Cinématique et dynamique des machines

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L02 Newton & Lagrange

Course objectives

Given input → Mechanical system → Predict the output

Equation of motion

→ Write down the equations of motion

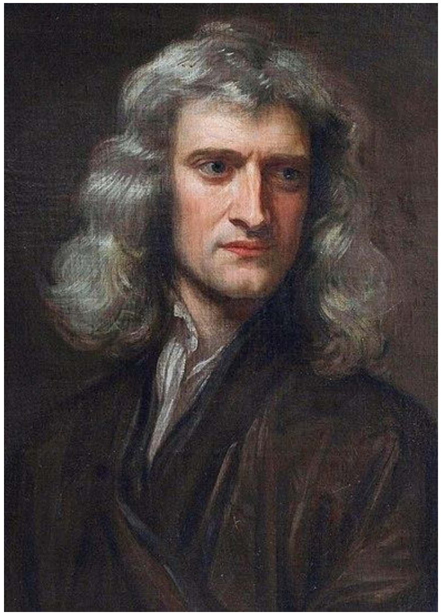
$$m\ddot{x} + c\dot{x} + kx = f(t)$$

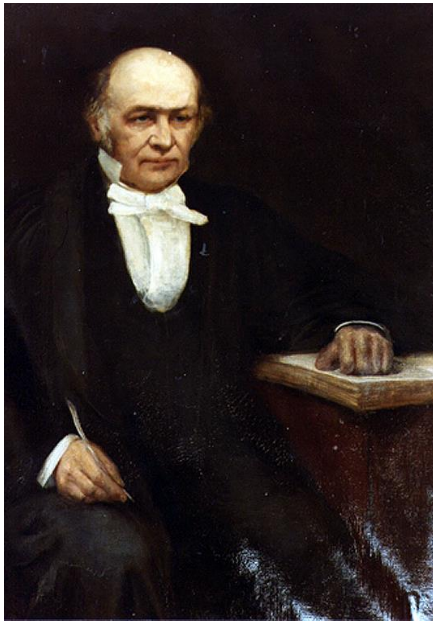
→ Calculate the response analytically or numerically

$$x = 0.19 \sin 2.3t$$

→ Is the structure safe ?

Outline






Théorème de la quantité de mouvement

When a body is acted upon by a force, the time rate of change of its momentum equals the force

Linear momentum (translation)

$$\boxed{m\frac{d\overline{v}_G}{dt} = \sum_{i=1}^{N} \overline{F}_{ext,h}}$$

…written in an inertial frame !...

Digression

What did Richard Feynman mean about the Second Law of Motion? Where was the error?

JANUARY 17, 2021 / FRANCES48 / 0 COMMENTS

Richard Feynman writes about Newton's Second Law of Motion in his work "Lectures on Physics" (Chapter 15):

„For over 200 years the equations of motion enunciated by Newton were believed to describe nature correctly, and the first time that an error in these laws was discovered, the way to correct it was also discovered. Both the error and its correction were discovered by Einstein in 1905.

Digression

Newton's Second Law, which we have expressed by the equation

$$F = d(mv)/dt$$

was stated with the tacit assumption that m is a constant, but we now know that this is not true, and that the mass of a body increases with velocity. In Einstein's corrected formula m has the value

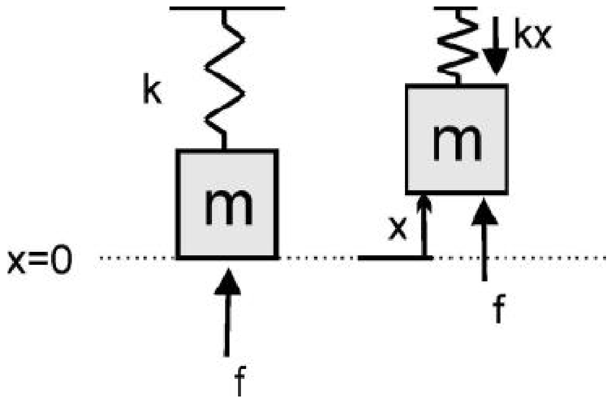
$$m = \frac{m_0}{\sqrt{1 - v^2 \;/\; c^2}}$$

where the rest mass represents the mass of a body that is not moving and c is the speed of light […].

Spring-mass system: a 1DOF system

Linear momentum (translation)

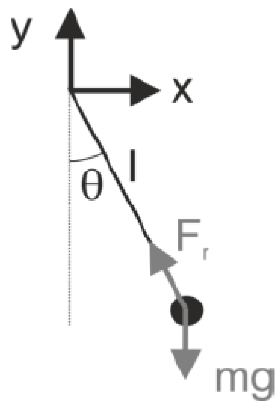
$$\boxed{m\frac{d\overline{v}_G}{dt} = \sum_{i=1}^{N} \overline{F}_{ext,h}}$$



- Spring force: $-kx$
- External force f acting on the mass.

$$m\ddot{x} = \sum F_x \longrightarrow \boxed{m\ddot{x} + kx = f}$$

Pendulum: a 1DOF system



$$m\ddot{x} = -F_r \sin\theta \longrightarrow F_r = \frac{-m\ddot{x}}{\sin\theta}$$

$$m\ddot{y} = F_r \cos\theta - mg \longrightarrow m\ddot{y} = \frac{-m\ddot{x}}{\sin\theta}\cos\theta - mg$$

$$\begin{aligned} x &= l\sin\theta & \dot{x} &= l\dot{\theta}\cos\theta \\ y &= -l\cos\theta & \dot{y} &= l\dot{\theta}\sin\theta \end{aligned} \longrightarrow \begin{aligned} \ddot{x} &= l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta \\ \ddot{y} &= l\ddot{\theta}\sin\theta + l\dot{\theta}^2\cos\theta \end{aligned}$$

$$\longrightarrow ml\ddot{\theta} + mg\sin\theta = 0$$

Small displacements

$$\longrightarrow \boxed{ml\ddot{\theta} + mg\theta = 0}$$

Théorème du moment cinétique

Angular momentum (rotation)

$$\boxed{\frac{d\overline{M}_A}{dt} = m\overline{v}_G \times \overline{v}_A + \overline{m}_{ext,A}}$$

- $G = A$
- $\overline{v}_G = 0$
- $\overline{v}_A = 0$
- $\overline{v}_G \parallel \overline{v}_A$

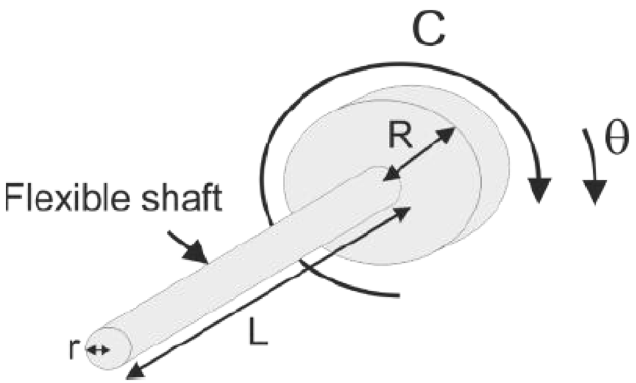
$$\longrightarrow \quad \boxed{\frac{d\overline{M}_A}{dt} = \overline{m}_{ext,A}}$$

Flexible shaft

At the center of gravity

$$\boxed{\frac{d\overline{M}_G}{dt} = \overline{m}_{ext,G}}$$

$$\overline{M}_G = I\dot{\theta} \longrightarrow \boxed{I\ddot{\theta} + K\theta = C}$$



$$M_{shaft} = \frac{GI_p}{L}\theta = \frac{E}{2(1+\nu)L}\frac{\pi D^4}{32}\theta = \frac{E\pi R^4}{4(1+\nu)L}\theta = K\theta$$

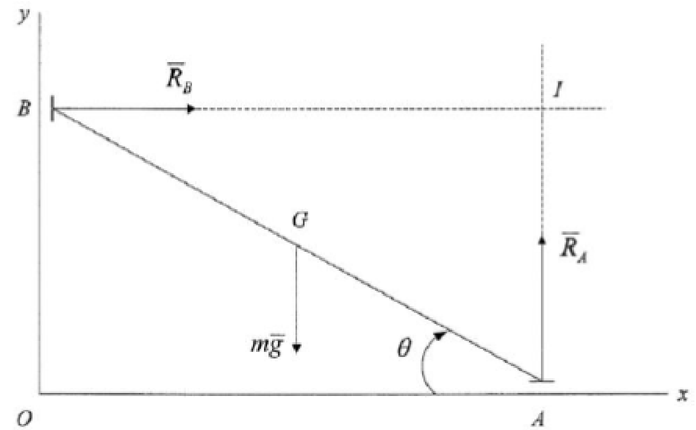
Sliding bar: linear momentum

$$\boxed{m\frac{d\overline{v}_G}{dt} = \sum_{i=1}^{N} \overline{F}_{ext,h}}$$

$$\overline{r}_G = \frac{L}{2}(\cos\theta\overline{1}_x + \sin\theta\overline{1}_y)$$

$$\overline{v}_G = \frac{L\dot{\theta}}{2}(-\sin\theta\overline{1}_x + \cos\theta\overline{1}_y)$$

$$\overline{a}_G = \frac{L}{2}\left((-\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta)\overline{1}_x + (\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)\overline{1}_y\right)$$



$$\frac{mL}{2}(-\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta) = \underline{R_B}$$

$$\frac{mL}{2}(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) = \underline{R_A} - mg$$

