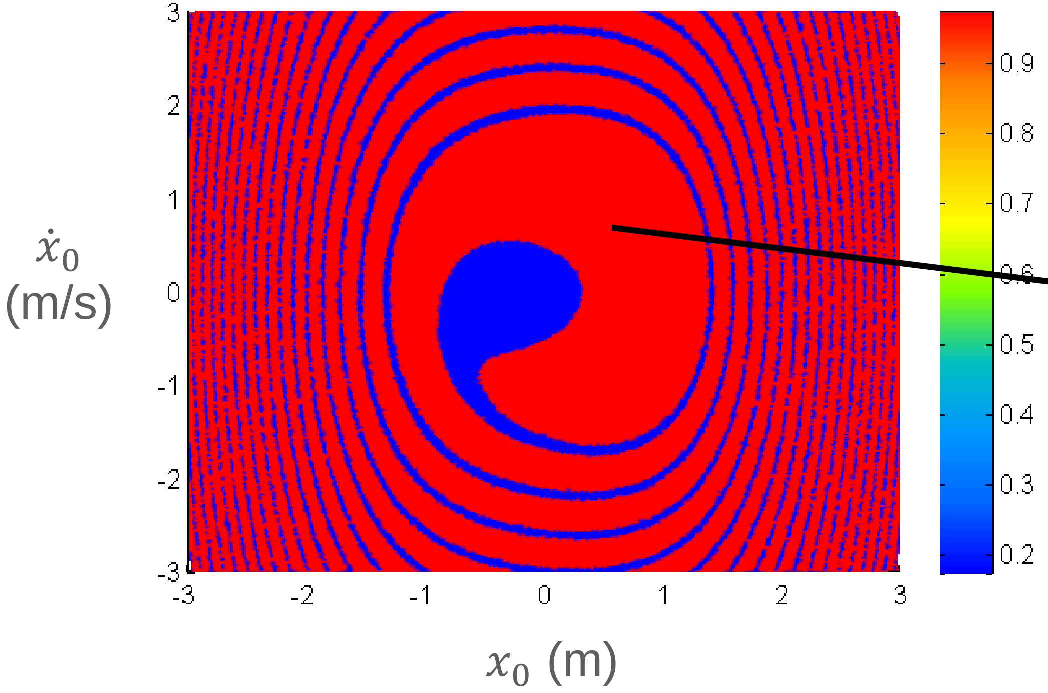
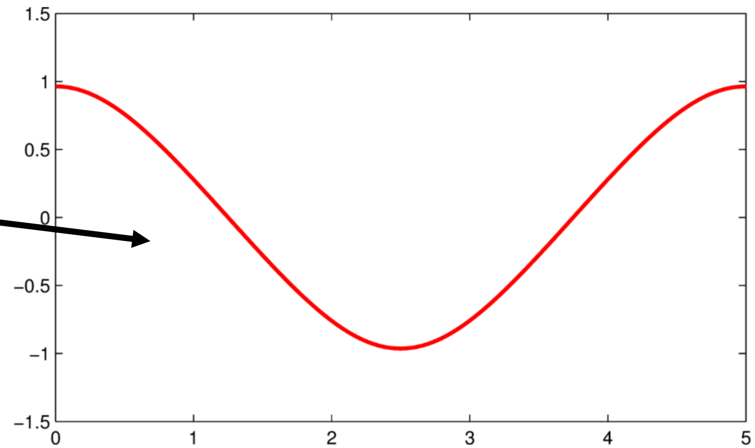




Basins of Attraction of Periodic Solutions

The set of initial conditions leading to a stable periodic solution is called region of attraction or basin of attraction.





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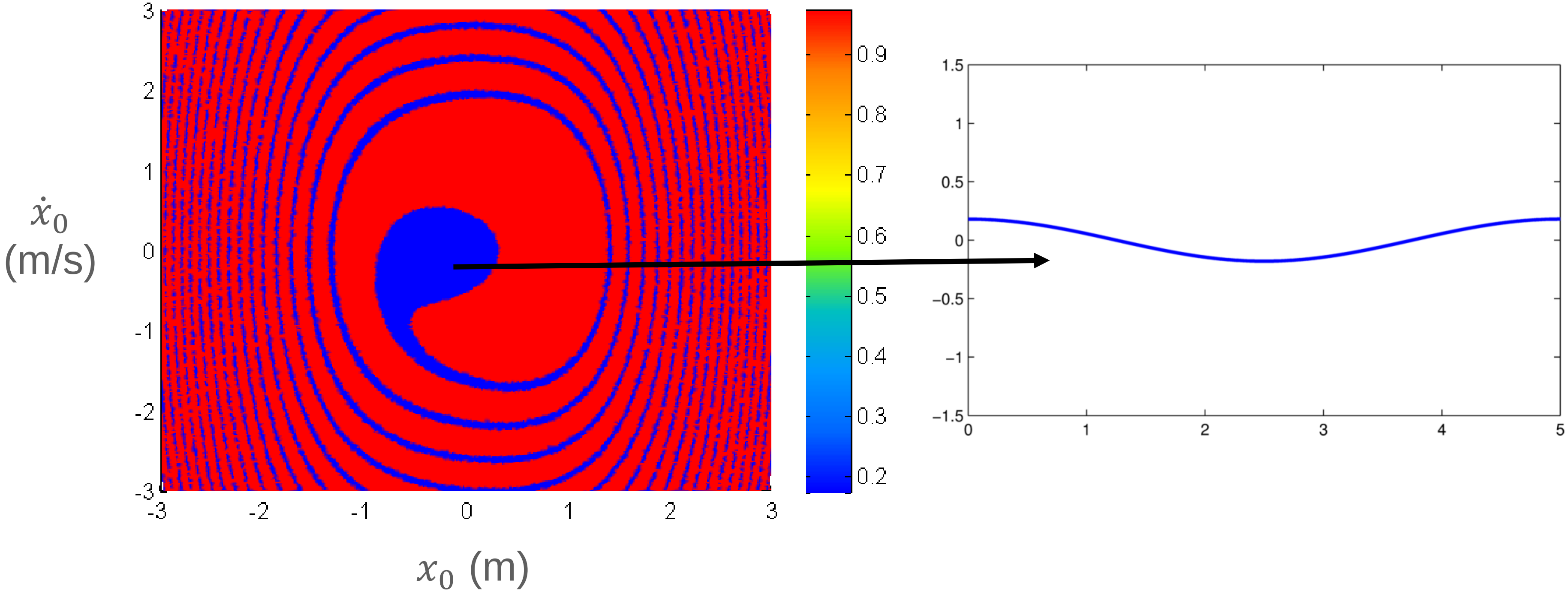

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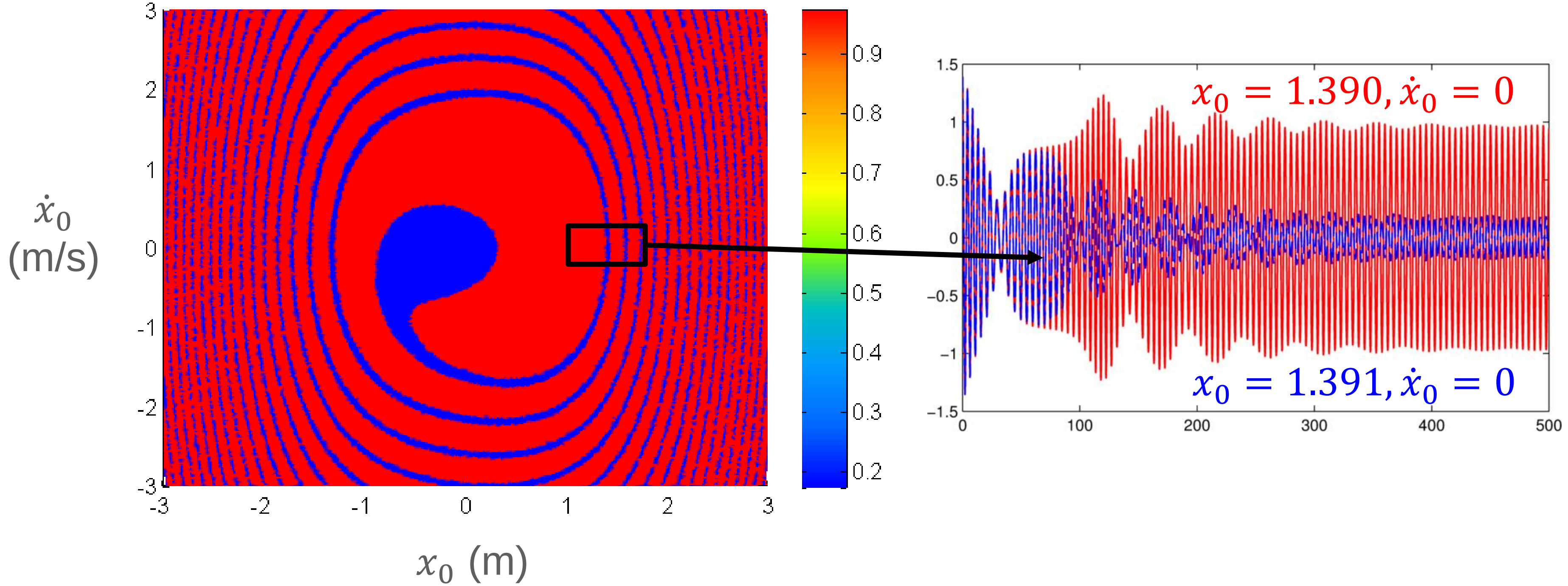
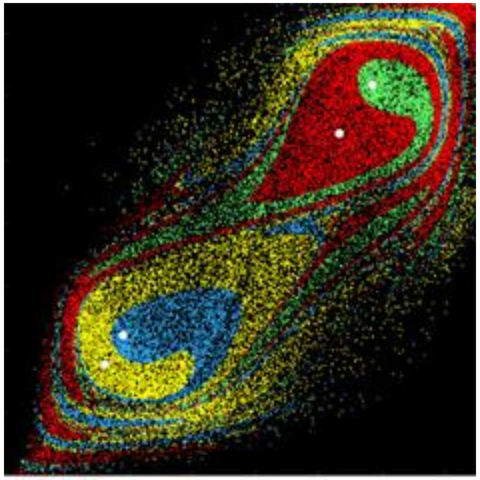


How to Construct Basins of Attraction?

- Numerically

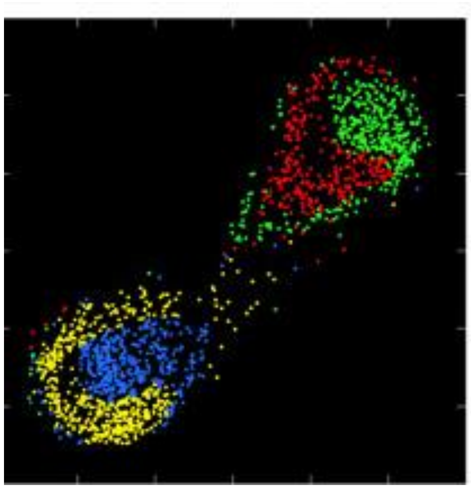
  Cell mapping ;

  Parallel time integrations.



- Experimentally

  Stochastic interrogation.



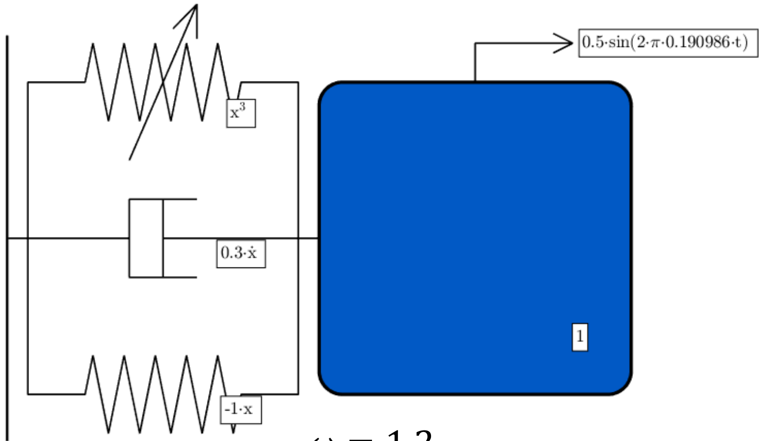
Basins of Attraction in a Nutshell

Basins of attraction can have complicated structures (for example, interlaced with fractal boundaries).

Their dimensions increase with the number of DOFs.

There exist different types of attractor:

- Equilibrium points
- Periodic solutions
- Quasiperiodic solutions
- Strange attractors (chaos)



$\omega = 1.2$

$f = 0.2 - 0.29 - 0.37 - 0.5$

Concluding Remarks on Stability Analysis

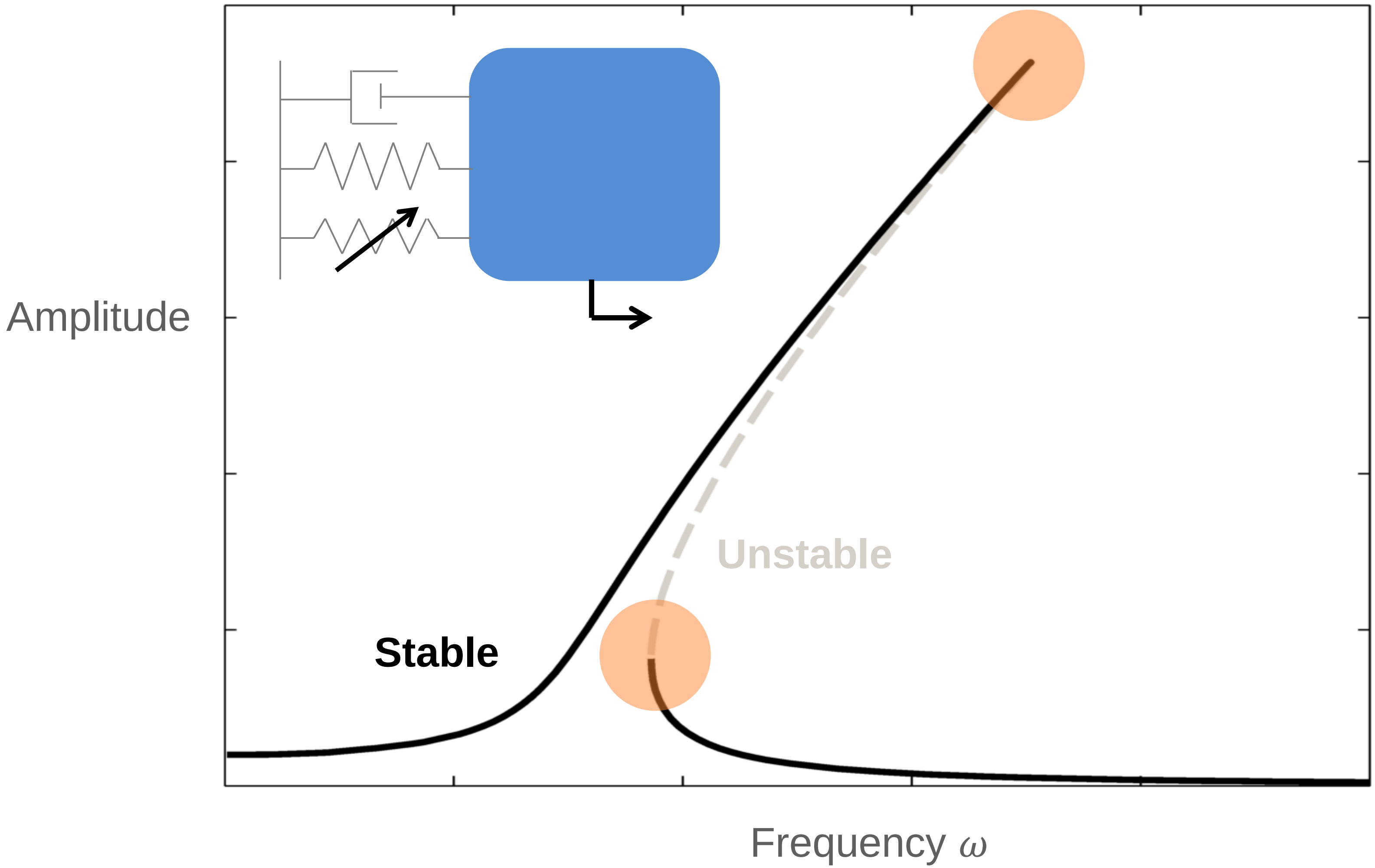
Stability of periodic solutions can be assessed by analysing the associated Floquet exponents or multipliers.

When the harmonic balance method is employed, stability analysis is preferably performed through the Hill's matrix.