Reaction forces

2 equations/
3 unknowns

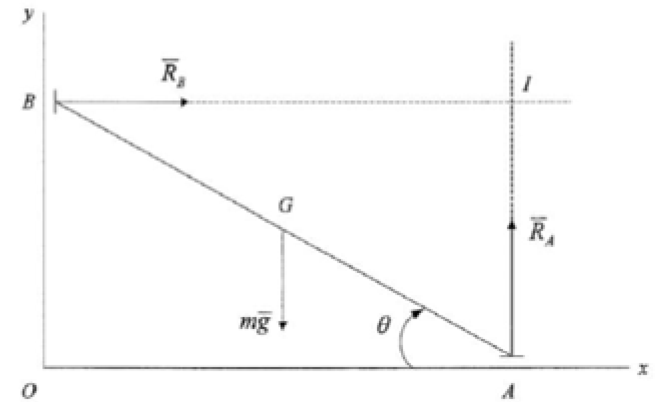
Sliding bar: angular momentum

Choice of point I allows to eliminate the reaction forces RA and RB

$$\overline{v}_G \parallel \overline{v}_I \quad \Rightarrow \quad \boxed{\frac{\mathrm{d}\overline{M}_I}{\mathrm{d}t} = mg\frac{L}{2}\cos\theta\,\overline{1}_z}$$

$$\overline{M}_I = \overline{M}_G + m\overline{IG}\times\overline{v}_G = -\frac{mL^2}{3}\dot{\theta}\,\overline{1}_z$$

$$\longrightarrow \quad \boxed{-\frac{mL^2}{3}\ddot{\theta} = mg\frac{L}{2}\cos\theta}$$



Once θ is known the reaction forces can be calculated.

Sliding bar

$$\vec{M}_G = \left(0, 0, \vec{\vec{I}}\,\vec{\omega}\right) = \left(0, 0, \frac{mL^2}{12}(-\dot{\theta})\right)$$

$$\vec{IG} = \left(\frac{-L}{2}\cos\theta, \frac{-L}{2}\sin\theta, 0\right)$$

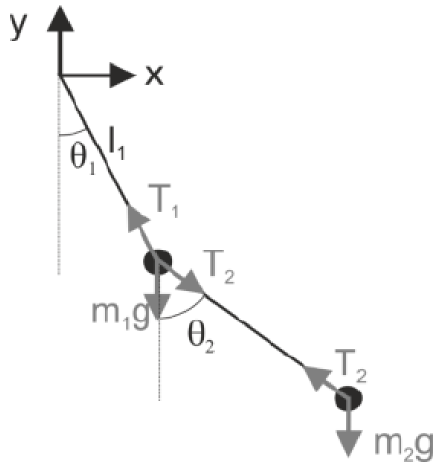
$$\vec{v}_G = \left(\frac{-L\dot{\theta}}{2}\sin\theta, \frac{L}{2}\dot{\theta}\cos\theta, 0\right)$$

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

$$\rightarrow \quad \vec{IG} \times \vec{v}_G = \begin{bmatrix} 0 \\ 0 \\ \frac{-L^2}{4}\dot{\theta}\cos^2\theta - \frac{L^2}{4}\dot{\theta}\sin^2\theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{-L^2}{4}\dot{\theta} \end{bmatrix}$$

$$\vec{M}_I = \begin{bmatrix} 0 \\ 0 \\ -\frac{mL^2}{12}\dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{mL^2}{4}\dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{mL^2}{3}\dot{\theta} \end{bmatrix}$$

Double pendulum



Linear momentum (translation)

$$m_1\ddot{x}_1 = -T_1 \sin\theta_1 + T_2 \sin\theta_2$$
$$m_1\ddot{y}_1 = T_1 \cos\theta_1 - T_2 \cos\theta_2 - m_1 g$$

$$m_2\ddot{x}_2 = -T_2 \sin\theta_2$$
$$m_2\ddot{y}_2 = T_2 \cos\theta_2 - m_2 g$$

Accelerations

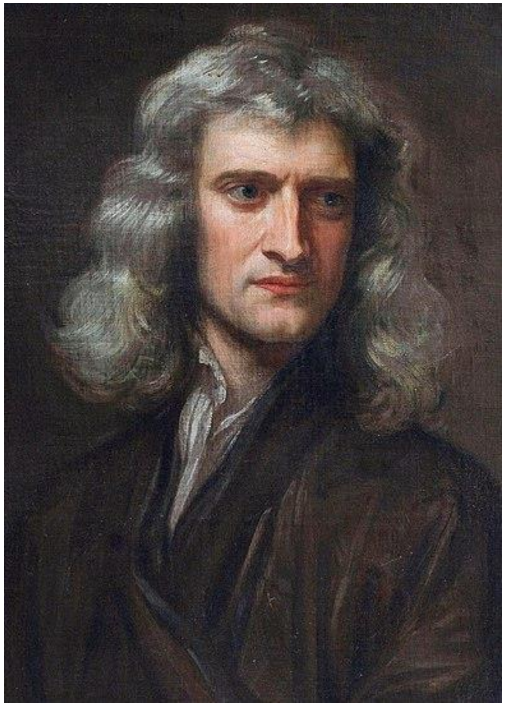
$$x_1 = l_1 \sin\theta_1 \qquad y_1 = -l_1 \cos\theta_1$$
$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2 \qquad y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2$$

Compute $\dot{x}_1, \ddot{x}_1, \dot{x}_2, \ddot{x}_2$ And replace in equilibrium equations ....

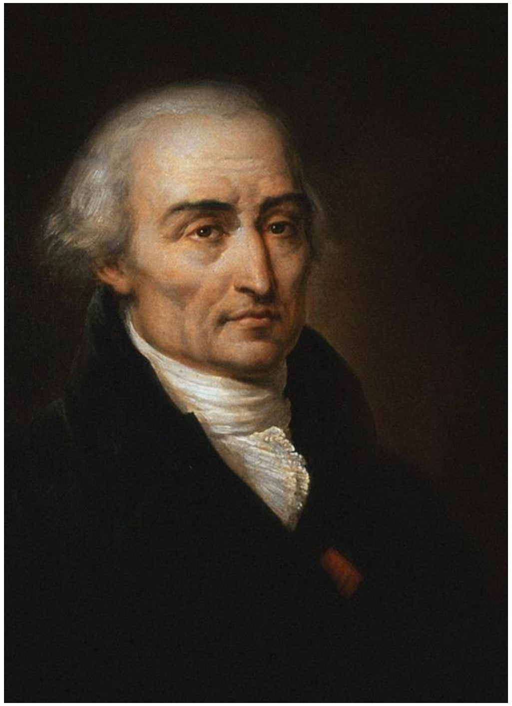
So what ?

- Newton's second law must be applied to each solid separately
- It introduces (unknown) reaction forces
- For multiple solids, it generally leads to lengthy calculations

Our objective for today



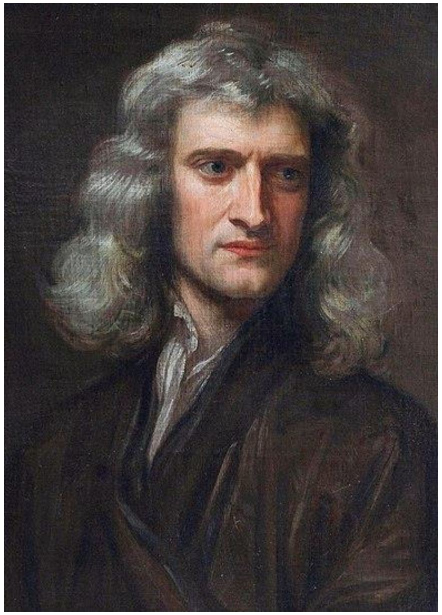
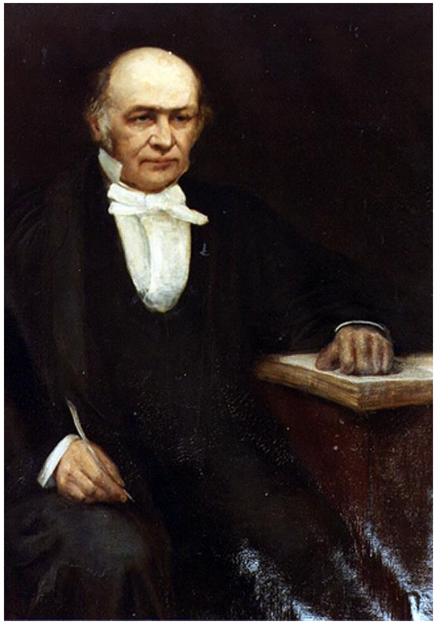
Newton (vectorial mechanics, force and acceleration)



Lagrange (analytical mechanics, work and energy)

… through the virtual work principle…

Outline

Statics

Historical perspective

Bernoulli systematized the virtual work principle (VWP) during the 18th century but was imagined before. The virtual work principle is a statement of the static equilibrium and is the first variational principle in mechanics.