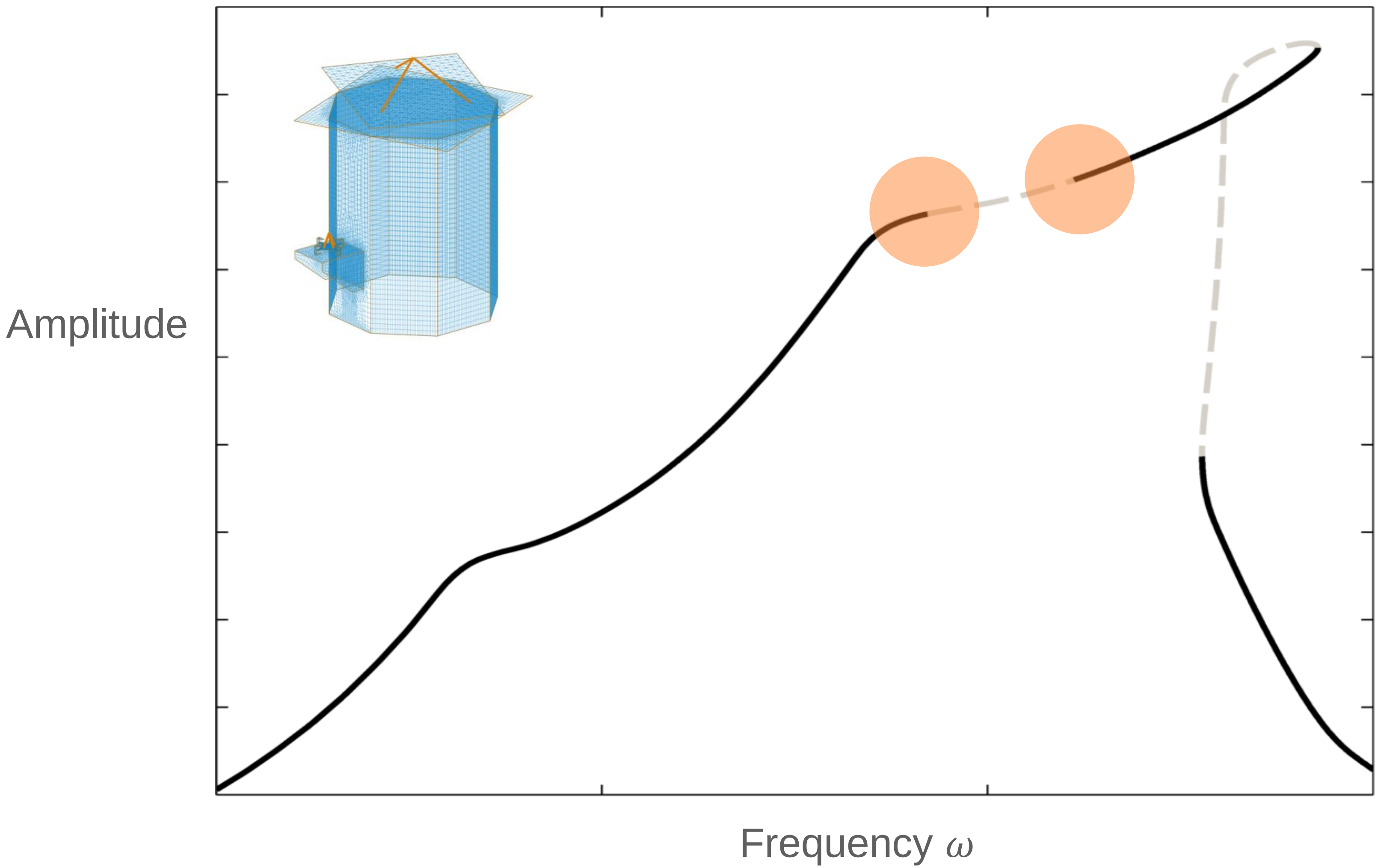
Global analysis (basins of attraction) provides important and intuitive stability information, but its usefulness is limited to small systems.

Bifurcation Analysis and Tracking

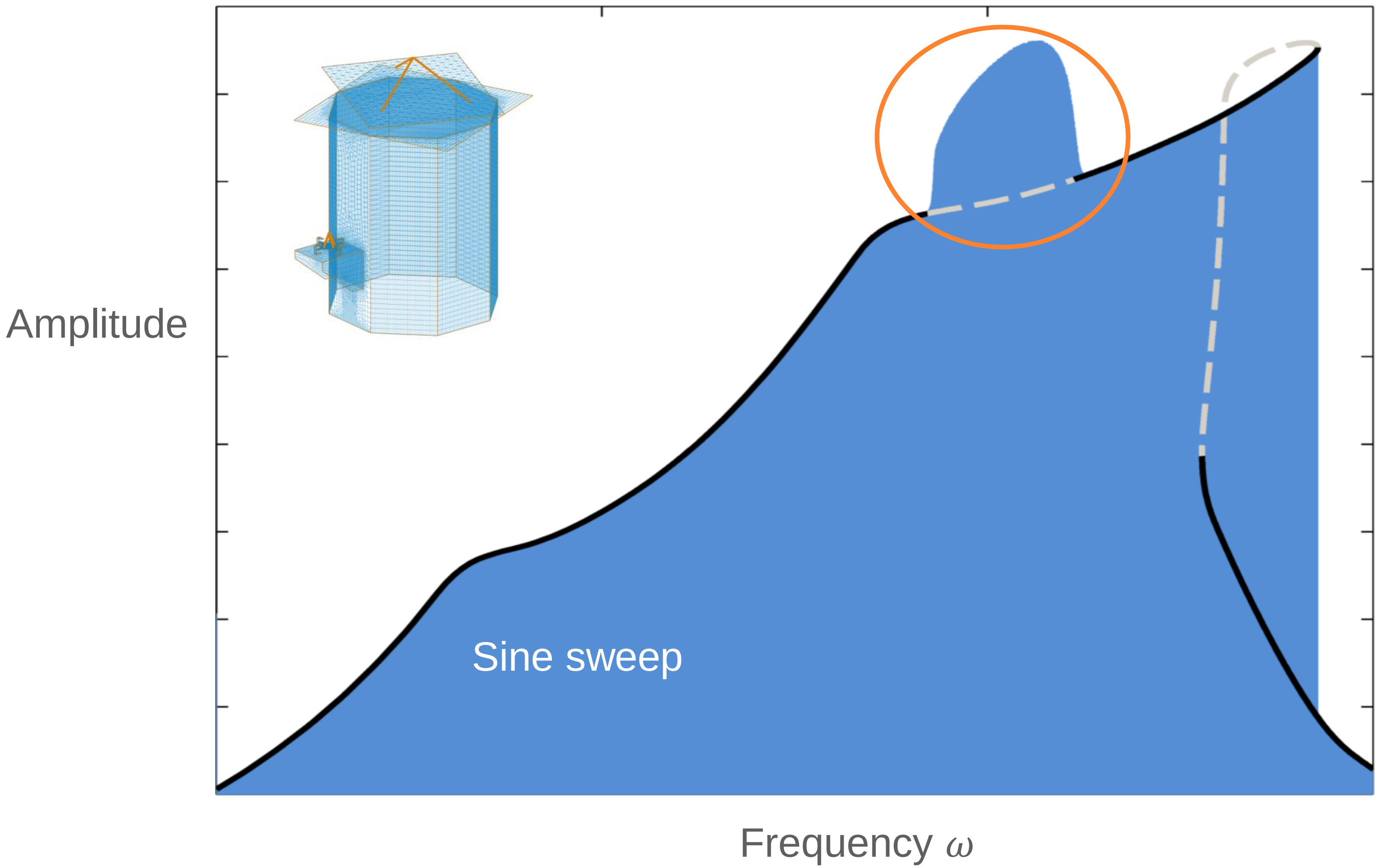
Stability Changes Occur not only near Turning Points …



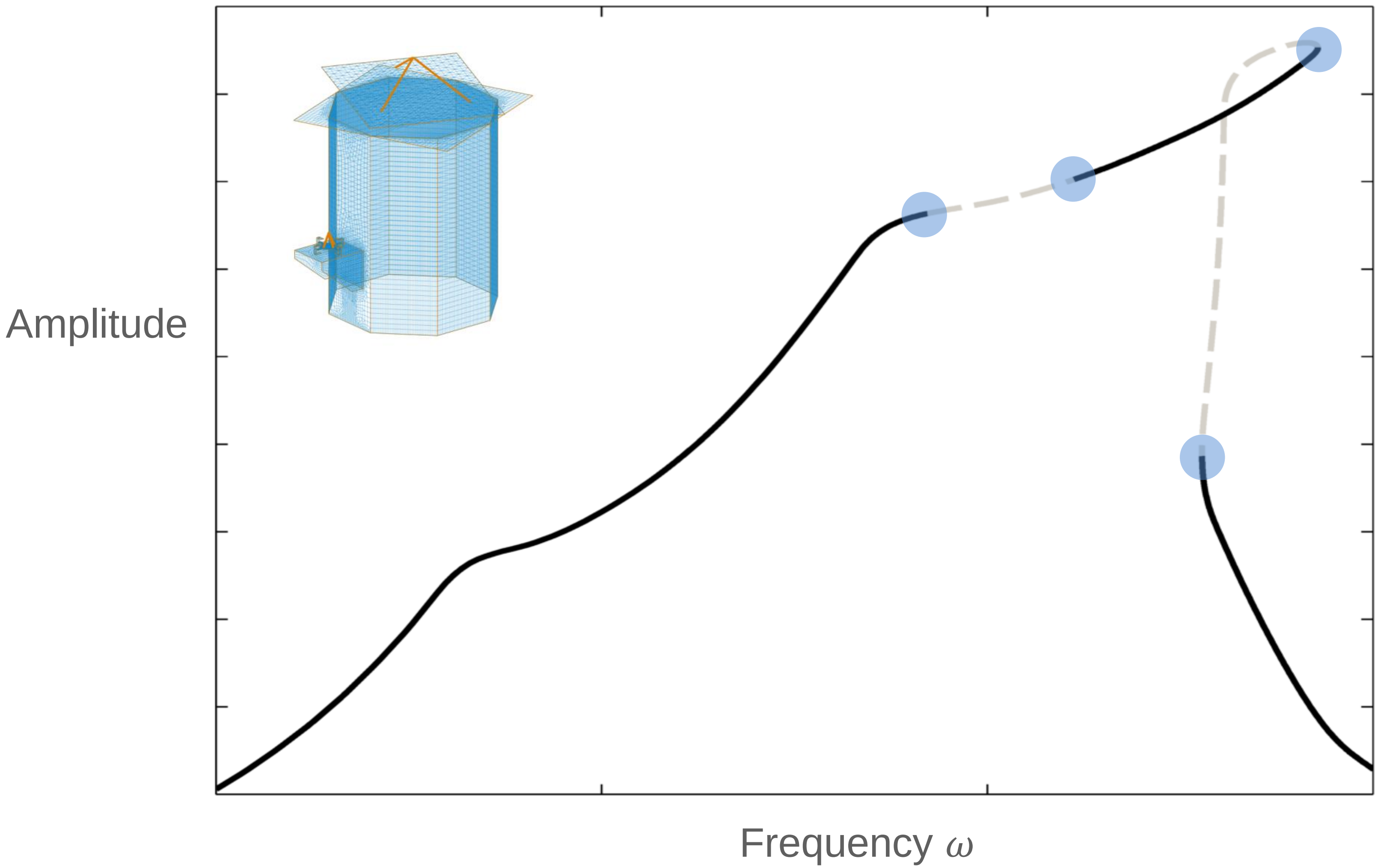
… but also at Unexpected Locations



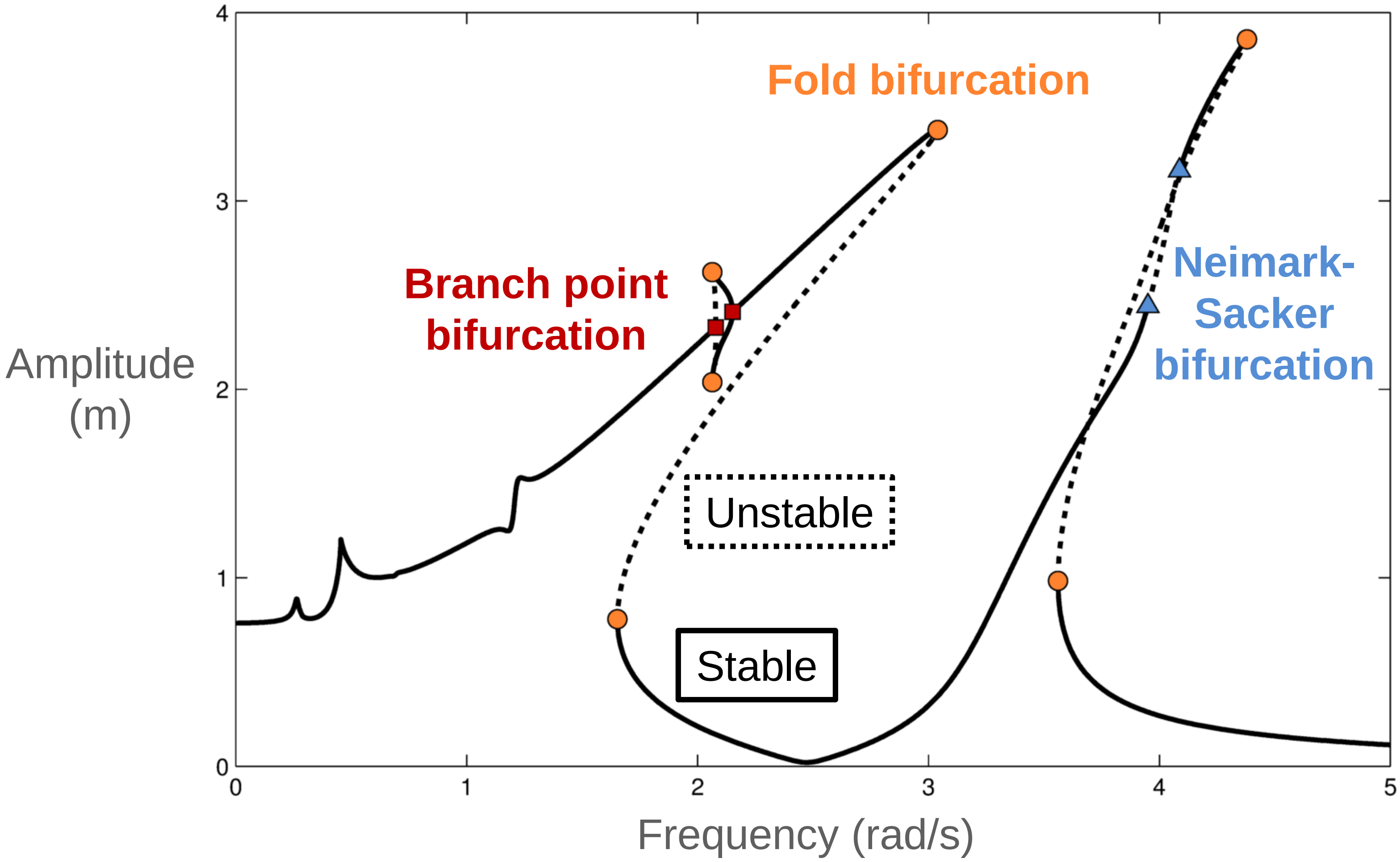
They Strongly Influence Dynamic Behaviours …