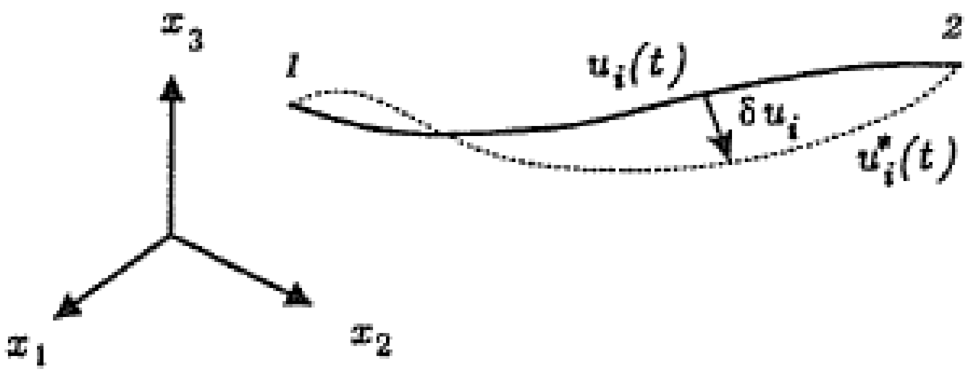
In 1743, d'Alembert extends the VWP to dynamics through the introduction of inertia forces. The VWP becomes equivalent to Newton's second law.

When they can be applied, the VWP and d'Alembert's principle are absolutely general. They state the necessary and sufficient conditions either for equilibrium or for motion.

In 1788, Lagrange proposed the VWP as the basis of analytical mechanics and applies the generalized coordinates to the VWP.

Virtual displacement concept

A virtual displacement $\overrightarrow{\delta u}$ shows how the mechanical system's trajectory can *hypothetically* (hence the term *virtual*) deviate very slightly from the actual trajectory.



A virtual displacement ...

is arbitrary but compatible with the kinematic constraints.

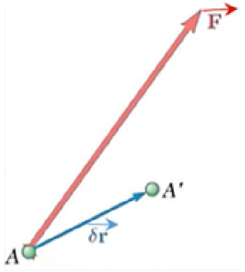
takes place instantaneouly ($\delta t=0$).

is infinitesimal; it obeys the rules of differential calculus.

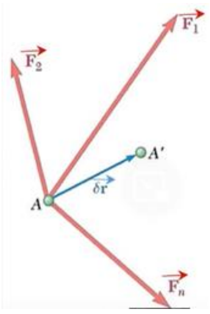
coincides with the real displacements at the extremities of the time interval $\delta u_i(t_1) = \delta u_i(t_2) = 0$

VWP for a material point (statics)

The virtual work of the applied force
(e.g., gravitational or magnetic) is $\delta W = \vec{F}.\overrightarrow{\delta r}$



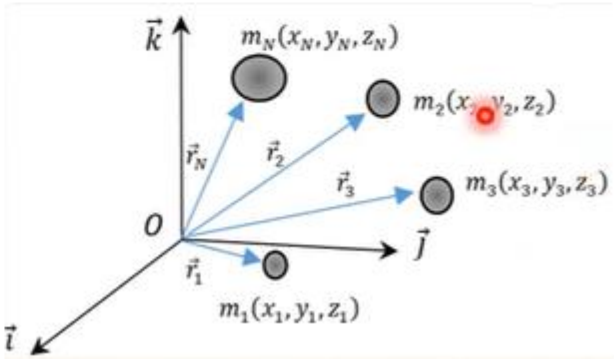
$$\begin{aligned}\delta W &= \vec{F}_1.\overrightarrow{\delta r} + \cdots + \vec{F}_N.\overrightarrow{\delta r} \\ &= \left(\vec{F}_1 + \cdots + \vec{F}_N\right).\overrightarrow{\delta r} = \vec{R}.\overrightarrow{\delta r}\end{aligned}$$



A material point is in equilibrium if the virtual work of the applied forces is zero for any virtual displacement of this point.

VWP for a set of material points

Virtual displacements $\overrightarrow{\delta r_1}, \dots, \overrightarrow{\delta r_N}$



Each point is subjected to a force

$$\vec{R}_i = \vec{F}_i + \vec{f}_i$$

Constraint force (e.g., move on a surface)

Equilibrium: each point is at rest, the total force is zero

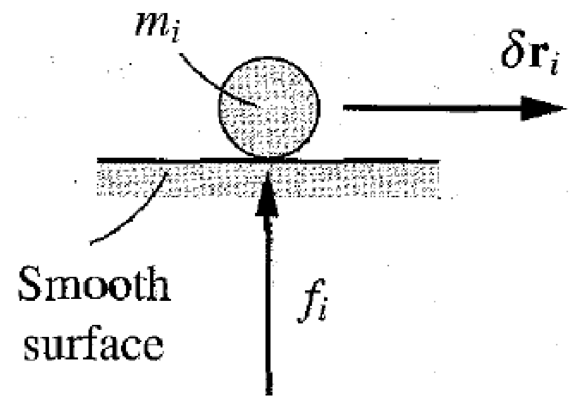
$$\vec{R}_i = \vec{F}_i + \vec{f}_i = 0 \Rightarrow \overrightarrow{\delta W_i} = \vec{R}_i . \overrightarrow{\delta r_i} = 0$$

$$\delta W = \sum_{i=1}^{N} \delta W_i = \sum_{i=1}^{N} \vec{F}_i . \overrightarrow{\delta r_i} + \sum_{i=1}^{N} \vec{f}_i . \overrightarrow{\delta r_i} = 0$$

Constraint forces

$$\delta W = \sum_{i=1}^{N} \delta W_i = \sum_{i=1}^{N} \vec{F}_i . \overrightarrow{\delta r_i} + \sum_{i=1}^{N} \vec{f}_i . \overrightarrow{\delta r_i} = 0$$

The constraint forces, which force the particle to move on, e.g., a surface, are normal to the surface (no friction).



So the virtual work of the constraint forces is zero for any virtual displacement:

$$\delta W = \sum_{i=1}^{N} \delta W_i = \sum_{i=1}^{N} \vec{F}_i . \overrightarrow{\delta r_i} + \cancel{\sum_{i=1}^{N} \vec{f}_i . \overrightarrow{\delta r_i}} = 0$$

Virtual work principle

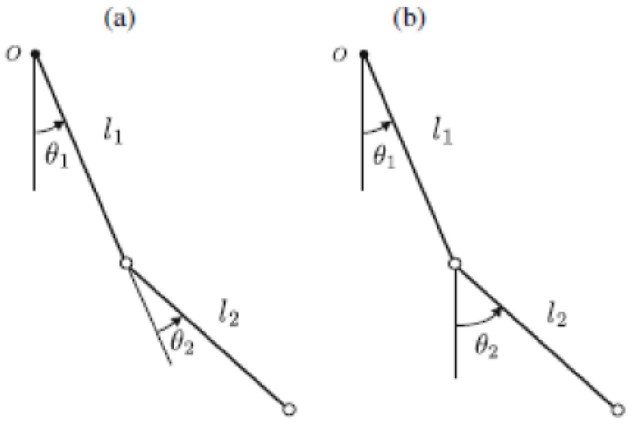
$$\boxed{\delta W = \sum_{i=1}^{N} \vec{F}_i . \overrightarrow{\delta r_i} = 0}$$

For a system to be in static equilibrium, the virtual work of the forces *applied* to the system must be zero for virtual displacements compatible with the system's constraints.

DOFs and generalized coordinates

Degrees of freedom N : minimum number of coordinates necessary to provide the full geometric description of the system

Generalized coordinates : $q_1, q_2, ...q_n$



Minimum set of coordinates: $n = N$

$n > N$ ⟶ Need for kinematic constraints

Kinematic constraints

The number of DOFs decreases:

$$f(q_1, ..., q_n, t) = 0$$

Holonomic constraints

$$f(q_1, ..., q_n) = 0$$

Scleronomic constraints

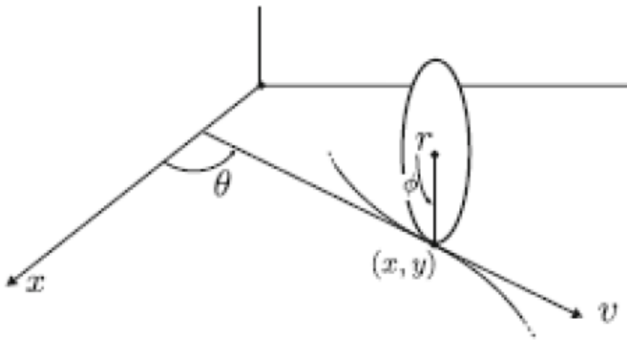
Two masses connected by a rigid bar

The number of DOFs does not decrease:

$$\sum_i a_i dq_i + a_0 dt = 0 \quad \text{or} \quad \sum_i a_i dq_i = 0$$

Non holonomic constraints

Example of nonholonomic constraints



- 4 generalized coordinates

  x, y, θ, ϕ

- All positions and orientations of the disk are possible

But, time derivatives of coordinates are not independent

$$v = r\dot{\phi}$$
$$\dot{x} = v\cos\theta$$
$$\dot{y} = v\sin\theta$$

$$\boxed{\begin{aligned} dx - r\cos\theta d\phi = 0 \\ dy - r\sin\theta d\phi = 0 \end{aligned}}$$

Not all paths are possible to go from one configuration to another

Independent generalized coordinates

An underlying assumption is that the virtual displacements must be independent.