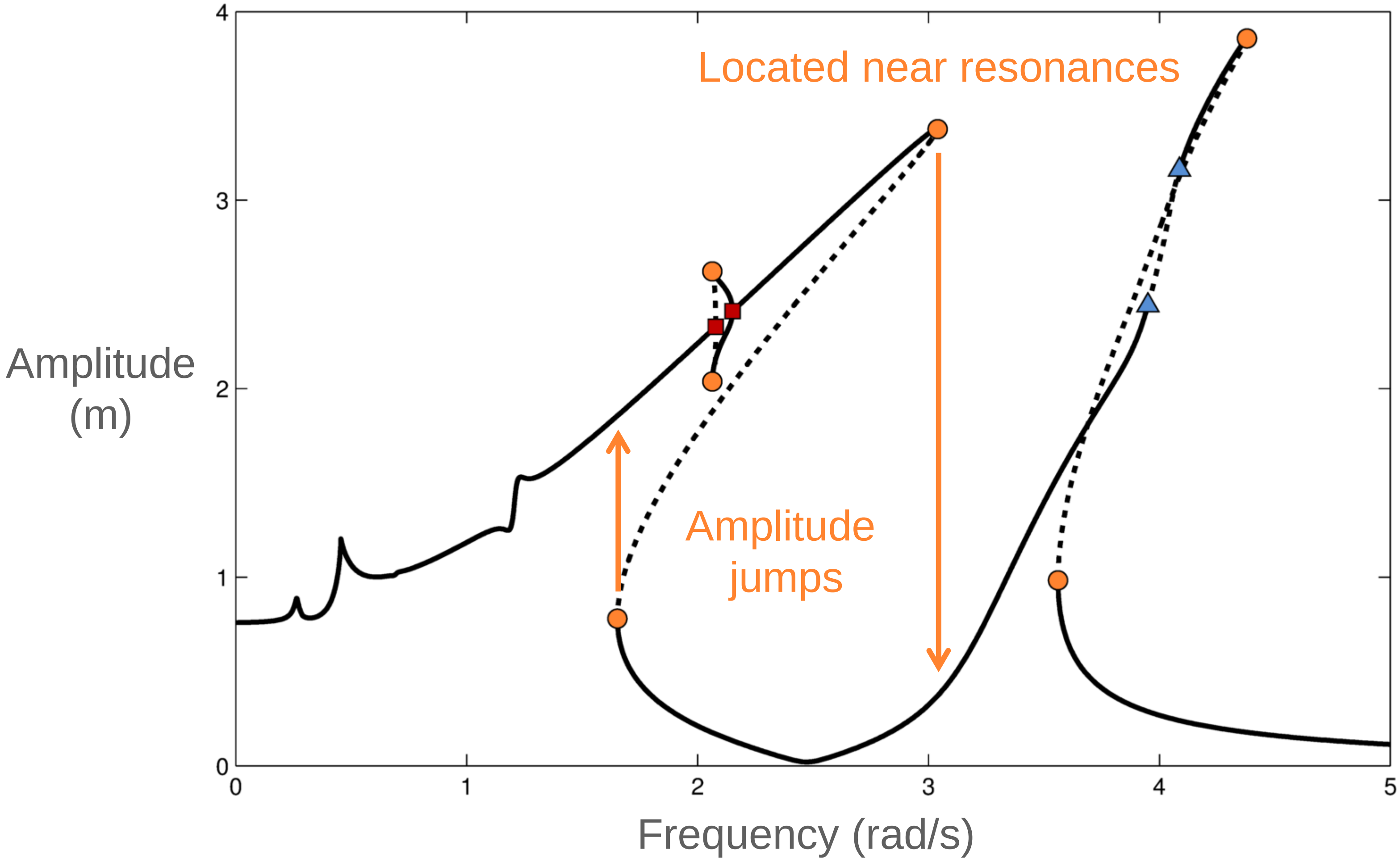
… and Are Related to the Presence of Bifurcations



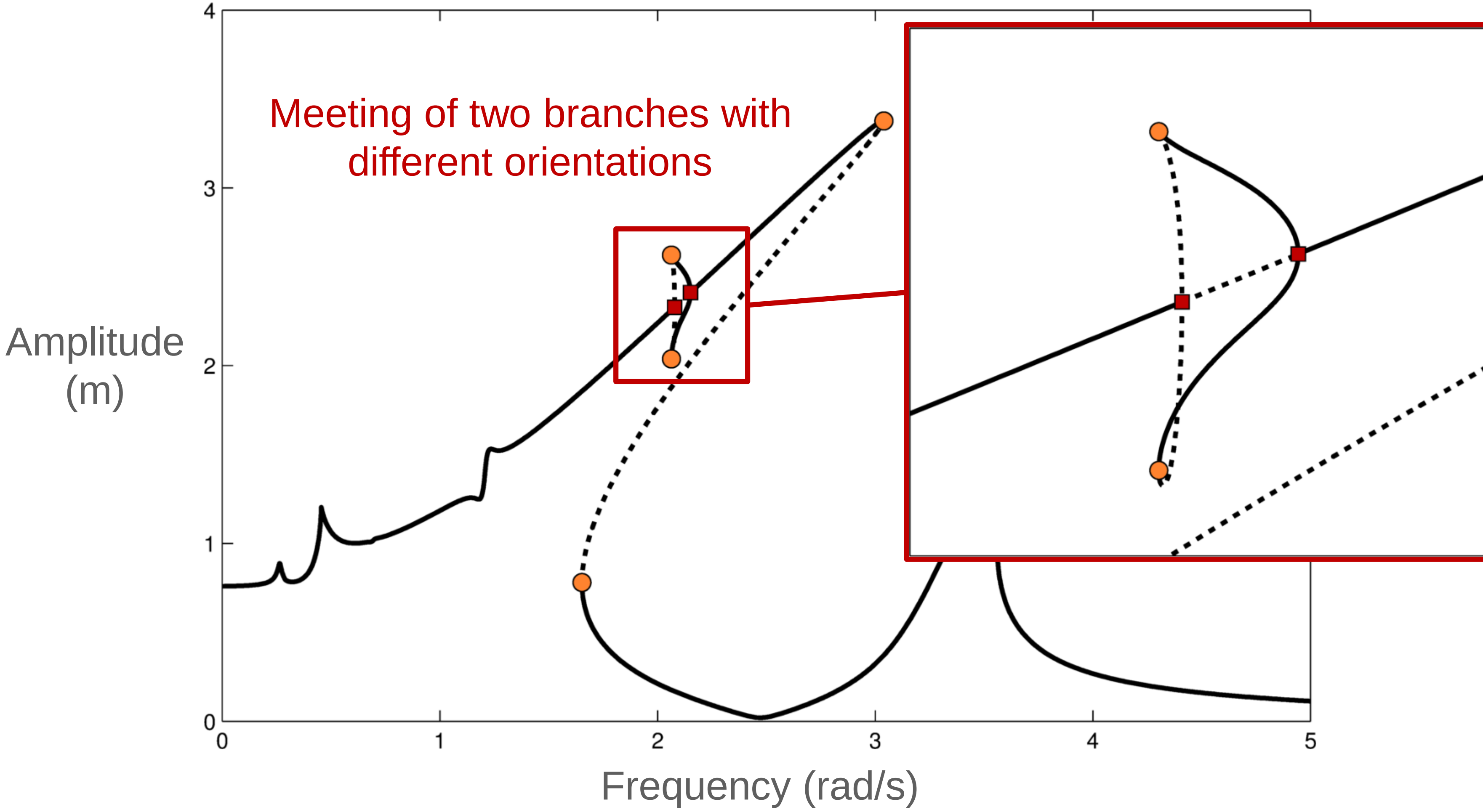
Each Stability Change Scenario Defines a Bifurcation



Fold (F) Bifurcations

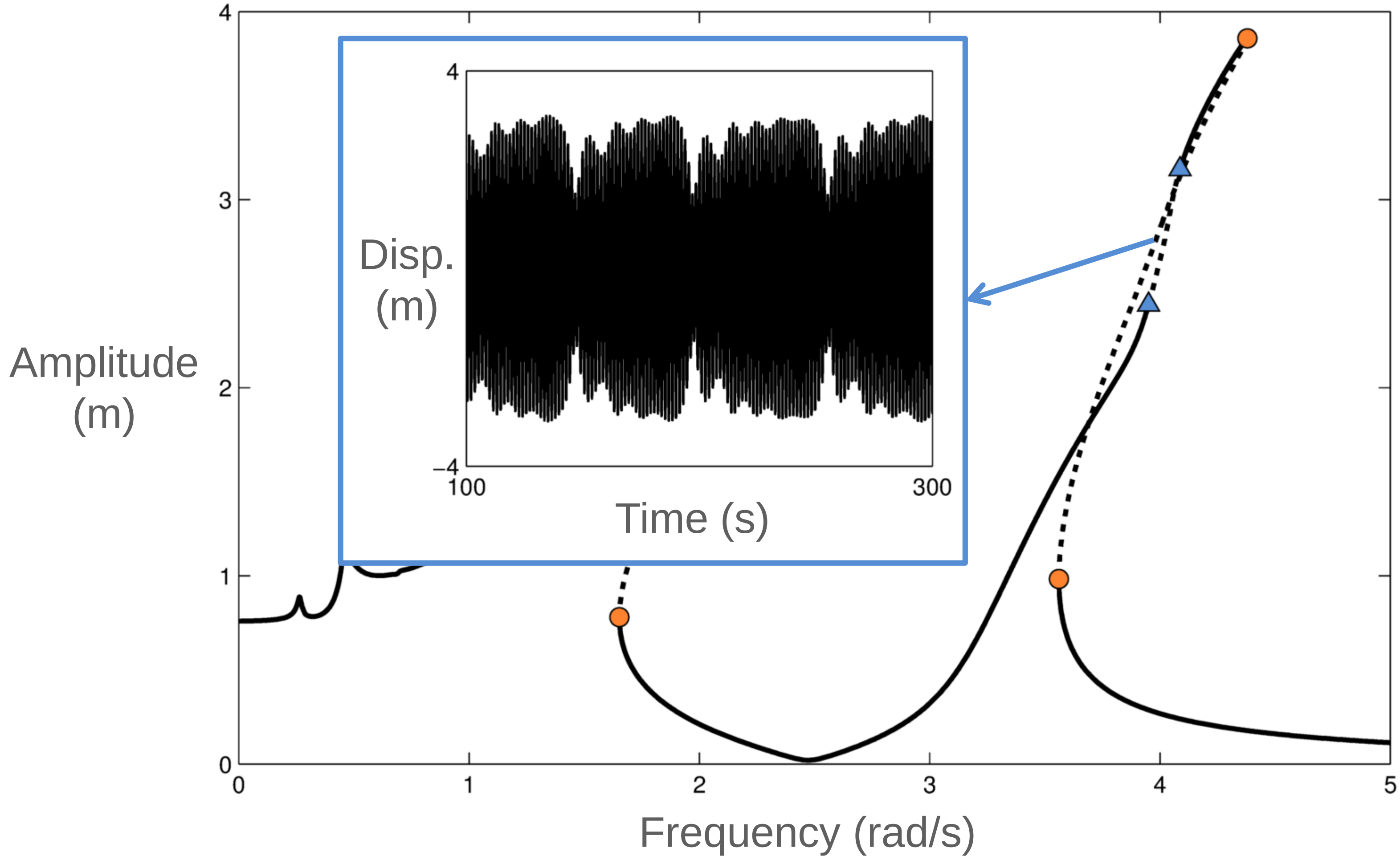


Branch-point (BP) Bifurcations



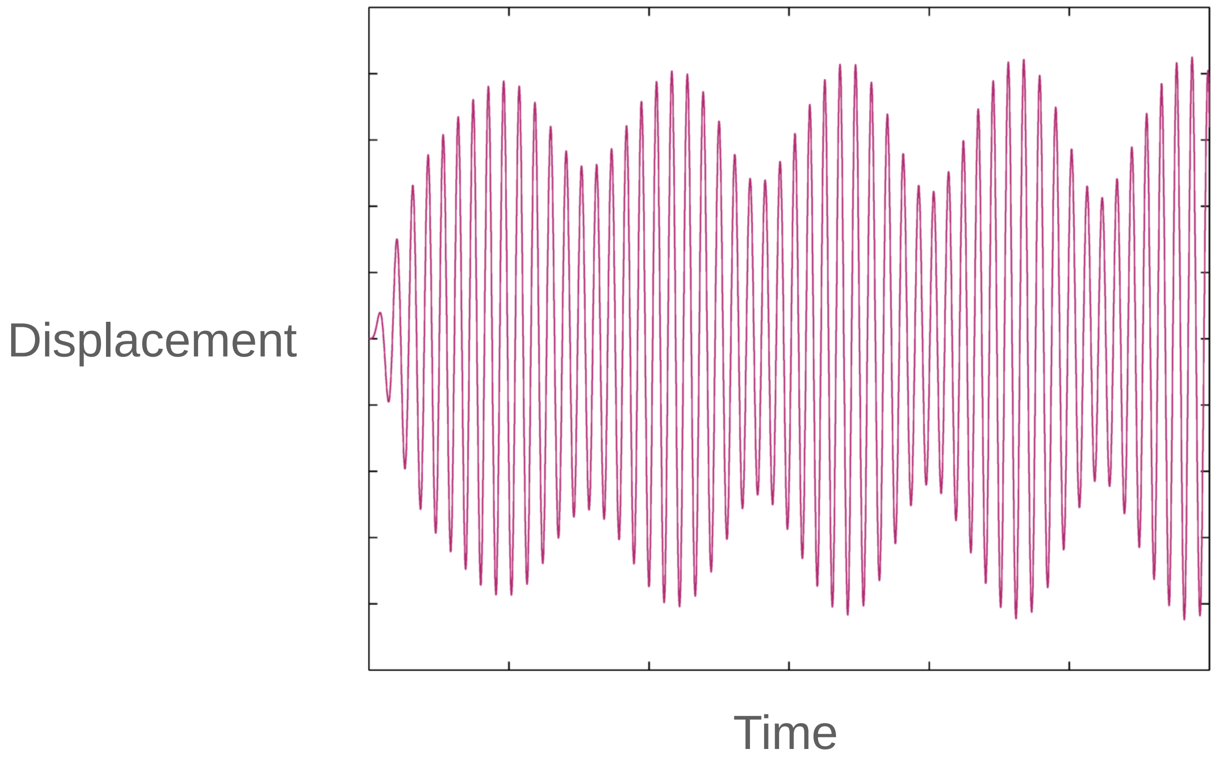
Neimark-Sacker (NS) Bifurcations

Another type of oscillations emanates: quasiperiodic (QP) oscillations



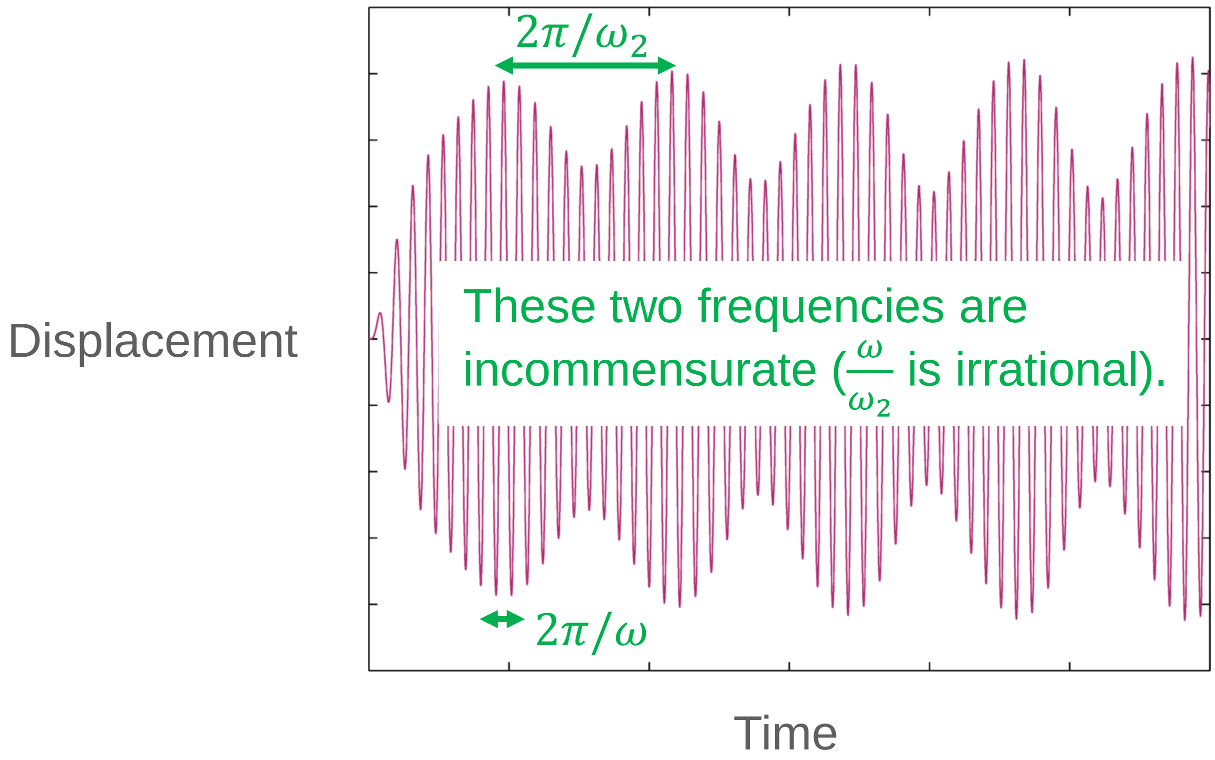
Quasiperiodic Oscillations

Quasiperiodic oscillations are not periodic, and this phenomenon is different from linear beating.