If this is not the case, we should consider independent generalized coordinates such that

$$\overrightarrow{\delta r_i} = \sum_{k=1}^{n} \frac{\partial \vec{r_i}}{\partial q_k} \delta q_k$$

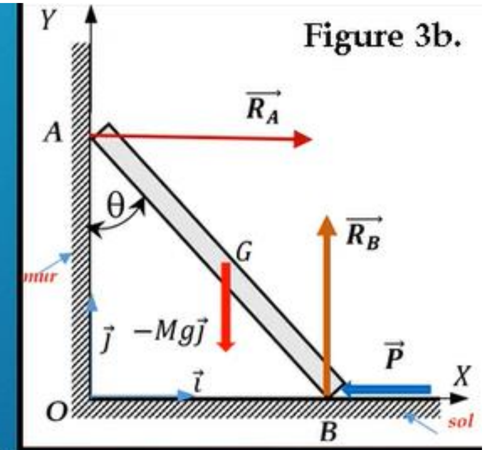
$$\delta W = \sum_{i=1}^{N} \vec{F}_i . \overrightarrow{\delta r_i} = \sum_{i=1}^{N} \vec{F}_i . \sum_{k=1}^{n} \frac{\partial \vec{r_i}}{\partial q_k} \delta q_k = \sum_{k=1}^{n} \left(\sum_{i=1}^{N} \vec{F}_i . \frac{\partial \vec{r_i}}{\partial q_k} \right) \delta q_k = 0$$

$$\delta W = \sum_{k=1}^{n} Q_k \delta q_k = 0 \longrightarrow \boxed{Q_k = 0}$$

Generalized forces

Static equilibrium

Sliding bar example



Exemple 1 *: étude d'une barre glissante*

➢ La barre est soumise aux **forces données** suivantes:

- Le poids $-Mg\vec{j}$ appliqué au point G, sachant que le vecteur position du point G est $\vec{r}_G\begin{pmatrix}x_G\\ y_G\end{pmatrix}$
- La force de maintient $\vec{P}$ appliqué au point B, sachant que la position du point B est $\vec{r}_B\begin{pmatrix}x_B\\ y_B\end{pmatrix}$
- La réaction $\overrightarrow{R_B}$ du sol en B, appliqué aussi au niveau du point B
- La réaction $\overrightarrow{R_A}$ du mur en A, appliqué au point A dont la position est définie par $\vec{r}_A\begin{pmatrix}x_A\\ y_A\end{pmatrix}$

➢ Si la barre glisse un petit peu, alors on va enregistrer une rotation élémentaire $\delta\theta$ qui va lui correspondre des déplacements élémentaires des points A, G et B, soient : $\overrightarrow{\delta r_A}\begin{pmatrix}\delta x_A\\ \delta y_A\end{pmatrix}$, $\overrightarrow{\delta r_G}\begin{pmatrix}\delta x_G\\ \delta y_G\end{pmatrix}$ et $\delta\vec{r}_B\begin{pmatrix}\delta x_B\\ \delta y_B\end{pmatrix}$.

➢ Dans le triangle rectangle OAB, on peut écrire que: $x_B = l\sin\theta,\ y_B = 0,\ x_G = 0.5\, l\sin\theta,\ y_G = 0.5\, l\cos\theta,$

➢ En conséquence de la rotation $\delta\theta$ compatible avec les contacts, on a $\delta x_B = l\cos\theta\,\delta\theta,\ \delta y_B = 0,$ $\delta x_G = 0.5\ \cos\theta\,\delta\theta,\ \delta y_G = -0.5\, l\sin\theta\delta\theta,$

➢ D'où: $\overrightarrow{\delta r_G}\begin{pmatrix}0.5\ \cos\theta\,\delta\theta\\ -0.5\, l\sin\theta\delta\theta\end{pmatrix}$ $\delta\vec{r}_B\begin{pmatrix}l\cos\theta\,\delta\theta\\ 0\end{pmatrix}$

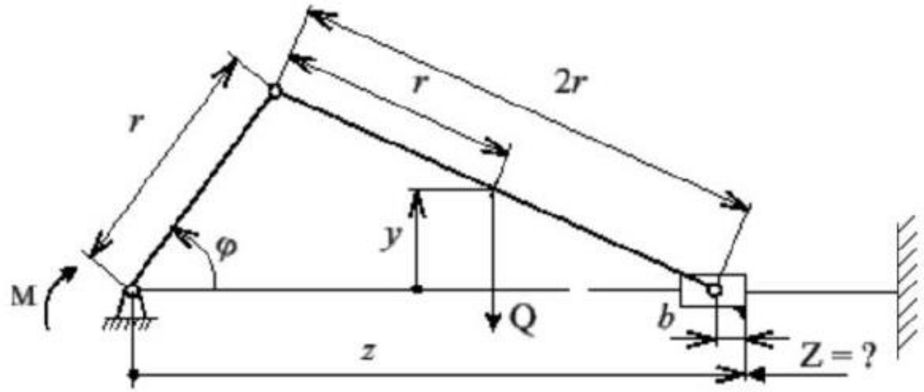
➢ En l'absence des frottements les deux réactions $\overrightarrow{R_B}$ et $\overrightarrow{R_A}$ ne travaillent pas, elles sont toutes les deux perpendiculaires aux vecteurs déplacements des points d'application. Il reste à déterminer le travail du poids et le force de maintien et selon le PTV, le travail virtuel total δW doit être nul : $\delta W = (-Mg\vec{j}).\overrightarrow{\delta r_G} + (-P\vec{i}).\delta\vec{r}_B = 0$

$$\begin{pmatrix}0\\ -Mg\end{pmatrix}.\begin{pmatrix}0.5\ \cos\theta\,\delta\theta\\ -0.5\, l\sin\theta\delta\theta\end{pmatrix} + \begin{pmatrix}-P\\ 0\end{pmatrix}.\begin{pmatrix}l\cos\theta\,\delta\theta\\ 0\end{pmatrix} = 0 \Rightarrow (0.5\, Mg\ \sin\theta\text{-}\ P\cos\theta)\, l\,\delta\theta = 0$$

la force à appliquer pour maintenir la barre à l'équilibre est $\boxed{P = 0.5\, M\, g\tan\theta.}$

Much simpler than Newtonian mechanics

Problem: Find *Z* to maintain the slider-crank mechanism in static equilibrium with $\phi = 30°$.



M = 50 Nm
Q = 35 N
r = 0.1 m

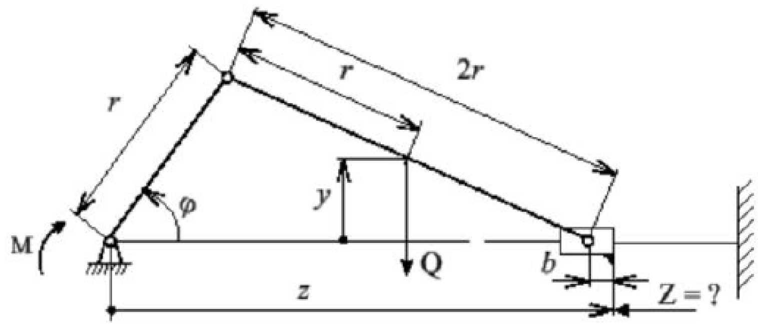
Newtonian mechanics : 2 equations for translation, 1 equation for rotation X 2 bodies = 6 equations

→ 5 reaction forces + Z = 6 unknowns

Much simpler than Newtonian mechanics

Forces: Q, Z; Moment: M

$$q_1 = \varphi$$



Virtual work:

$$-M\delta\varphi - Q\delta y - Z\delta z$$

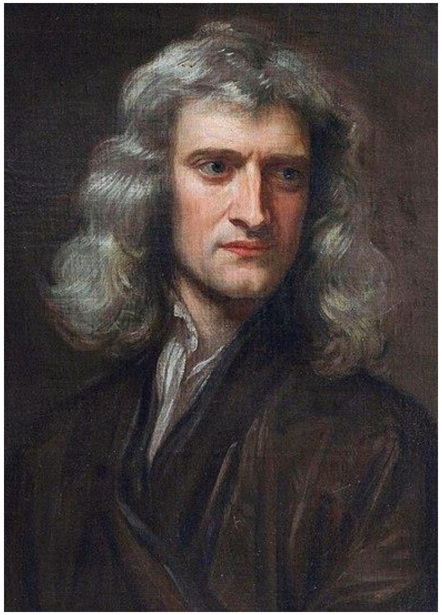
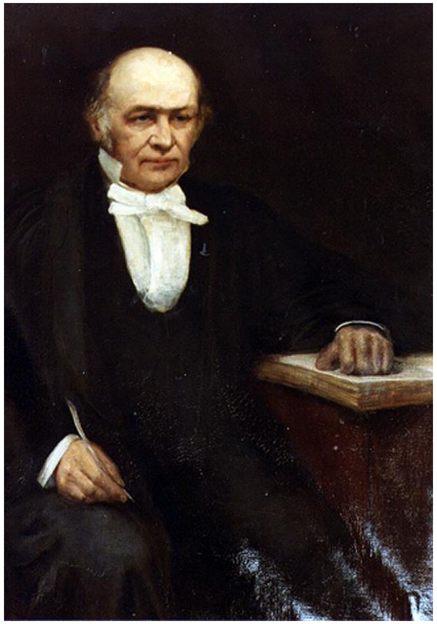
$$y = \frac{r}{2}\sin\varphi \longrightarrow \delta y = \frac{r}{2}\cos\varphi\delta\varphi$$

$$z = r\cos\varphi + r\sqrt{4-\sin^2\varphi} + b \longrightarrow \delta z = -r\sin\varphi\left(1 + \frac{\cos\varphi}{\sqrt{4-\sin^2\varphi}}\right)\delta\varphi$$

$$-M\delta\varphi - Q\delta y - Z\delta z = 0$$

$$\varphi = 30^\circ \longrightarrow Z = 711.92\ N$$

Outline

Statics → Dynamics

D’Alembert’s principle

We now include inertia forces: $\vec{R}_i = \vec{F}_i + \vec{f}_i - m_i \vec{\ddot{r}}_i$