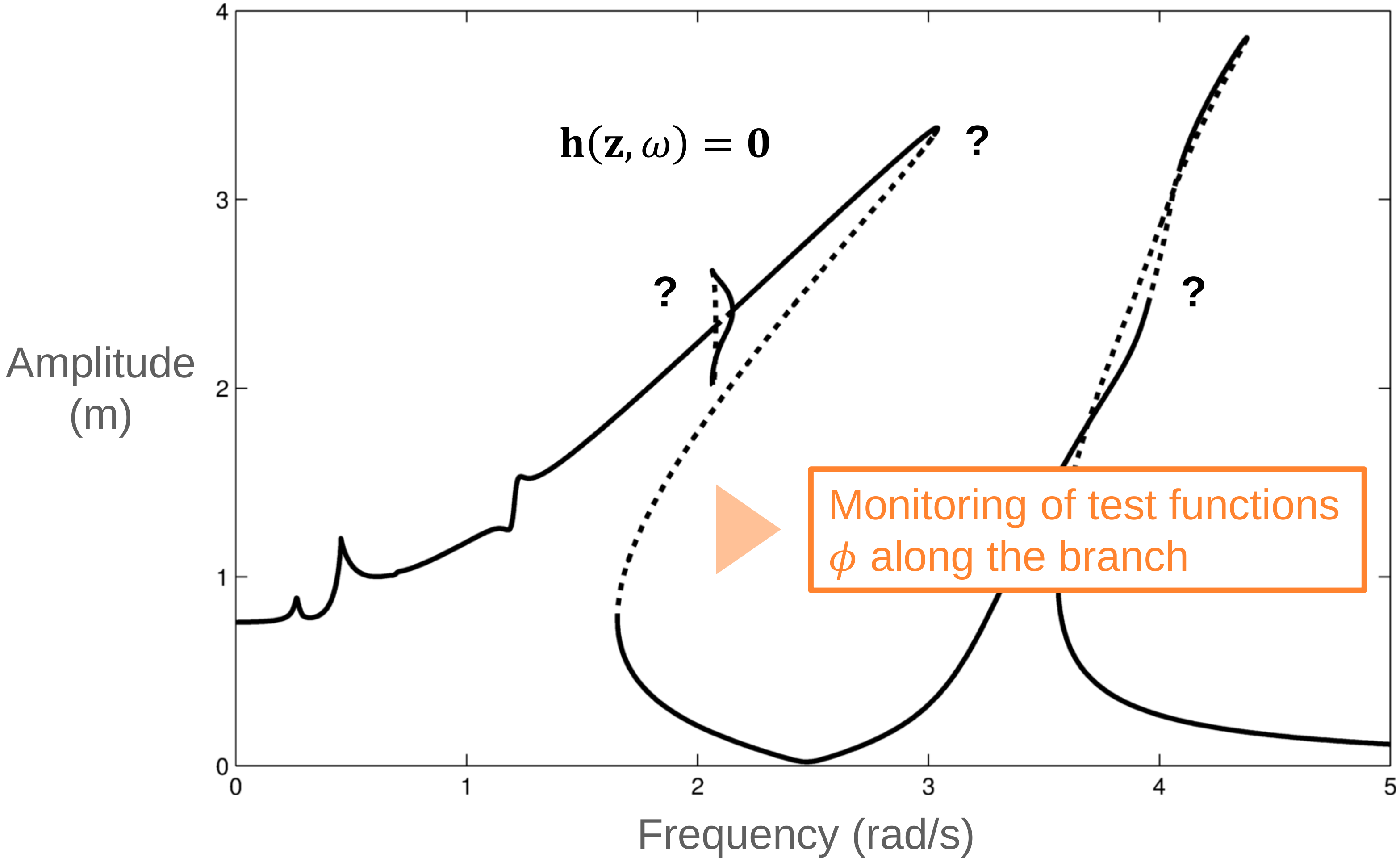


Quasiperiodic Oscillations

Quasiperiodic oscillations contain the forcing frequency ω (forcing), and at least another frequency ω_2 (envelope).



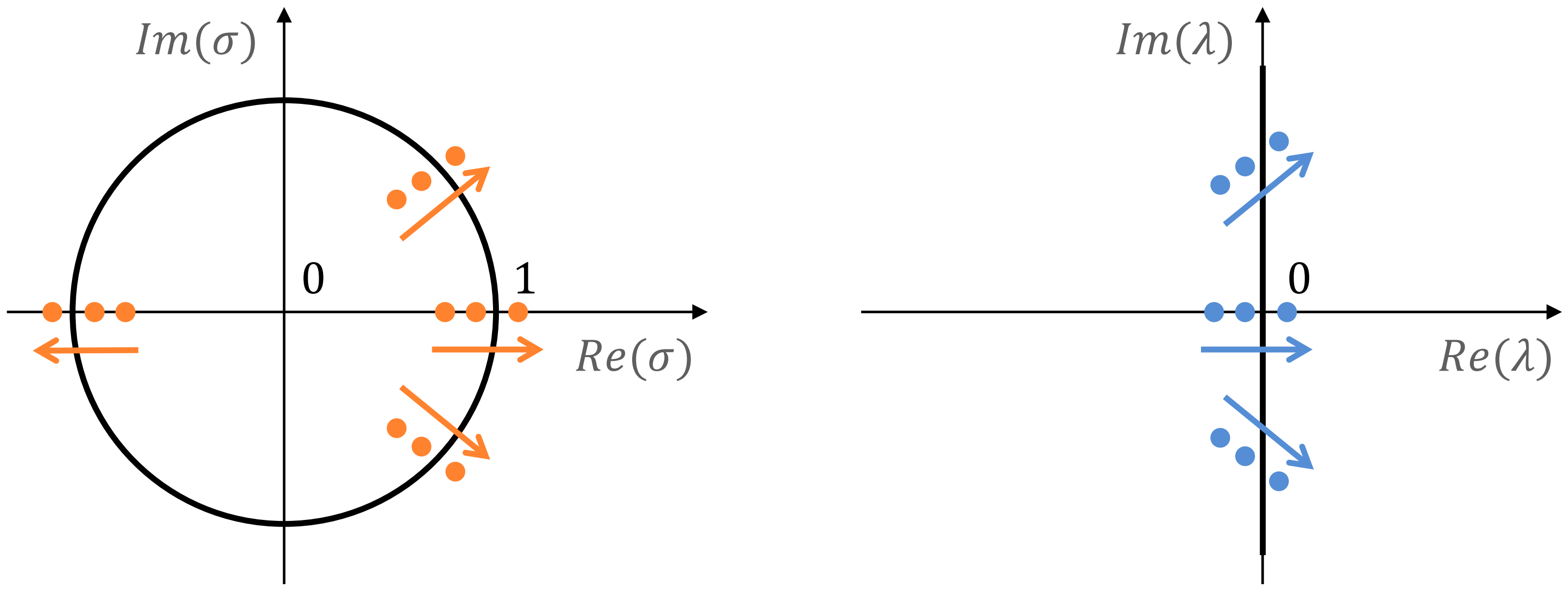
How to Detect Bifurcations using the HB Formalism?



Monitoring of test functions ϕ along the branch

Mechanisms for Losing Stability

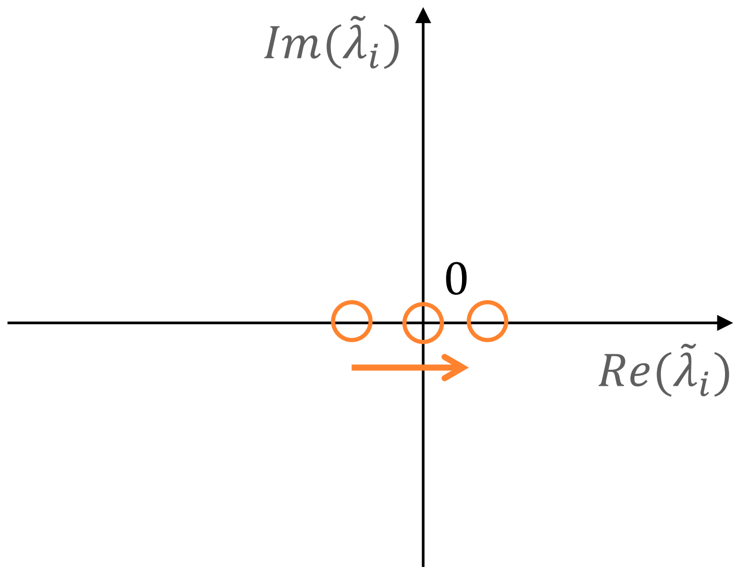
There are only a few different scenarios for periodic solutions to lose stability.



Each mechanism is associated with a type of bifurcation.

Detection of Fold Bifurcations: 2 Conditions

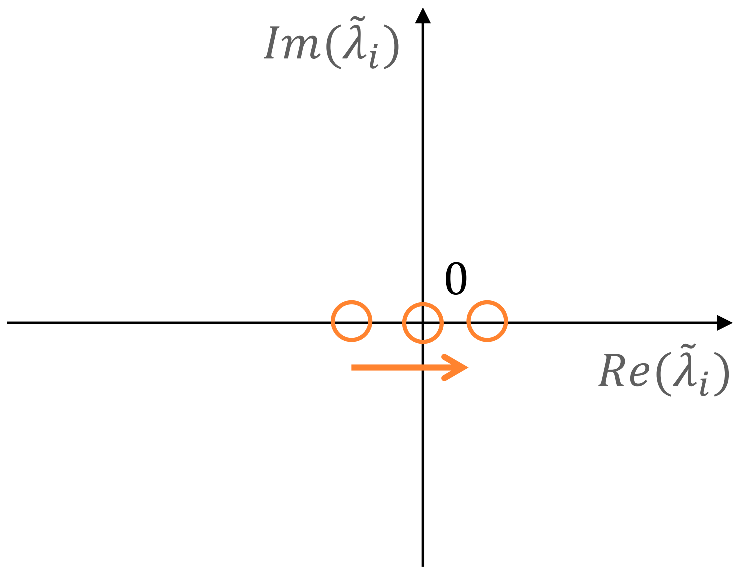
1. A Floquet exponent crosses the imaginary axis through 0.



- $\det(\widetilde{\mathbf{B}}) = 0$
- $\phi_F = \det(\mathbf{h_z}) = 0$

Detection of Fold Bifurcations: 2 Conditions

1. A Floquet exponent crosses the imaginary axis through 0.



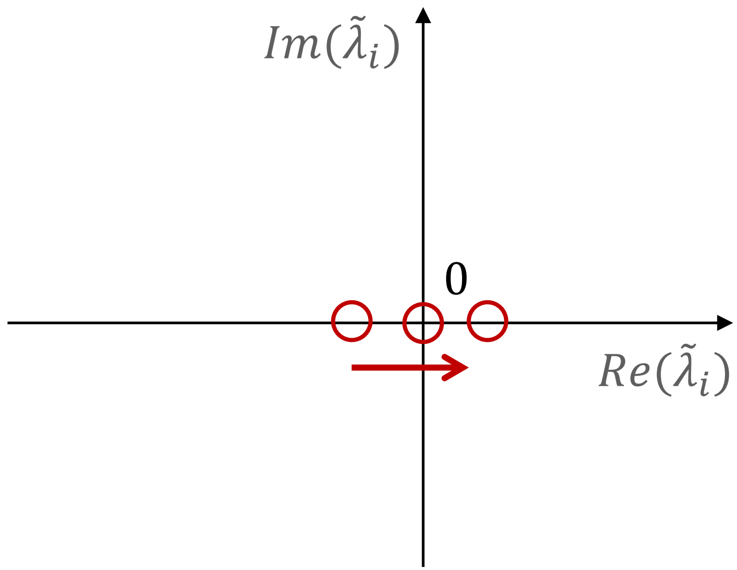
- $\det(\widetilde{\mathbf{B}}) = 0$
- $\phi_F = \det(\mathbf{h}_\mathbf{z}) = 0$

2. $[\mathbf{h}_\mathbf{z} \;\; \mathbf{h}_\omega]$ has full rank.

$$\det\begin{pmatrix} \mathbf{h}_\mathbf{z} & \mathbf{h}_\omega \\ \mathbf{t}^T & \end{pmatrix} \neq 0$$

Detection of Branch-point Bifurcations: 2 Conditions

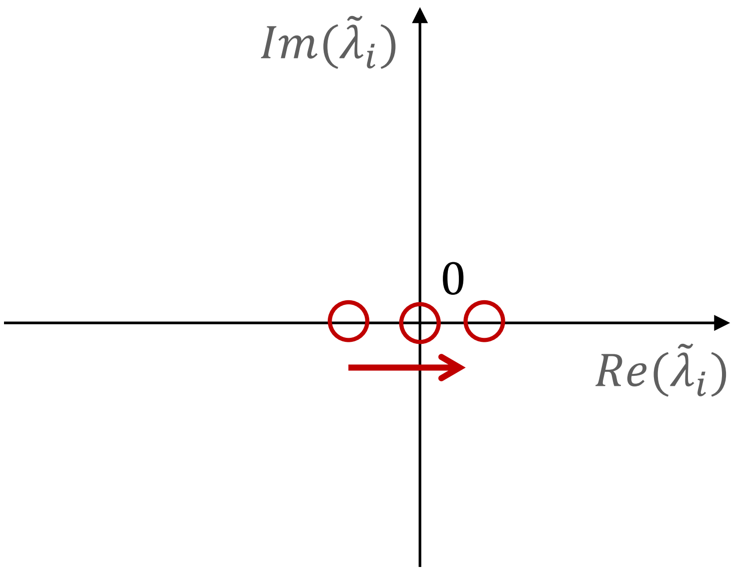
1. A Floquet exponent crosses the imaginary axis through 0.



- $\det\left(\widetilde{\mathbf{B}}\right) = 0$
- $\det(\mathbf{h_z}) = 0$

Detection of Branch-point Bifurcations: 2 Conditions

1. A Floquet exponent crosses the imaginary axis through 0.



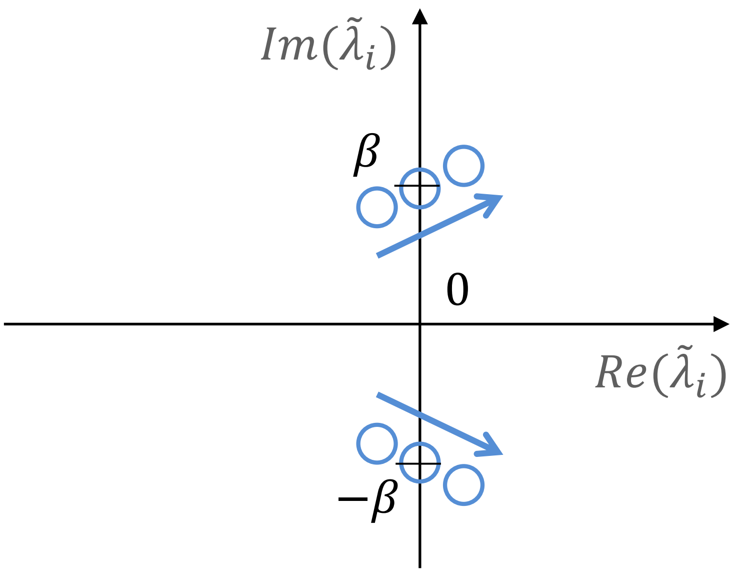
- $\det(\widetilde{\mathbf{B}}) = 0$
- $\det(\mathbf{h_z}) = 0$

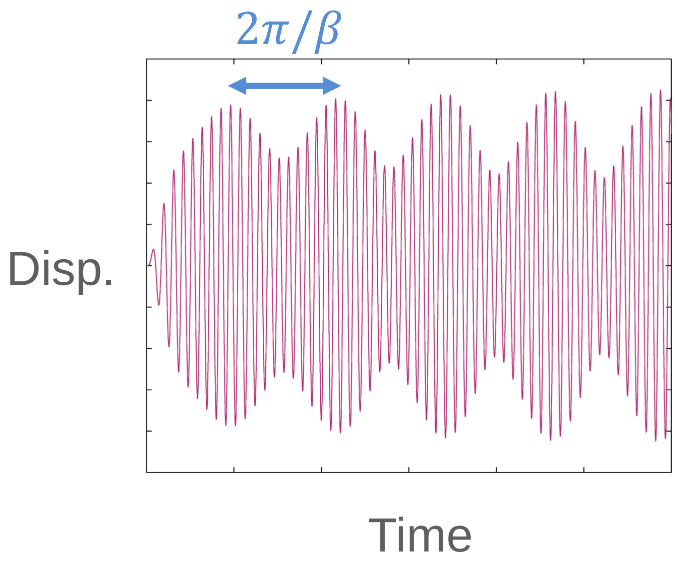
2. $[\mathbf{h_z} \;\; \mathbf{h}_\omega]$ is rank-deficient.

$$\phi_{BP} = \det\begin{pmatrix} \mathbf{h_z} & \mathbf{h}_\omega \\ \mathbf{t}^T & \end{pmatrix} = 0$$

Detection Condition for Neimark-Sacker Bifurcations

A pair of Floquet exponents crosses the imaginary axis as complex conjugates at $\tilde{\lambda}_i = \pm i\beta$.