The VWP becomes

$$\sum_{i=1}^{N} \left(\vec{F}_i - m_i \vec{\ddot{r}}_i\right) . \overrightarrow{\delta r_i} = 0$$

$$\overrightarrow{\delta r_i}(t_1) = \overrightarrow{\delta r_i}(t_2) = 0$$

If $\overrightarrow{\delta r_i} = \vec{\dot{r}}_i dt$, i.e., no explicit depend. on time in constraints

$$\sum_{i=1}^{N} \vec{F}_i . \overrightarrow{\delta r_i} - m_i \vec{\ddot{r}}_i . \vec{\dot{r}}_i dt = 0$$

Conservation of total energy

$$\sum_{i=1}^{N} \vec{F}_i.\overrightarrow{\delta r_i} - m_i\overrightarrow{\ddot{r}_i}.\overrightarrow{\dot{r}_i}dt = 0$$

The forces can be expressed as the gradient of a potential V

$$\sum_{i=1}^{N} \vec{F}_i.\overrightarrow{\delta r_i} = -dV$$

T is the kinetic energy

$$m_i\overrightarrow{\ddot{r}_i}.\overrightarrow{\dot{r}_i}dt = \frac{d}{dt}\left(\frac{1}{2}\sum_{i=1}^{N} m_i\overrightarrow{\dot{r}_i}.\overrightarrow{\dot{r}_i}\right)dt = dT$$

$$d(T+V) = 0 \Rightarrow \boxed{T + V = E}$$

Hamilton’s principle

We derive the equations of motion from three scalar quantities, namely the kinetic energy, the potential energy and the virtual work of the nonconservative forces.

It is derived from the generalized d’Alembert’s principle.

$$\sum_{i=1}^{N}\left(\vec{F}_i - m_i\vec{\ddot{r}}_i\right).\overrightarrow{\delta r_i} = 0$$

Hamilton's principle: Integral form of the VWP

$$\sum_{i=1}^{N}\left(\vec{F}_i - m_i\vec{\ddot{r}}_i\right).\overrightarrow{\delta r_i} = 0$$

$$\delta W = \sum_{i=1}^{N}\vec{F}_i.\overrightarrow{\delta r_i}$$

Virtual work of external forces

$$\vec{\ddot{r}}_i.\overrightarrow{\delta r_i} = \frac{d}{dt}\left(\vec{\dot{r}}_i.\overrightarrow{\delta r_i}\right) - \delta\frac{1}{2}\left(\vec{\dot{r}}_i.\vec{\dot{r}}_i\right)$$

$$\sum_{i=1}^{N} m_i\vec{\ddot{r}}_i.\overrightarrow{\delta r_i} = \sum_{i=1}^{N} m_i\frac{d}{dt}\left(\vec{\dot{r}}_i.\overrightarrow{\delta r_i}\right) - \delta T$$

$$\delta W + \delta T = \sum_{i=1}^{N} m_i\frac{d}{dt}\left(\vec{\dot{r}}_i.\overrightarrow{\delta r_i}\right)$$

Hamilton's principle: Integral form of the VWP

$$\delta W + \delta T = \sum_{i=1}^{N} m_i \frac{d}{dt}\left(\vec{\dot{r}_i}.\overrightarrow{\delta r_i}\right)$$

Integration over a time interval $[t_1, t_2]$ with $\overrightarrow{\delta r_i}(t_1)=\overrightarrow{\delta r_i}(t_2) = 0$

$$\boxed{\int_{t_1}^{t_2} (\delta W + \delta T) dt = 0}$$

Hamilton's principle

Conservative forces

$$\delta W = -\delta V + \delta W_{nc}$$

Non-conservative forces

Example from Meirovitch

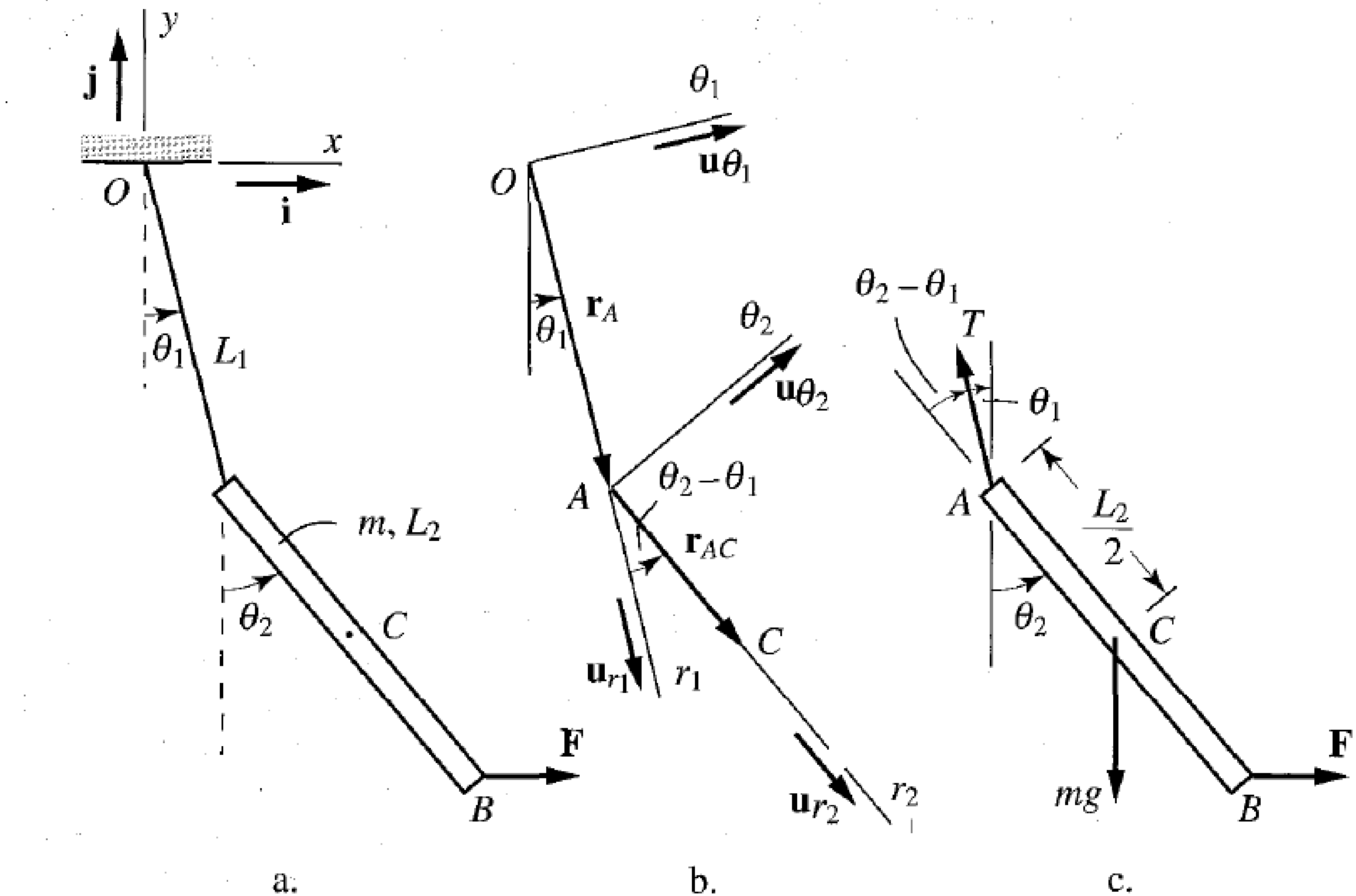


FIGURE 1.10

a. Rigid bar on a string, b. Position of the mass center, c. Free-body diagram

Example from Meirovitch

Example 1.6. A uniform rigid bar of total mass m and length L_2, suspended at point O by a string of length L_1, is acted upon by the horizontal force F, as shown in Fig. 1.10a. Use the angular displacements θ_1 and θ_2 to define the position, velocity and acceleration of the mass center C in terms of body axes and then derive the equations of motion for the translation of C and the rotation about C.