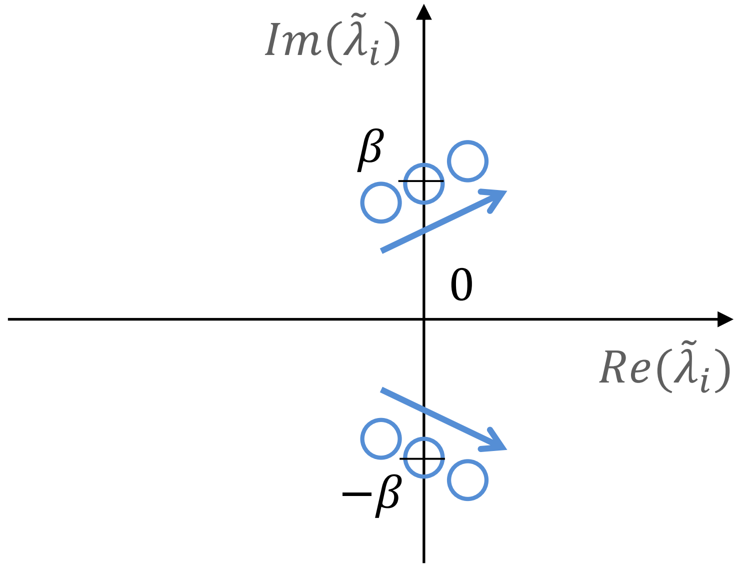




The imaginary part of the Floquet exponents that cross the imaginary axis provides the envelope pulsation (in rad/s) of the quasiperiodic oscillations in the vicinity of the bifurcation.

Detection Condition for Neimark-Sacker Bifurcations

A pair of Floquet exponents crosses the imaginary axis as complex conjugates at $\tilde{\lambda}_i = \pm i\beta$.



$$\phi_{NS} = \det\left(\widetilde{\mathbf{B}}_{\odot}\right) = 0$$

Bialternate
matrix product

Computational challenge: How to calculate and manipulate determinants of large systems?

Bordering Technique Applied to the HB Formalism

Instead of dealing with $\det(\mathbf{G})$, compute g from

$$\begin{bmatrix} \mathbf{G} & \mathbf{p} \\ \mathbf{q}^* & 0 \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ g \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}$$

so that $g = 0 \Leftrightarrow \det(\mathbf{G}) = 0$

$\phi_F = g$

with

$\mathbf{G} = \mathbf{h_z}$

$\phi_{BP} = g$

with

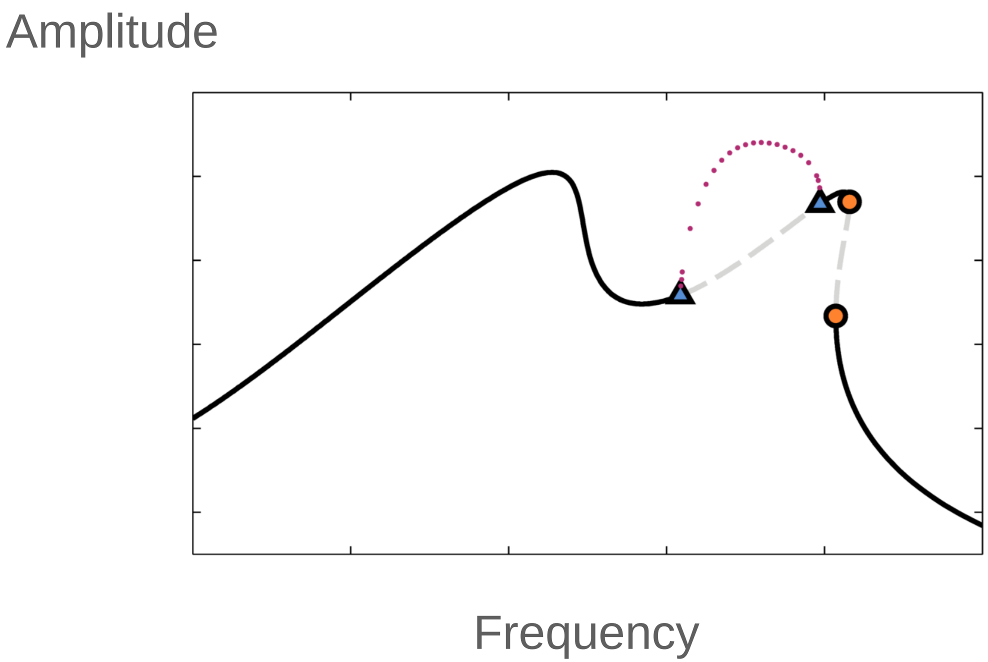
$$\mathbf{G} = \begin{bmatrix} \mathbf{h_z} & \mathbf{h}_\omega \\ \mathbf{t}^T & \end{bmatrix}$$

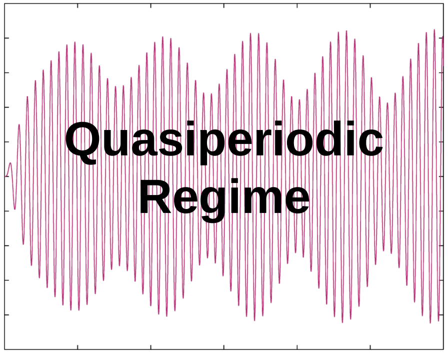
$\phi_{NS} = g$

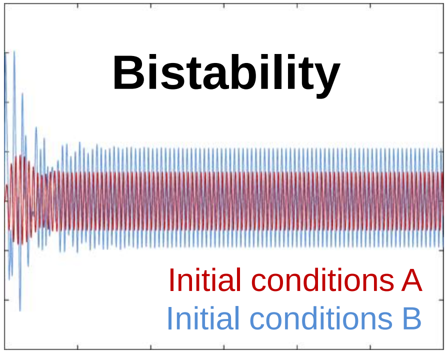
with

$\mathbf{G} = \widetilde{\mathbf{B}}_\odot$

Bifurcations Contain Key Dynamic Information

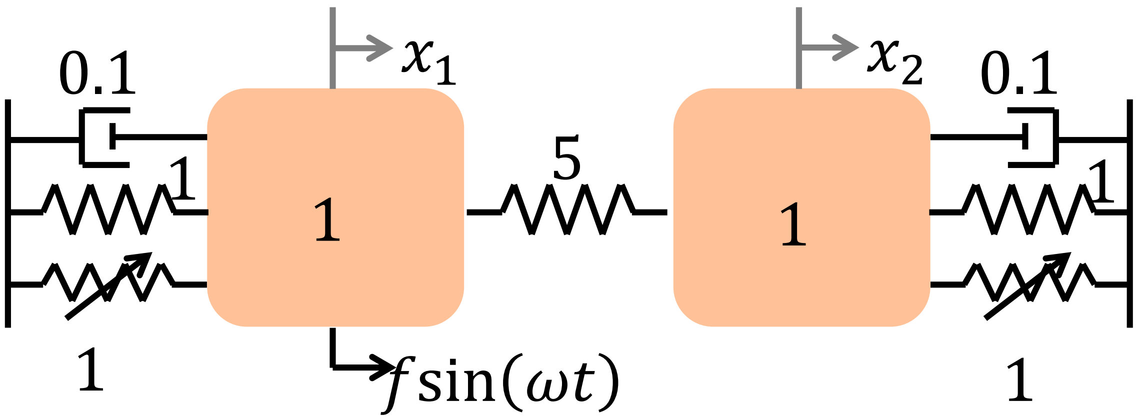






How do bifurcations vary with system parameters?

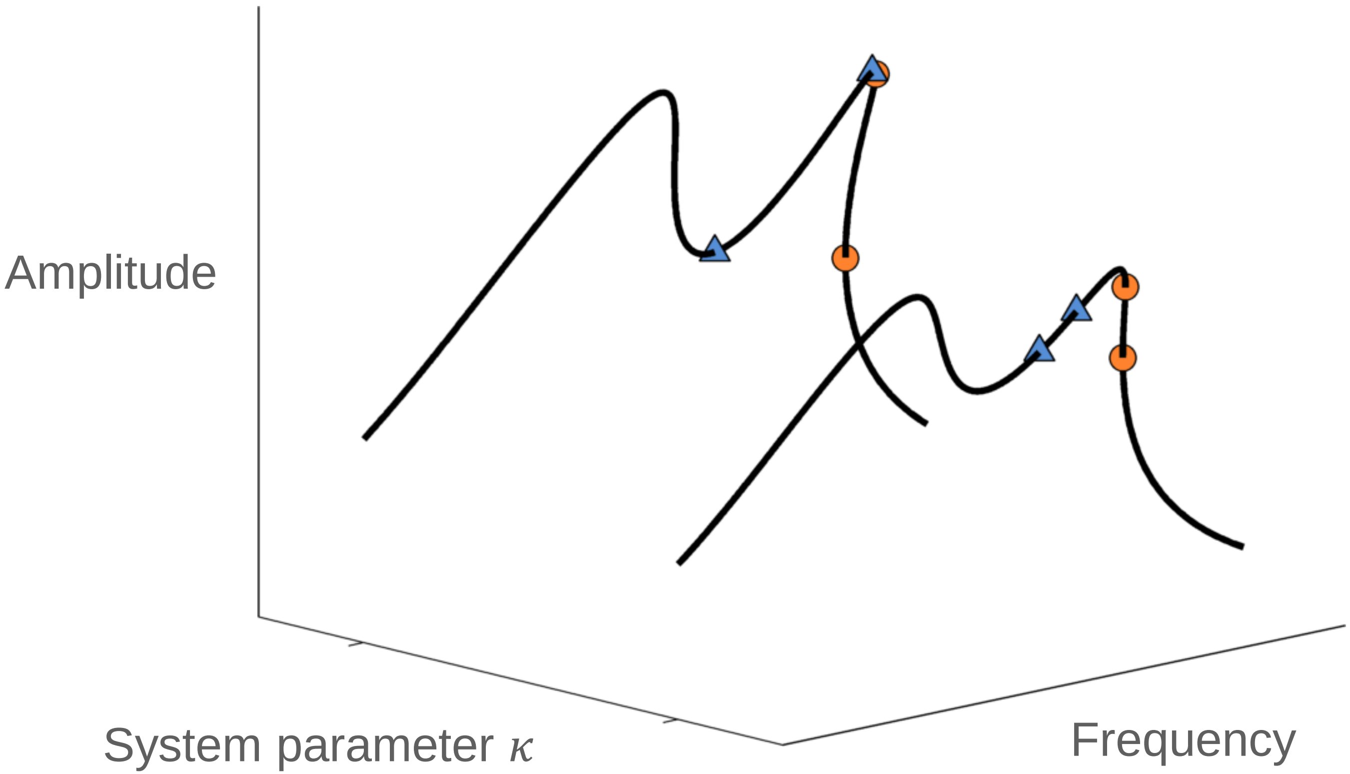
DEMO



Mode	Natural frequency (rad/s)	Damping ratio (%)
1	1.00	5.00
2	3.32	1.51

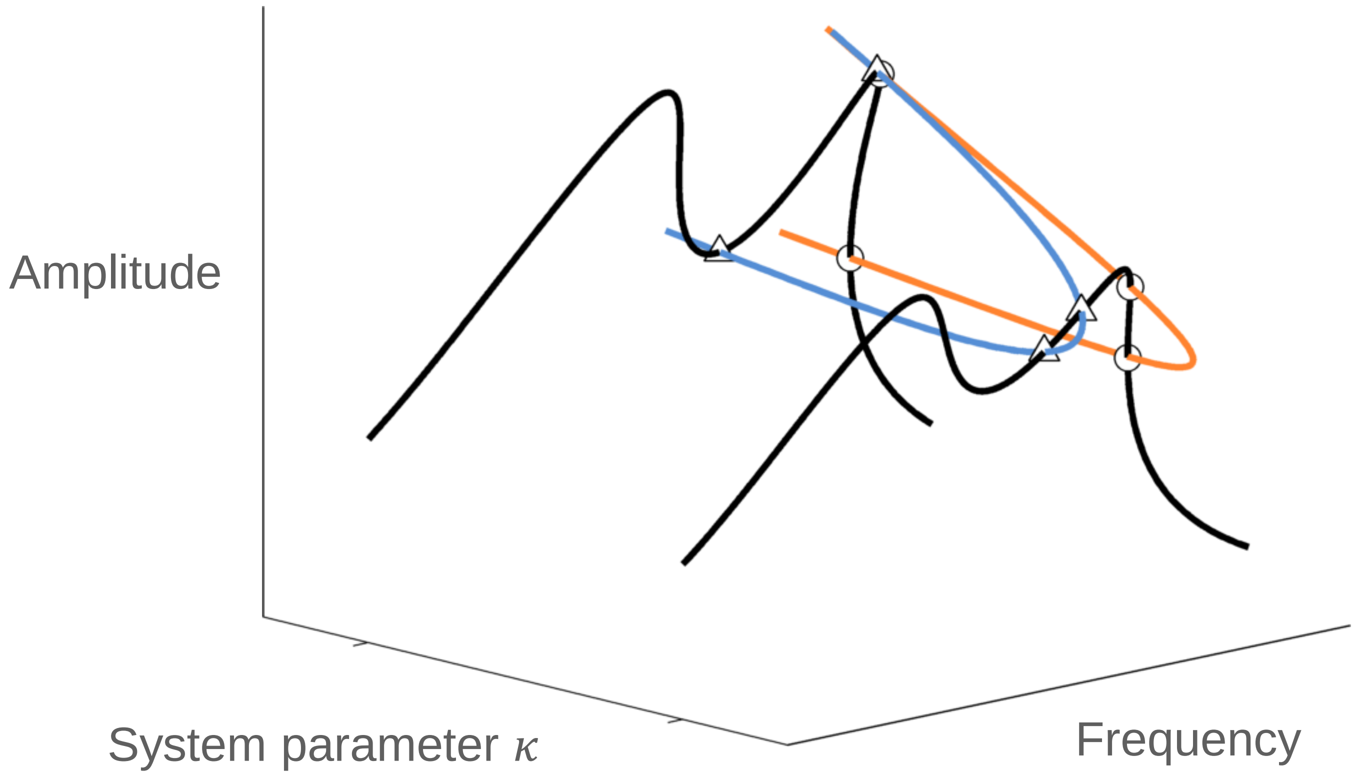
Influence of System Parameters on Bifurcations

Constructing different frequency responses takes time …



Influence of System Parameters on Bifurcations

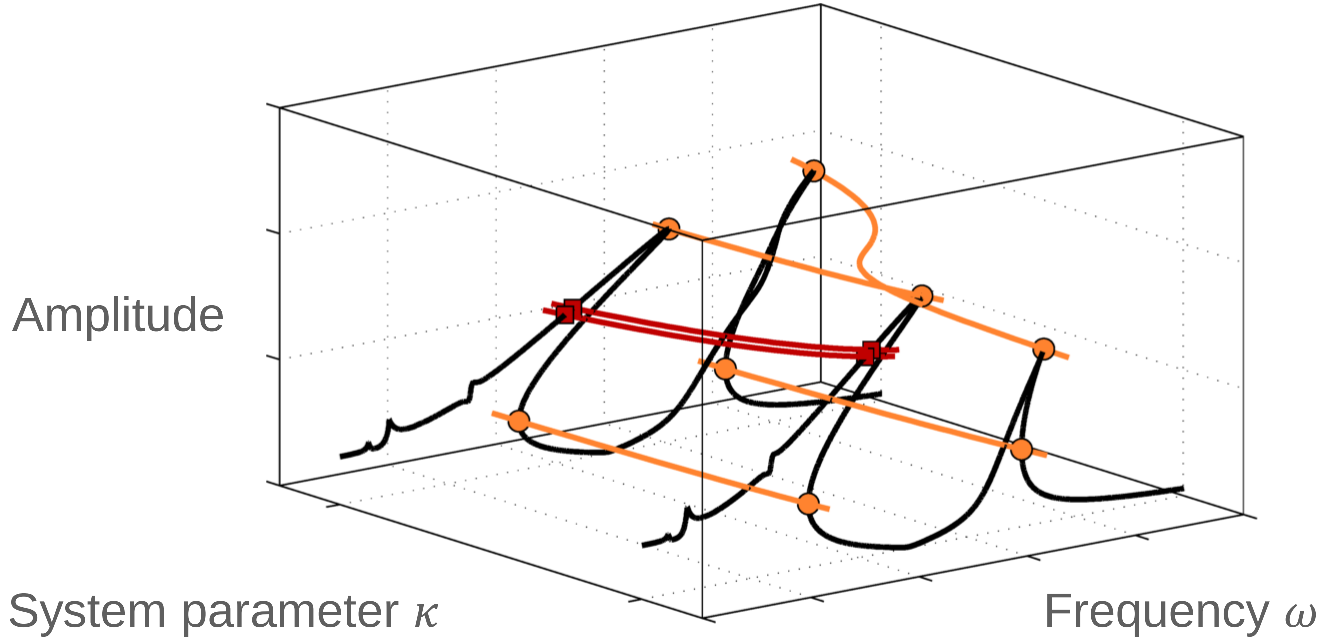
… while bifurcation tracking quickly provides useful information.