Referring to Fig. 1.10b, we can write the position, velocity and acceleration of the mass center C in the form

$$\mathbf{r}_C = \mathbf{r}_A + \mathbf{r}_{AC} = L_1\mathbf{u}_{r1} + \frac{L_2}{2}\mathbf{u}_{r2} \tag{a}$$

$$\mathbf{v}_C = \mathbf{v}_A + \mathbf{v}_{AC} = L_1\dot{\theta}_1\mathbf{u}_{\theta 1} + \frac{L_2}{2}\dot{\theta}_2\mathbf{u}_{\theta 2} \tag{b}$$

and

$$\mathbf{a}_C = \mathbf{a}_A + \mathbf{a}_{AC} = -L_1\dot{\theta}_1^2\mathbf{u}_{r1} + L_1\ddot{\theta}_1\mathbf{u}_{\theta 1} - \frac{L_2}{2}\dot{\theta}_2^2\mathbf{u}_{r2} + \frac{L_2}{2}\ddot{\theta}_2\mathbf{u}_{\theta 2} \tag{c}$$

Example from Meirovitch

respectively. Equations (a) - (c) are in terms of two sets of unit vectors. To obtain expressions in terms of the body axes r_2, θ_2, we observe from Fig. 1.10b that the two sets of unit vectors are related by

$$\begin{aligned} \mathbf{u}_{r1} &= \cos(\theta_2-\theta_1)\mathbf{u}_{r2} - \sin(\theta_2-\theta_1)\mathbf{u}_{\theta 2} \\ \mathbf{u}_{\theta 1} &= \sin(\theta_2-\theta_1)\mathbf{u}_{r2} + \cos(\theta_2-\theta_1)\mathbf{u}_{\theta 2} \end{aligned} \tag{d}$$

Inserting Eqs. (d) into Eqs. (a) - (c), we obtain the position, velocity and acceleration of the mass center C in terms of components along the body axes, as follows:

$$\mathbf{r}_C = \left[L_1\cos(\theta_2-\theta_1) + \frac{L_2}{2}\right]\mathbf{u}_{r2} - L_1\sin(\theta_2-\theta_1)\mathbf{u}_{\theta 2} \tag{e}$$

$$\mathbf{v}_C = L_1\dot{\theta}_1\sin(\theta_2-\theta_1)\mathbf{u}_{r2} + \left[L_1\dot{\theta}_1\cos(\theta_2-\theta_1) + \frac{L_2}{2}\dot{\theta}_2\right]\mathbf{u}_{\theta 2} \tag{f}$$

and

$$\begin{aligned} \mathbf{a}_C = &\left[L_1\ddot{\theta}_1\sin(\theta_2-\theta_1) - L_1\dot{\theta}_1^2\cos(\theta_2-\theta_1) - \frac{L_2}{2}\dot{\theta}_2^2\right]\mathbf{u}_{r2} \\ &+ \left[L_1\ddot{\theta}_1\cos(\theta_2-\theta_1) + L_1\dot{\theta}_1^2\sin(\theta_2-\theta_1) + \frac{L_2}{2}\ddot{\theta}_2\right]\mathbf{u}_{\theta 2} \end{aligned}$$

Example from Meirovitch

The bar translates and rotates, with the velocity of the mass center C being given by Eq. (f) of Example 1.6 and the velocity of rotation being $\dot{\theta}_2$. Hence, the kinetic energy consists of two parts, one due to translation of C and one due to rotation about C. Inserting Eq. (f) of Example 1.6 into Eq. (1.71), we obtain the kinetic energy of translation

$$T_{\text{tr}} = \frac{1}{2} m \mathbf{v}_C \cdot \mathbf{v}_C = \frac{1}{2} m \left\{ L_1 \dot{\theta}_1 \sin(\theta_2 - \theta_1) \mathbf{u}_{r2} + \left[L_1 \dot{\theta}_1 \cos(\theta_2 - \theta_1) + \frac{L_2}{2} \dot{\theta}_2 \right] \mathbf{u}_{\theta 2} \right\}$$

$$\cdot \left\{ L_1 \dot{\theta}_1 \sin(\theta_2 - \theta_1) \mathbf{u}_{r2} + \left[L_1 \dot{\theta}_1 \cos(\theta_2 - \theta_1) + \frac{L_2}{2} \dot{\theta}_2 \right] \mathbf{u}_{\theta 2} \right\}$$

$$= \frac{1}{2} m \left[L_1^2 \dot{\theta}_1^2 + L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_2 - \theta_1) + \frac{L_2^2}{4} \dot{\theta}_2^2 \right] \quad \text{(a)}$$

On the other hand, using Eq. (1.72) and recalling that the mass moment of inertia of a thin uniform bar about C is $I_C = mL^2/12$, the kinetic energy of rotation about C is simply

$$T_{\text{rot}} = \frac{1}{2} I_C \omega^2 = \frac{1}{2} \frac{m L_2^2}{12} \dot{\theta}_2^2 \quad \text{(b)}$$

Example from Meirovitch

$$T = T_{tr} + T_{rot} = \frac{1}{2}m\left[L_1^2\dot{\theta}_1^2 + L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1) + \frac{L_2^2}{4}\dot{\theta}_2^2\right] + \frac{1}{2}\frac{mL_2^2}{12}\dot{\theta}_2^2$$

$$= \frac{1}{2}m\left[L_1^2\dot{\theta}_1^2 + L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1) + \frac{L_2^2}{3}\dot{\theta}_2^2\right] \quad \text{(a)}$$

$$V = mg\left[L_1(1-\cos\theta_1) + \frac{L_2}{2}(1-\cos\theta_2)\right]$$

$$\overline{\delta W}_{nc} = \mathbf{F}\cdot\delta\mathbf{r}_B = F\mathbf{i}\cdot\delta\left[(L_1\sin\theta_1 + L_2\sin\theta_2)\mathbf{i} - (L_1\cos\theta_1 + L_2\cos\theta_2)\mathbf{j}\right]$$

$$= F(L_1\cos\theta_1\delta\theta_1 + L_2\cos\theta_2\delta\theta_2) = \Theta_1\delta\theta_1 + \Theta_2\delta\theta_2 = Q_1\delta q_1 + Q_2\delta q_2 \quad \text{(e)}$$

$$Q_1 = \Theta_1 = FL_1\cos\theta_1,\ \ Q_2 = \Theta_2 = FL_2\cos\theta_2$$

represent the generalized nonconservative forces.

Example from Meirovitch

$$\boxed{\begin{aligned}\delta T =& mL_1^2\dot\theta_1\delta\dot\theta_1 + \frac{mL_1L_2}{2}[\dot\theta_2\cos(\theta_2-\theta_1)\delta\dot\theta_1 + \dot\theta_1\cos(\theta_2-\theta_1)\delta\dot\theta_2 \\ & \boxed{-\dot\theta_1\dot\theta_2\sin(\theta_2-\theta_1)\delta(\theta_2-\theta_1)]} + \frac{mL_2^2}{3}\dot\theta_2\delta\dot\theta_2\end{aligned}}$$

$$\begin{aligned}=& \frac{mL_1L_2}{2}\dot\theta_1\dot\theta_2\sin(\theta_2-\theta_1)\delta\theta_1 - \frac{mL_1L_2}{2}\dot\theta_1\dot\theta_2\sin(\theta_2-\theta_1)\delta\theta_2 \\ & + mL_1\left[L_1\dot\theta_1 + \frac{L_2}{2}\dot\theta_2\cos(\theta_2-\theta_1)\right]\delta\dot\theta_1 + mL_2\left[\frac{L_1}{2}\dot\theta_1\cos(\theta_1-\theta_1) + \frac{L_2}{3}\dot\theta_2\right]\delta\dot\theta_2\end{aligned} \tag{g}$$

and the variation in the potential energy is simply

$$\delta V = mg\left(L_1\sin\theta_1\delta\theta_1 + \frac{L_2}{2}\sin\theta_2\delta\theta_2\right) \tag{h}$$

Example from Meirovitch

Inserting Eqs. (e)–(g) into Eq. (6.31) and collecting terms, we have

$$\int_{t_1}^{t_2} (\delta T - \delta V + \delta W_{nc}) dt = \int_{t_1}^{t_2} \left\{ \left[\frac{mL_1L_2}{2} \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2 - \theta_1) - mgL_1 \sin\theta_1 \right.\right.$$
$$\left. + FL_1 \cos\theta_1 \right] \delta\theta_1 + \left[-\frac{mL_1L_2}{2} \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2 - \theta_1) - \frac{mgL_2}{2} \sin\theta_2 + FL_2 \cos\theta_2 \right] \delta\theta_2$$
$$\left. + mL_1 \left[L_1 \dot{\theta}_1 + \frac{L_2}{2} \dot{\theta}_2 \cos(\theta_2 - \theta_1) \right] \delta\dot{\theta}_1 + mL_2 \left[\frac{L_1}{2} \dot{\theta}_1 \cos(\theta_2 - \theta_1) + \frac{L_2}{3} \dot{\theta}_2 \right] \delta\dot{\theta}_2 \right\} dt = 0 \tag{i}$$