How to Track Bifurcations within the HB Formalism?

Through the addition of a bifurcation condition and a parameter κ.

$$\mathbf{h}_{aug}(\mathbf{z}, \omega, \kappa) \equiv \begin{cases} \mathbf{h}(\mathbf{z}, \omega, \kappa) = \mathbf{0} & \text{Amplitude equation (HB method)} \\ g(\mathbf{z}, \omega, \kappa) = 0 & \text{Bifurcation condition} \end{cases}$$



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Resolution of the bordered system

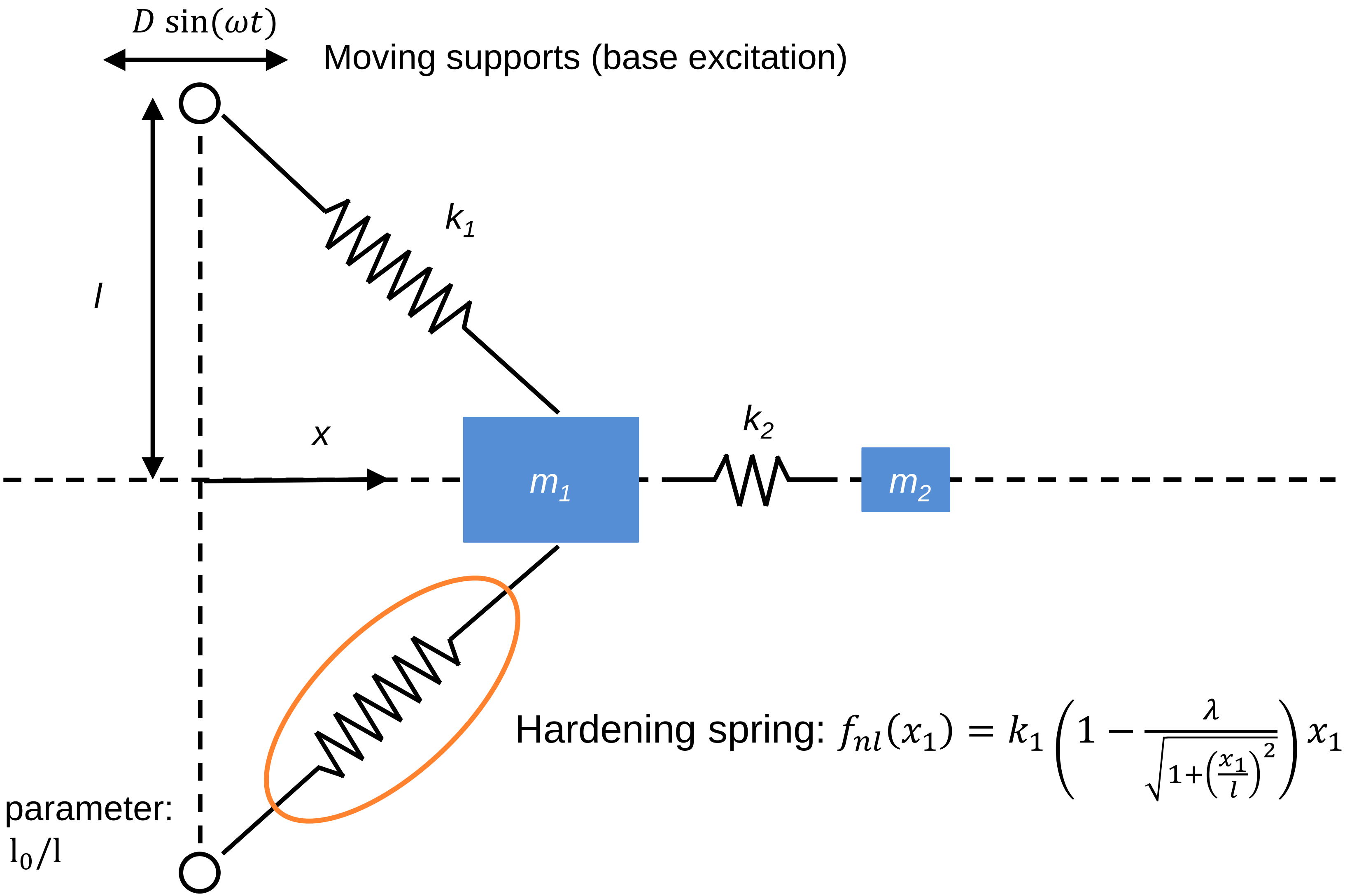
$\mathbf{G} = \mathbf{h_z}$

Folds & branch points

$\mathbf{G} = \widetilde{\mathbf{B}}_{\odot}$

Neimark-Sacker's

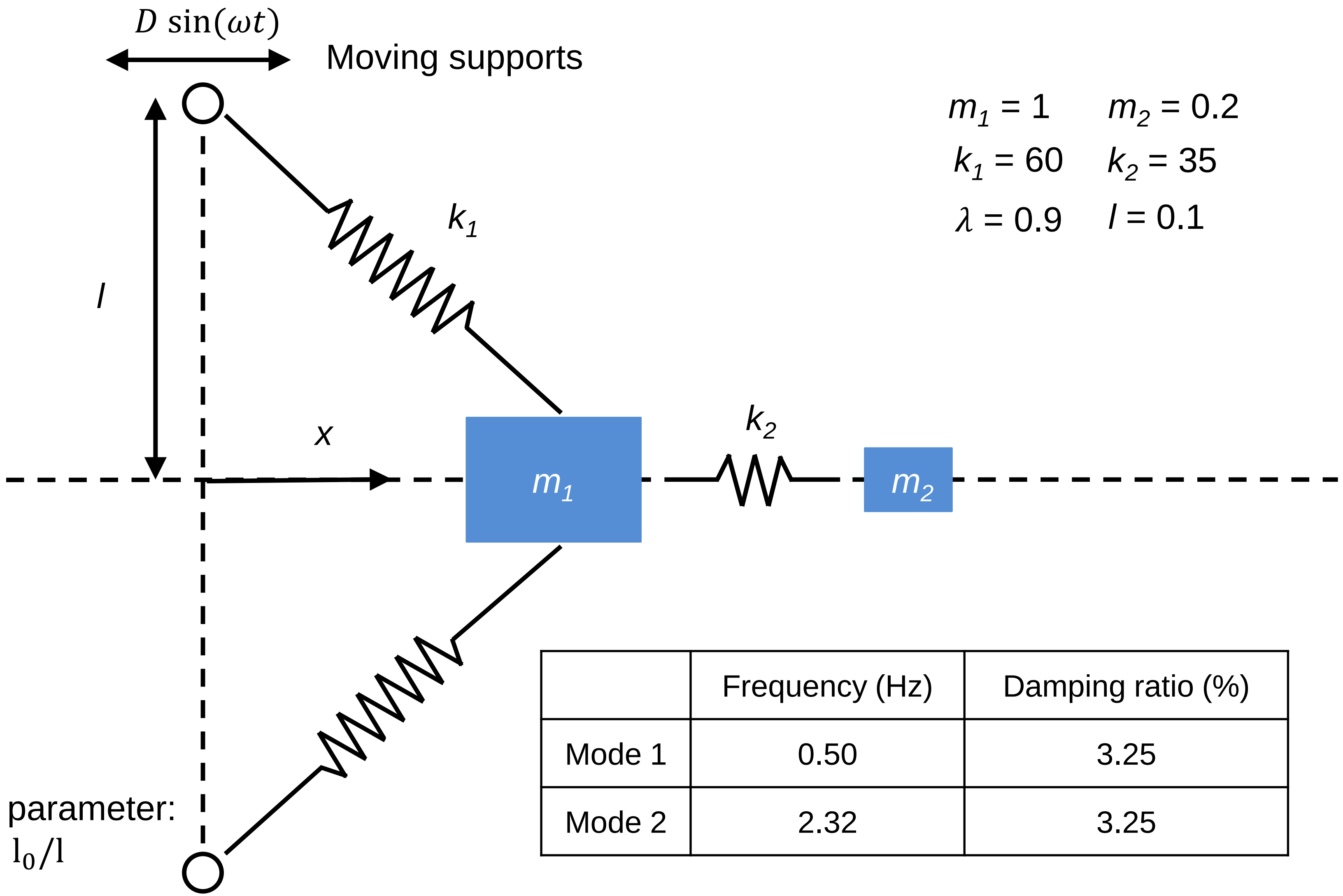
Application to a 2-DOF System



Hardening spring: $f_{nl}(x_1) = k_1\left(1 - \frac{\lambda}{\sqrt{1+\left(\frac{x_1}{l}\right)^2}}\right)x_1$