Example from Meirovitch

At this point, we observe that Eq. (i) involves both the virtual displacements $\delta\theta_1$ and $\delta\theta_2$ and the virtual velocities $\delta\dot{\theta}_1$ and $\delta\dot{\theta}_2$, and only the virtual displacements are arbitrary. Hence, before we can derive the equations of motion, we must transform the terms in $\delta\dot{\theta}_1$ and $\delta\dot{\theta}_2$ into terms in $\delta\theta_1$ and $\delta\theta_2$, respectively. To this end, we carry out the following integrations by parts:

$$\int_{t_1}^{t_2} mL_1\left[L_1\dot{\theta}_1 + \frac{L_2}{2}\dot{\theta}_2\cos(\theta_2-\theta_1)\right]\delta\dot{\theta}_1 dt = mL_1\left[L_1\dot{\theta}_1 + \frac{L_2}{2}\dot{\theta}_2\cos(\theta_2-\theta_1)\right]\delta\theta_1\Bigg|_{t_1}^{t_2}$$

$$-\int_{t_1}^{t_2} mL_1\frac{d}{dt}\left[L_1\dot{\theta}_1 + \frac{L_2}{2}\dot{\theta}_2\cos(\theta_2-\theta_1)\right]\delta\theta_1 dt$$

$$= -\int_{t_1}^{t_2} mL_1\left[L_1\ddot{\theta}_1 + \frac{L_2}{2}\ddot{\theta}_2\cos(\theta_2-\theta_1) - \frac{L_2}{2}\dot{\theta}_2(\dot{\theta}_2-\dot{\theta}_1)\sin(\theta_2-\theta_1)\right]\delta\theta_1\, dt$$

Example from Meirovitch

$$\int_{t_1}^{t_2} mL_2\left[\frac{L_1}{2}\dot{\theta}_1\cos(\theta_2-\theta_1)+\frac{L_2}{3}\dot{\theta}_2\right]\delta\dot{\theta}_2 dt = mL_2\left[\frac{L_1}{2}\dot{\theta}_1\cos(\theta_2-\theta_1)+\frac{L_2}{3}\dot{\theta}_2\right]\delta\theta_2\Big|_{t_1}^{t_2}$$

$$-\int_{t_1}^{t_2} mL_2\frac{d}{dt}\left[\frac{L_1}{2}\dot{\theta}_1\cos(\theta_2-\theta_1)+\frac{L_2}{3}\dot{\theta}_2\right]\delta\theta_2 dt$$

$$=-\int_{t_1}^{t_2} mL_2\left[\frac{L_1}{2}\ddot{\theta}_1\cos(\theta_2-\theta_1)-\frac{L_1}{2}\dot{\theta}_1(\dot{\theta}_2-\dot{\theta}_1)\sin(\theta_2-\theta_1)+\frac{L_2}{3}\ddot{\theta}_2\right]\delta\theta_2 dt$$

where we recalled the auxiliary conditions $\delta\theta_1=\delta\theta_2=0$ at $t=t_1,\ t_2$. Introducing Eqs. (j) in Eq. (i) and collecting terms, we can write

$$\int_{t_1}^{t_2}\left\{-\left[mL_1^2\ddot{\theta}_1+\frac{mL_1L_2}{2}\ddot{\theta}_2\cos(\theta_2-\theta_1)-\frac{mL_1L_2}{2}\dot{\theta}_2^2\sin(\theta_2-\theta_1)\right.\right.$$

$$\left.+mgL_1\sin\theta_1-FL_1\cos\theta_1\right]\delta\theta_1-\left[\frac{mL_1L_2}{2}\ddot{\theta}_1\cos(\theta_2-\theta_1)+\frac{mL_2^2}{3}\ddot{\theta}_2\right.$$

$$\left.\left.+\frac{mL_1L_2}{2}\dot{\theta}_1^2\sin(\theta_2-\theta_1)+\frac{mgL_2}{2}\sin\theta_2-FL_2\cos\theta_2\right]\delta\theta_2\right\}dt=0 \qquad \text{(k)}$$

Example from Meirovitch

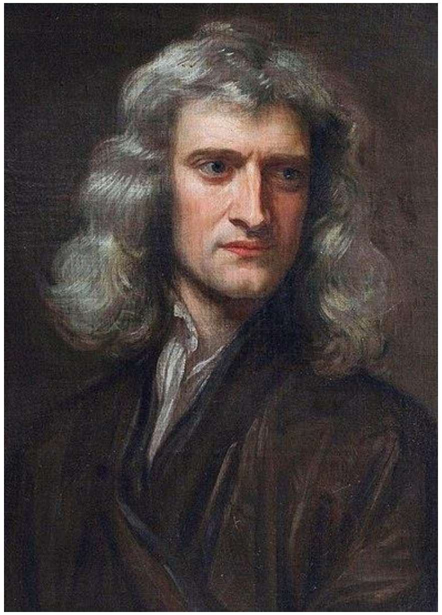
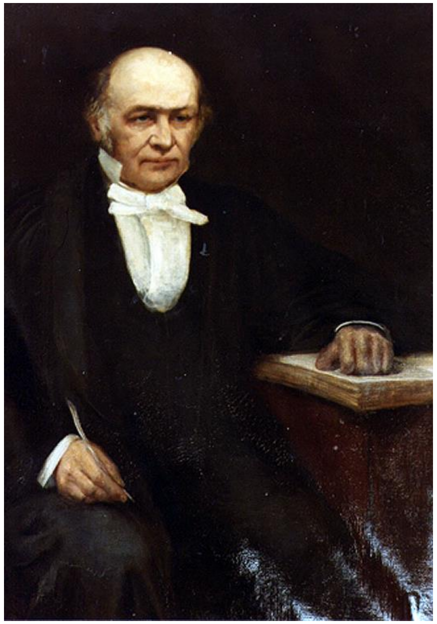
Finally, the integrand is in a form permitting the extraction of the equations of motion. To this end, we invoke the arbitrariness of $\delta\theta_1$ and $\delta\theta_2$, and assign different values to $\delta\theta_1$, while we set $\delta\theta_2 = 0$. Because the resulting equation must hold for all values of $\delta\theta_1$, we conclude that this is possible only if the coefficient of $\delta\theta_1$ is zero. A similar argument, but with the roles of $\delta\theta_1$ and $\delta\theta_2$ reversed, causes us to conclude that the coefficient of $\delta\theta_2$ must be zero as well. Hence, setting the coefficients of $\delta\theta_1$ and $\delta\theta_2$ equal to zero, we obtain the equations of motion

$$\begin{aligned} &mL_1^2\ddot{\theta}_1 + \frac{mL_1L_2}{2}\left[\ddot{\theta}_2\cos(\theta_2-\theta_1) - \dot{\theta}_2^2\sin(\theta_2-\theta_1)\right] + mgL_1\sin\theta_1 = FL_1\cos\theta_1 \\ &\frac{mL_1L_2}{2}\left[\ddot{\theta}_1\cos(\theta_2-\theta_1) + \dot{\theta}_1^2\sin(\theta_2-\theta_1)\right] + \frac{mL_2^2}{3}\ddot{\theta}_2 + \frac{mgL_2}{2}\sin\theta_2 = FL_2\cos\theta_2 \end{aligned} \tag{1}$$

Example from Meirovitch

We observe from Eqs. (l) that there are two equations of motion in the unknowns θ_1 and θ_2, as there should be for a two-degree-of-freedom system, and the equations are free of the string tension T. By contrast, Eqs. (k) of Example 1.6 are three in number and there are three unknowns, θ_1, θ_2 and T. Hence, the extended Hamilton's principle not only yields the correct number of equations of motion, but the equations themselves are not encumbered by quantities that may present no interest, such as internal forces and reaction forces. Of

Outline

The method of choice !

Lagrange’s equations

Lagrange’s equations can be derived from Hamilton’s principle. For the same generalized conditions, they yield identical equations of motion.

Lagrange’s equations are more expeditious. For instance, integration by parts can be avoided. Lagrange’s equations represent the method of choice.

Start from Hamilton's principle

$$\int_{t_1}^{t_2} (\delta L + \delta W_{nc}) dt = 0 \qquad\qquad L = T - V$$

$$\delta W_{nc} = \sum_{i=1}^{N} \vec{F}_i . \overrightarrow{\delta r_i} = \sum_{k=1}^{n} \left(\sum_{i=1}^{N} \vec{F}_i . \frac{\partial \vec{r_i}}{\partial q_k} \right) \delta q_k = \sum_{k=1}^{n} Q_k \delta q_k$$

$$Q_k = \sum_{i=1}^{N} \vec{F}_i . \frac{\partial \vec{r_i}}{\partial q_k}$$

Generalized coordinates: $\vec{r}_i = \vec{r}_i(q_1, \dots, q_n, t)$

Carry out some derivations

$$\int_{t_1}^{t_2} \left(\delta L + \sum_{k=1}^{n} Q_k \delta q_k \right) dt = 0$$

$$\Downarrow$$

$$\int_{t_1}^{t_2} \left(\sum_{k=1}^{n} \left(\frac{\partial L}{\partial q_k} \delta q_k + \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \right) + \sum_{k=1}^{n} Q_k \delta q_k \right) dt = 0$$

where $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \delta q_k \right) = \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k$

$$\Downarrow$$

$$\int_{t_1}^{t_2} \left(\sum_{k=1}^{n} \left(\frac{\partial L}{\partial q_k} \delta q_k + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \delta q_k \right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k \right) + \sum_{k=1}^{n} Q_k \delta q_k \right) dt = 0$$