Pre-stress parameter:
$\lambda = l_0/l$

Application to a 2-DOF System



$m_1 = 1$ $m_2 = 0.2$

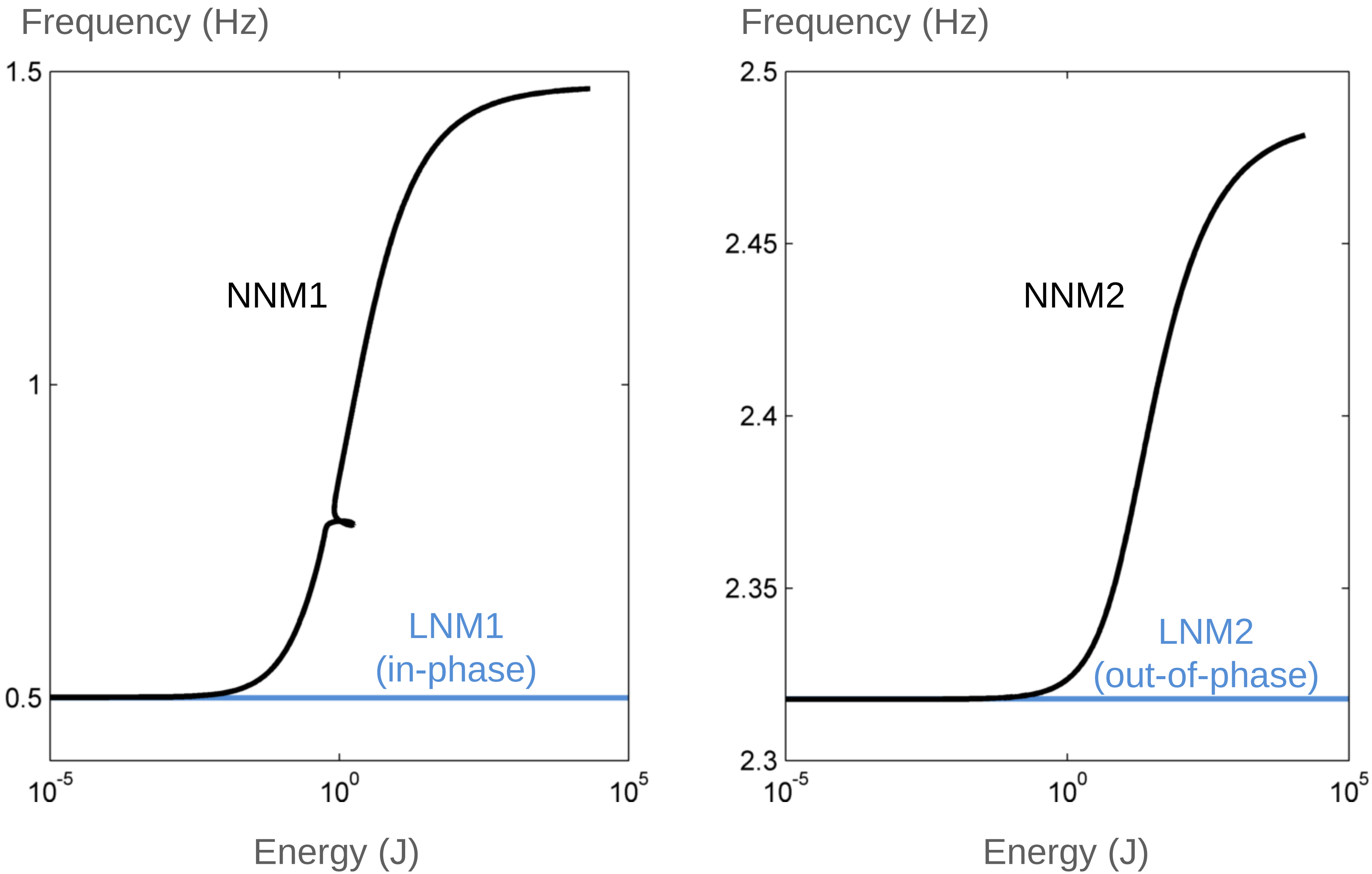
$k_1 = 60$ $k_2 = 35$

$\lambda = 0.9$ $l = 0.1$

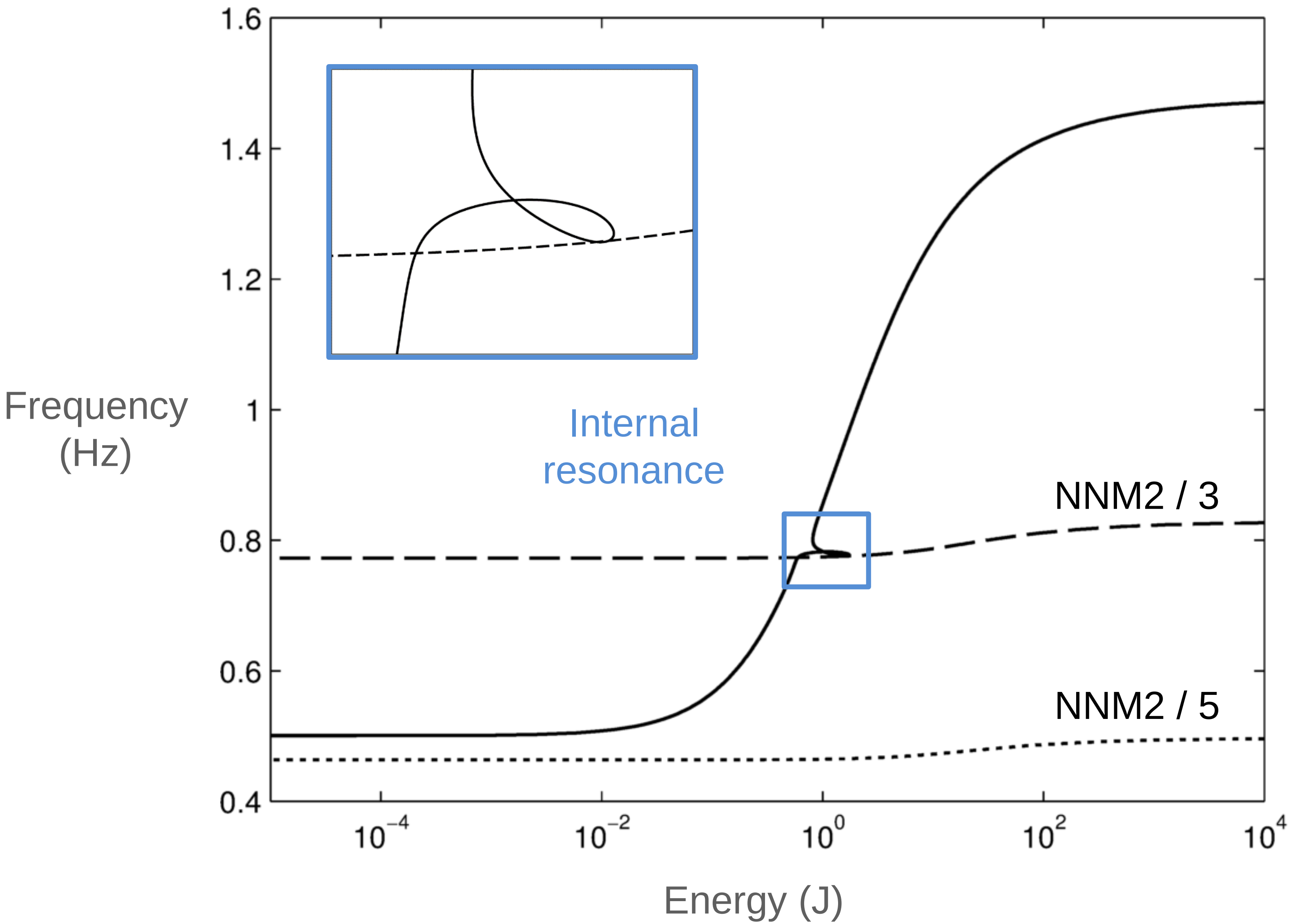
Pre-stress parameter:
$\lambda = l_0/l$

	Frequency (Hz)	Damping ratio (%)
Mode 1	0.50	3.25
Mode 2	2.32	3.25

FEPs of the Two Fundamental NNMs

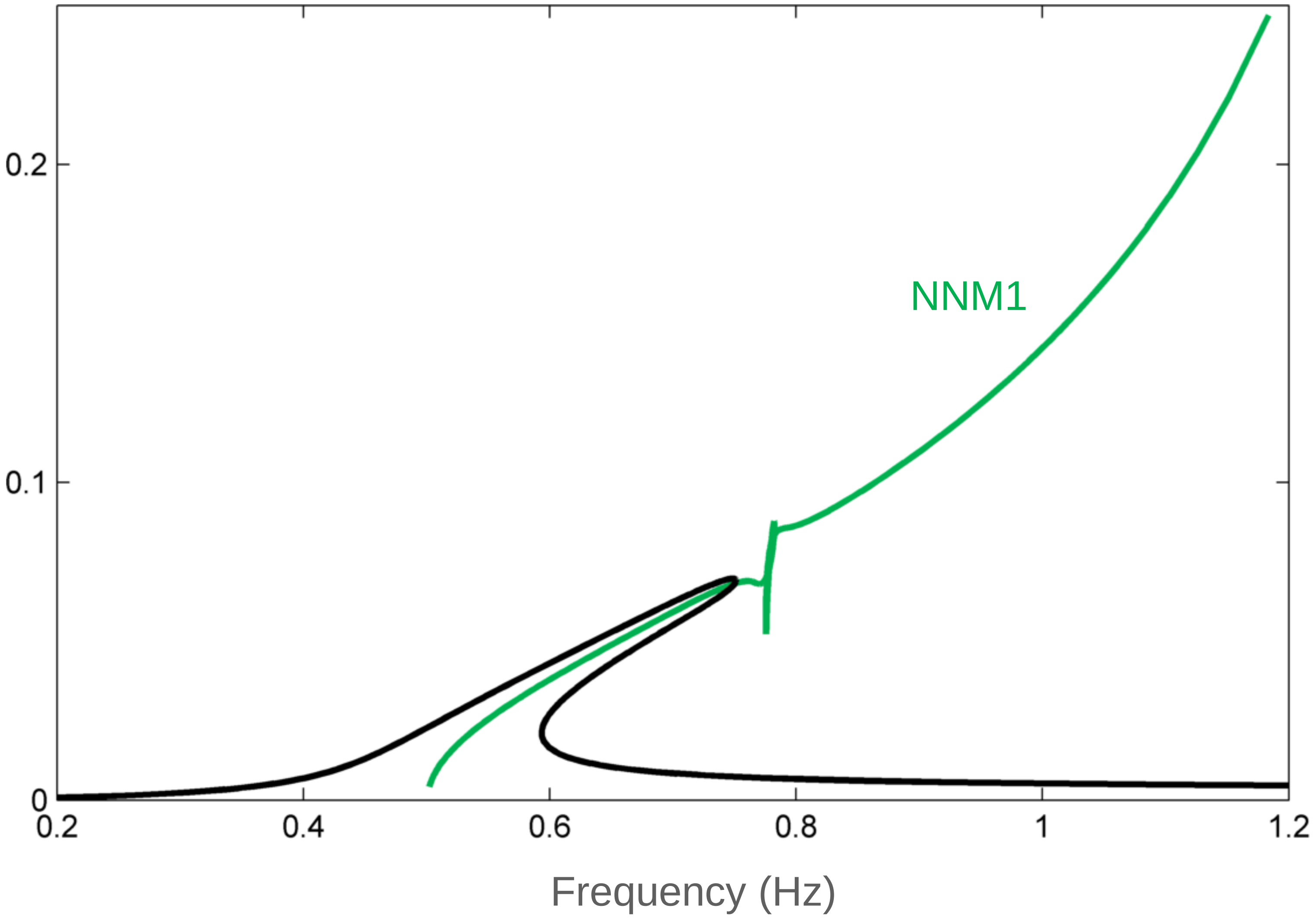


NNM1 Features an Alpha-loop due to 3:1 Resonance



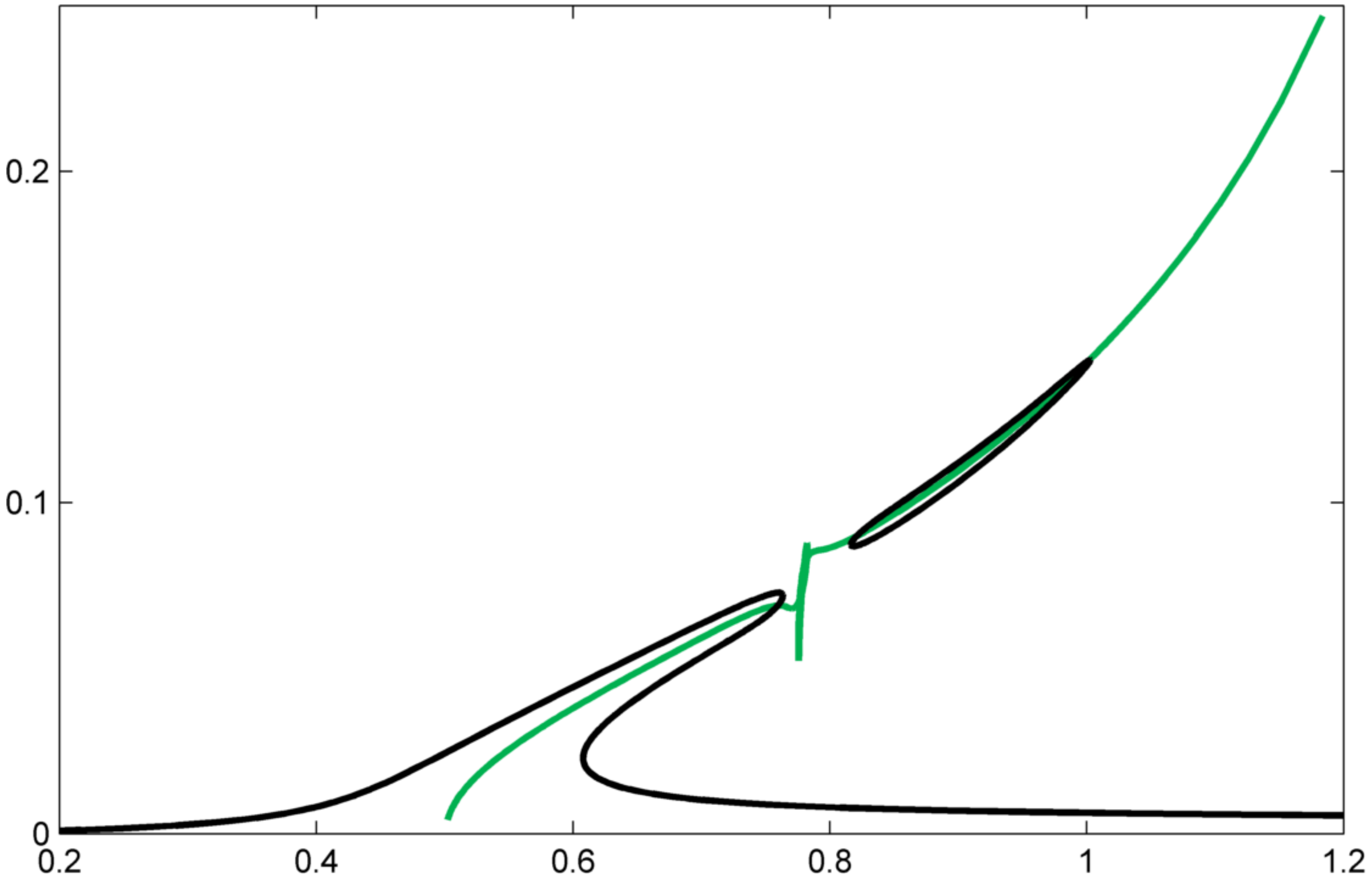
Computation of the NFRC

Disp. of mass 1 (m) – Base disp. D = 5 mm



An Isolated Response Branch Exists at D = 6 mm

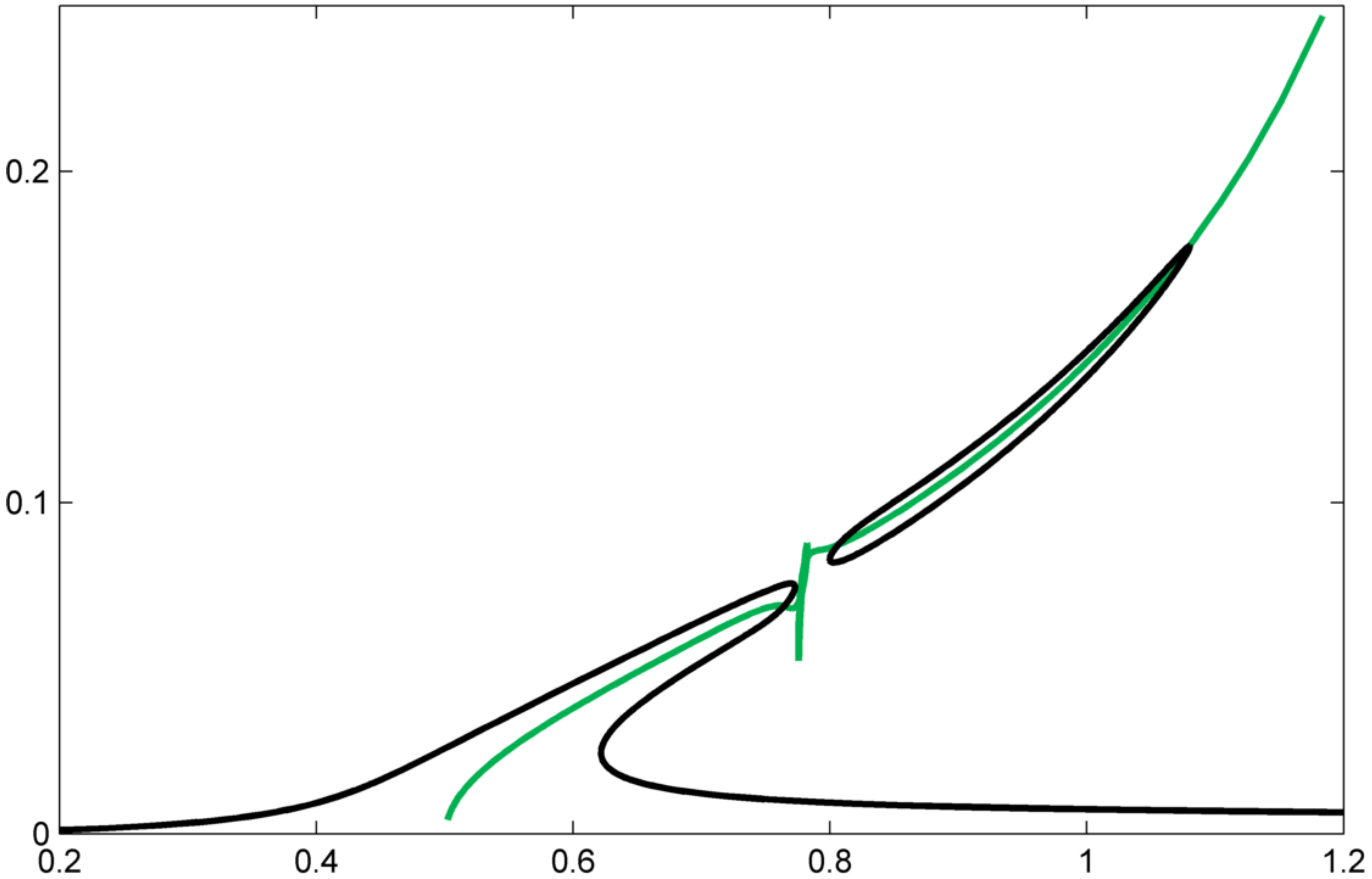
Disp. of mass 1 (m) – Base disp. D = 6 mm



Frequency (Hz)

An Increase in Forcing Enlarges the Isola Domain

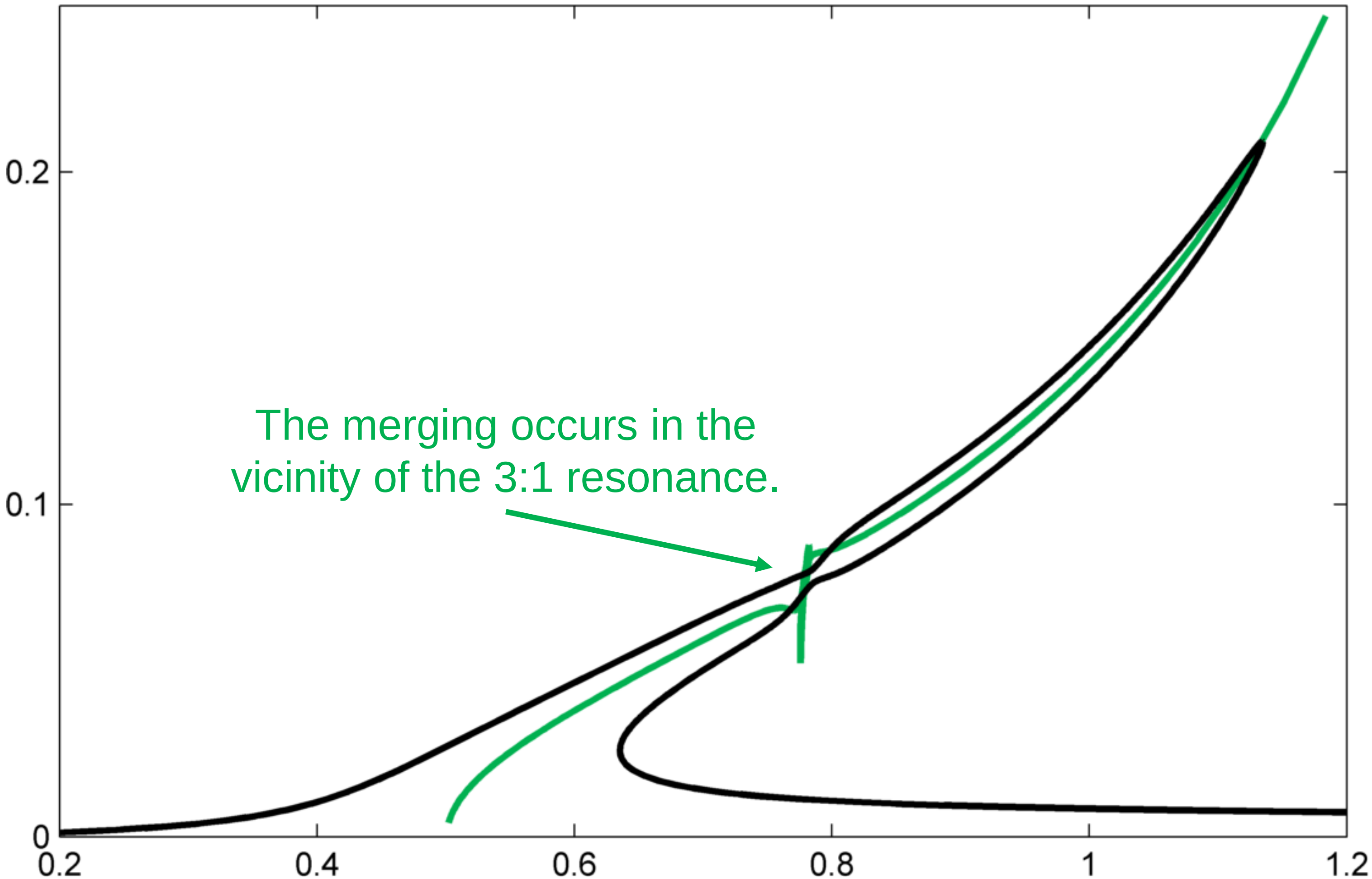
Disp. of mass 1 (m) – Base disp. D = 7 mm



Frequency (Hz)

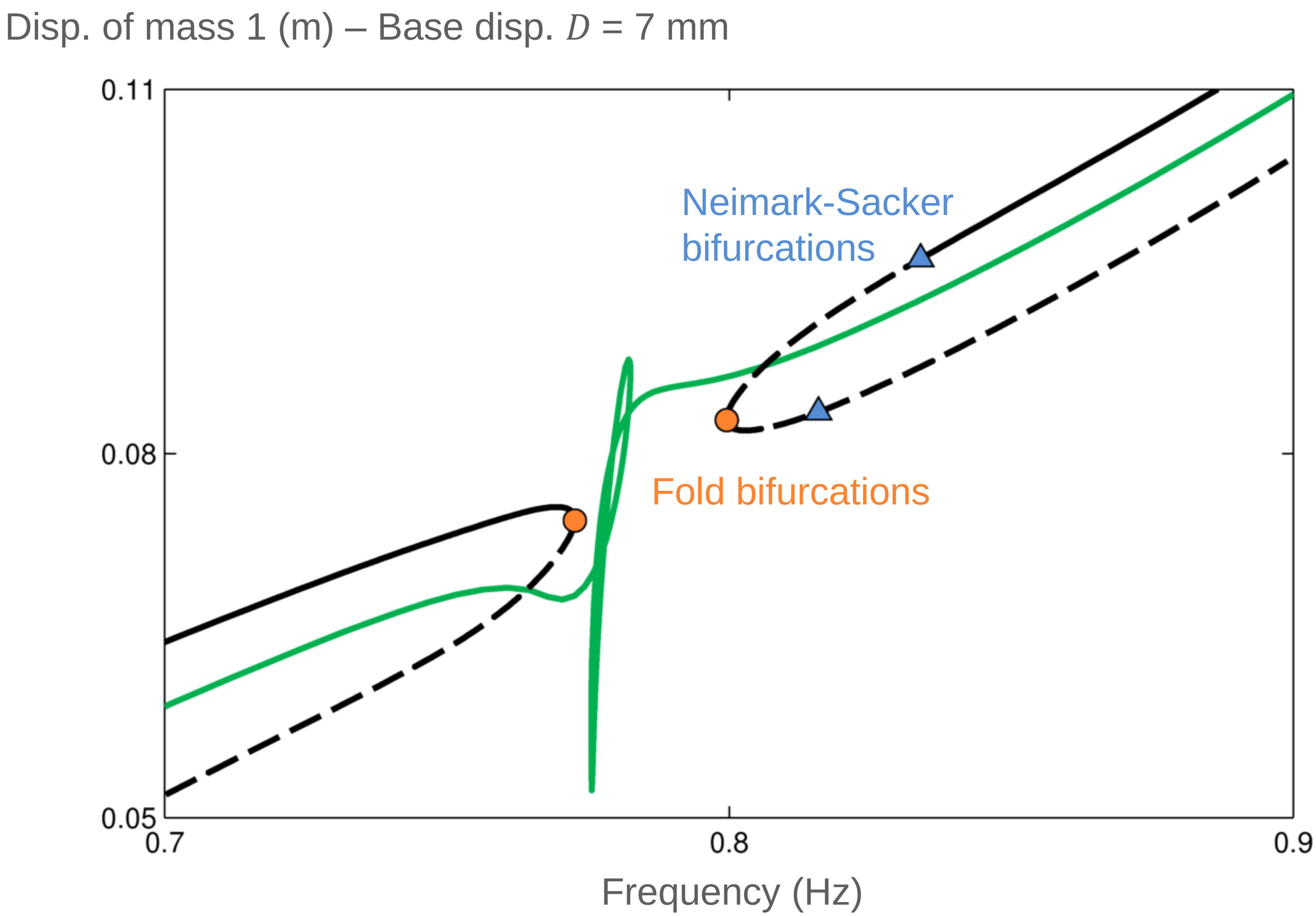
Increasing Forcing Further Leads to the Isola Merging

Disp. of mass 1 (m) – Base disp. D = 8 mm



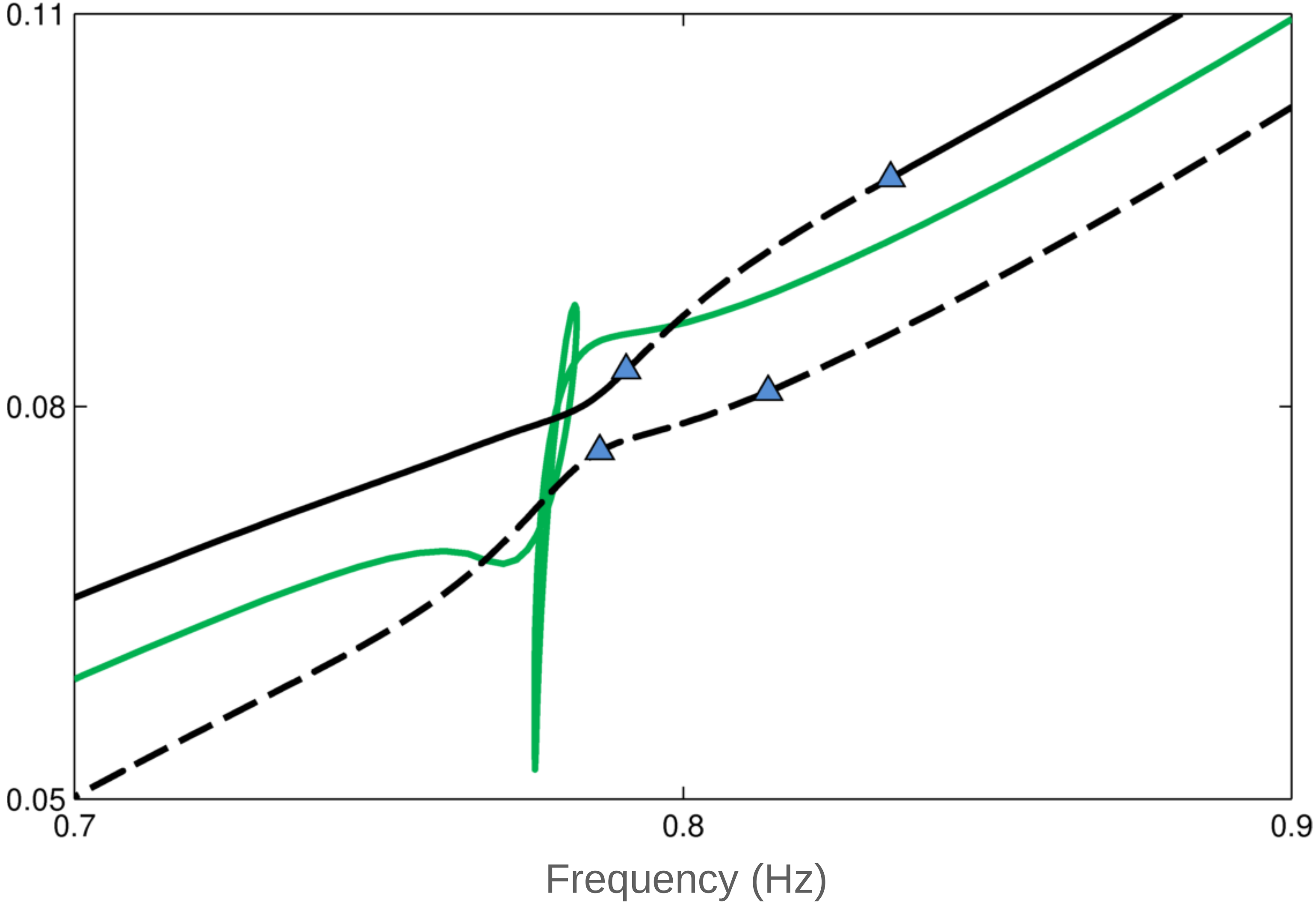
Frequency (Hz)

Merging Mechanism and Stability Information

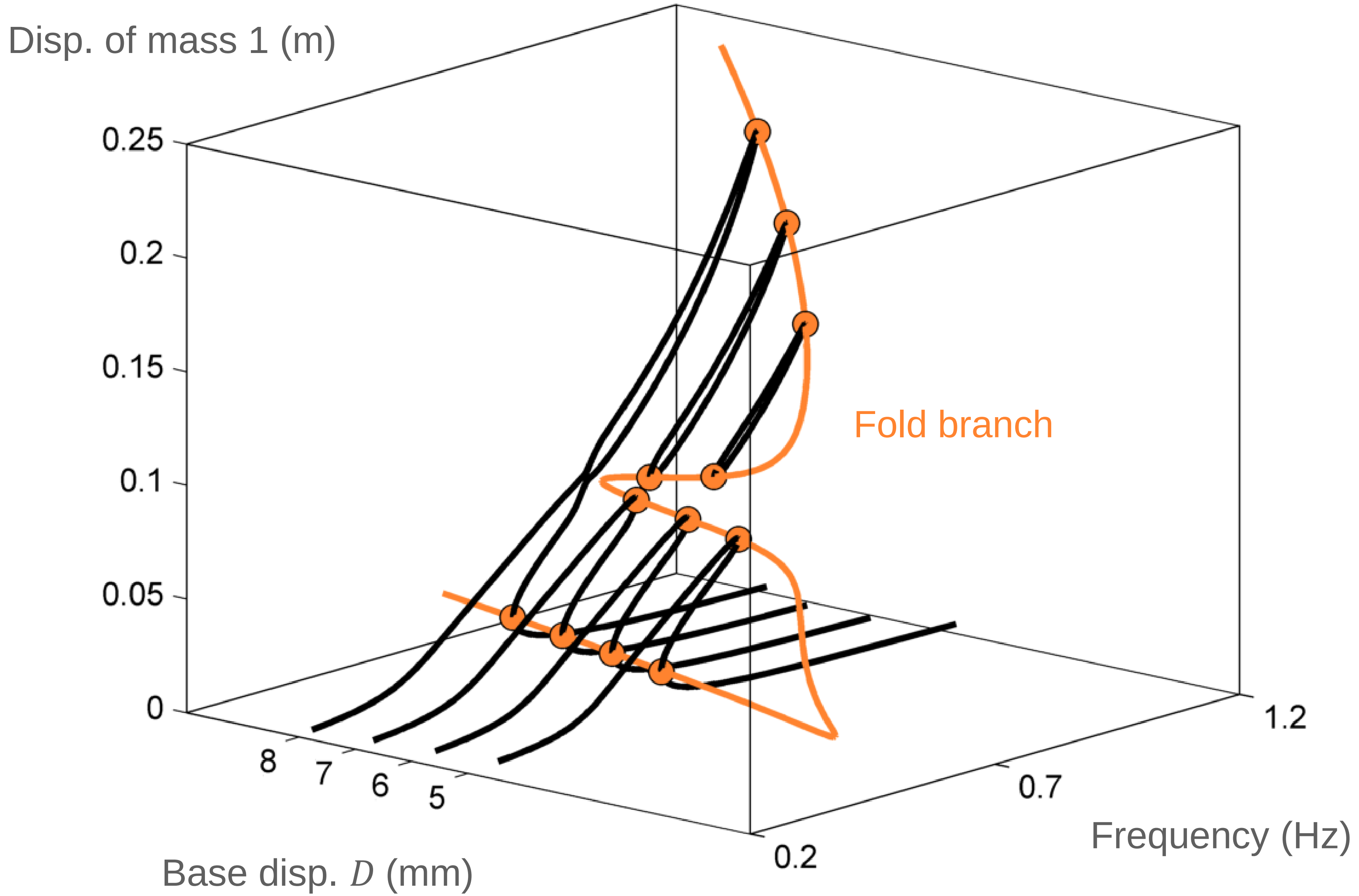


The Merging Occurs through the Elimination of 2 Folds

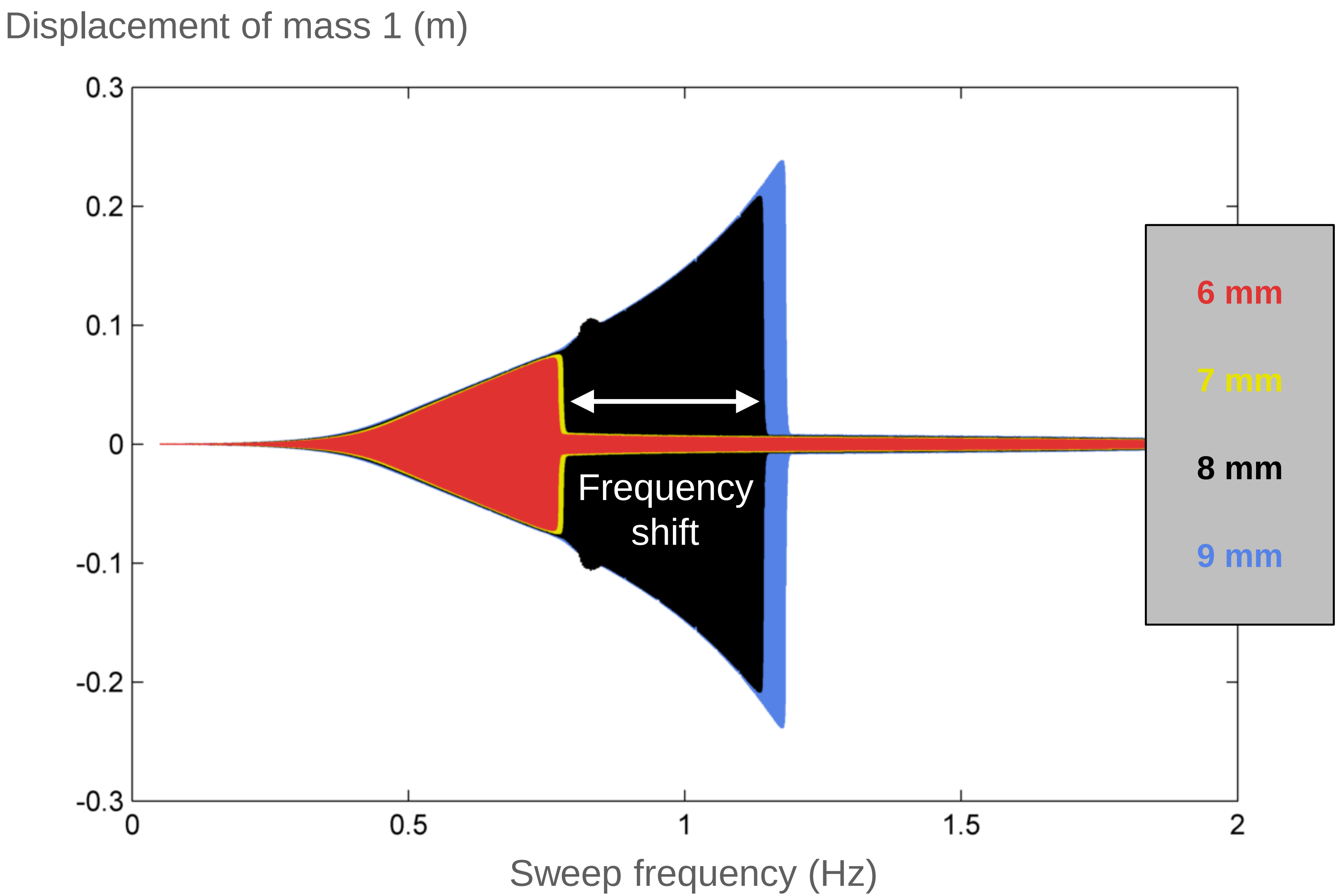
Disp. of mass 1 (m) – Base disp. D = 8 mm



The Fold Curve Reveals the Isola Dynamics



Merging Causes a 50% Rise of the Resonance Frequency



Concluding Remarks on Bifurcation Analysis

Bifurcation analysis is useful to understand nonlinear phenomena (amplitude jumps, quasiperiodic oscillations, etc.).

They can be monitored during continuation using test functions.

Bifurcation tracking is useful to predict nonlinear phenomena (appearance and merging of isolated solutions, appearance of quasiperiodic oscillations, etc.).

Further Readings

M. Peeters, R. Viguié, G. Sérandour, G. Kerschen, J. C. Golinval, **Nonlinear normal modes, Part II: Toward a practical computation using numerical continuation techniques**, Mechanical systems and signal processing, 23(1), 195-216, 2009.

L. Peletan, S. Baguet, M. Torkhani,G. Jacquet-Richardet, **A comparison of stability computational methods for periodic solution of nonlinear problems with application to rotordynamics**, Nonlinear Dynamics, 72(3), 671-682, 2013.

T. Detroux, L. Renson, L. Masset, G. Kerschen, **The harmonic balance method for bifurcation analysis of large-scale nonlinear mechanical systems**, Computer Methods in Applied Mechanics and Engineering, 296, 18-38, 2015.

T. Detroux, **Performance and Robustness of Nonlinear Systems Using Bifurcation Analysis**, PhD Thesis, University of Liège, 2016.