And finally Lagrange's equations

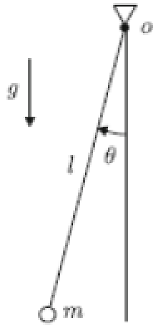
$$\int_{t_1}^{t_2} \left(\sum_{k=1}^{n} \left(\frac{\partial L}{\partial q_k} \delta q_k + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \delta q_k \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k \right) + \sum_{k=1}^{n} Q_k \delta q_k \right) dt = 0$$

$$\sum_{k=1}^{n} \underbrace{\left[\frac{\partial L}{\partial \dot{q}_k} \delta q_k \right]_{t_1}^{t_2}}_{=0} - \int_{t_1}^{t_2} \sum_{k=1}^{n} \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} - Q_k \right] \underbrace{\delta q_k}_{\text{arbitrary}} \, dt = 0$$

$$\boxed{\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = Q_k, \qquad k = 1, \ldots, n}$$

$$\boxed{\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} + \frac{\partial T}{\partial q_k} = Q_k, \qquad k = 1, \ldots, n}$$

Pendulum example: no reaction force !



$$T = \frac{1}{2}mv_m^2 = \frac{1}{2}ml^2\dot{\theta}^2$$

$$V = -mgy = -mgl\cos\theta$$

$$\boxed{L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta}$$

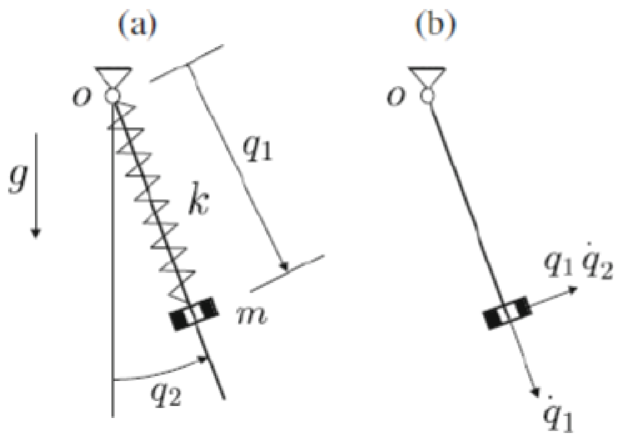
$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k$$

$$\longrightarrow \boxed{ml^2\ddot{\theta} + mgl\sin\theta = 0}$$

Pendulum with a sliding mass: 2 DOFs



angle

$$T = \frac{1}{2}m\left(\dot{q_1}^2 + q_1^2\dot{q_2}^2\right)$$

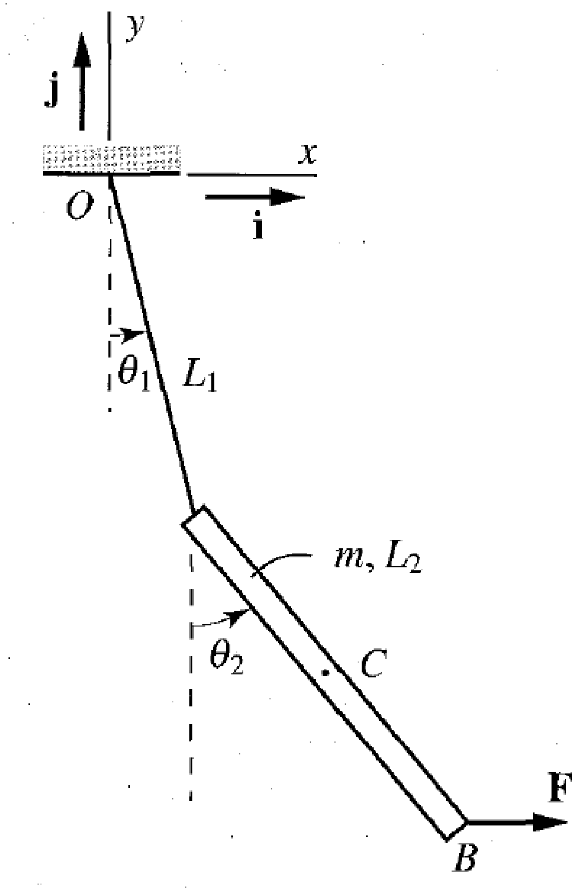
$$V = -mgq_1\cos q_2 + \frac{1}{2}kq_1^2$$

$$\boxed{L = T - V = \frac{1}{2}m\left(\dot{q_1}^2 + q_1^2\dot{q_2}^2\right) + mgq_1\cos q_2 - \frac{1}{2}kq_1^2}$$

$$q_1 \longrightarrow m\ddot{q}_1 - mq_1\dot{q_2}^2 - mg\cos q_2 + kq_1 = 0$$

$$q_2 \longrightarrow \frac{d}{dt}\left(mq_1^2\dot{q}_2\right) + mgq_1\sin q_2 = 0$$

Example from Meirovitch



Example from Meirovitch

As in Example 6.1, we use the angles θ_1 and θ_2 (see Fig. 1.10a) as generalized coordinates, $q_1 = \theta_1$, $q_2 = \theta_2$, so that Lagrange's equations, Eqs. (6.42), take the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_k}\right) - \frac{\partial T}{\partial \theta_k} + \frac{\partial V}{\partial \theta_k} = \Theta_k, \; k = 1, 2 \tag{a}$$

where Θ_k $(k = 1, 2)$ are the generalized nonconservative forces. From Example 6.1, we obtain the kinetic energy

$$T = \frac{1}{2} m \left[L_1^2 \dot{\theta}_1^2 + L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_2 - \theta_1) + \frac{L_2^2}{3} \dot{\theta}_2^2 \right] \tag{b}$$

the potential energy

$$V = mg \left[L_1 (1 - \cos\theta_1) + \frac{L_2}{2} (1 - \cos\theta_2) \right] \tag{c}$$

and the virtual work of the nonconservative forces

$$\overline{\delta W}_{nc} = F L_1 \cos\theta_1 \delta\theta_1 + F L_2 \cos\theta_2 \delta\theta_2$$

Example from Meirovitch

The derivatives with respect to the angular velocities are as follows:

$$\frac{\partial T}{\partial \dot{\theta}_1} = mL_1^2\dot{\theta}_1 + \frac{mL_1L_2}{2}\dot{\theta}_2\cos(\theta_2 - \theta_1)$$

$$\frac{\partial T}{\partial \dot{\theta}_2} = \frac{mL_1L_2}{2}\dot{\theta}_1\cos(\theta_2 - \theta_1) + \frac{mL_2^2}{3}\dot{\theta}_2 \tag{e}$$

so that

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) = mL_1^2\ddot{\theta}_1 + \frac{mL_1l_2}{2}\left[\ddot{\theta}_2\cos(\theta_2 - \theta_1) - \dot{\theta}_2(\dot{\theta}_2 - \dot{\theta}_1)\sin(\theta_2 - \theta_1)\right]$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) = \frac{mL_1L_2}{2}\left[\ddot{\theta}_1\cos(\theta_2 - \theta_1) - \dot{\theta}_1(\dot{\theta}_2 - \dot{\theta}_1)\sin(\theta_2 - \theta_1)\right] + \frac{mL_2^2}{3}\ddot{\theta}_2 \tag{f}$$

Moreover, the derivatives with respect to the angular displacements are

$$\frac{\partial T}{\partial \theta_1} = \frac{mL_1L_2}{2}\dot{\theta}_1\dot{\theta}_2\sin(\theta_2 - \theta_1), \quad \frac{\partial T}{\partial \theta_2} = -\frac{mL_1L_2}{2}\dot{\theta}_1\dot{\theta}_2\sin(\theta_2 - \theta_1)$$

$$\frac{\partial V}{\partial \theta_1} = mgL_1\sin\theta_1, \quad \frac{\partial V}{\partial \theta_2} = \frac{mgL_2}{2}\sin\theta_2 \tag{g}$$

Example from Meirovitch

In addition, the generalized nonconservative forces are recognized as the coefficients of $\delta\theta_1$ and $\delta\theta_2$ in the virtual work, Eq. (d), or

$$\Theta_1 = FL_1 \cos\theta_1, \ \Theta_2 = FL_2 \cos\theta_2 \tag{h}$$

Inserting Eqs. (f)–(h) into Eqs. (a), we obtain the desired Lagrange's equations

$$\begin{aligned} &mL_1^2\ddot{\theta}_1 + \frac{mL_1L_2}{2}[\ddot{\theta}_2\cos(\theta_2-\theta_1) - \dot{\theta}_2^2\sin(\theta_2-\theta_1)] + mgL_1\sin\theta_1 = FL_1\cos\theta_1 \\ &\frac{mL_1L_2}{2}[\ddot{\theta}_1\cos(\theta_2-\theta_1) + \dot{\theta}_1^2\sin(\theta_2-\theta_1)] + \frac{mL_2^2}{3}\ddot{\theta}_2 + \frac{mgL_2}{2}\sin\theta_2 = FL_2\cos\theta_2 \end{aligned} \tag{i}$$

We observe that Eqs. (i) just derived are identical to Eqs. (l) of Example 6.1, obtained by the extended Hamilton's principle, as was to be expected. Clearly, Lagrange's equations reduce the derivation of the equations of motion to a routine series of differentiations.

Lagrange equations with constraints

Consider *n* generalized coordinates which are not independent

Constraint equations

$$\sum_k a_{lk}\delta q_k = 0 \qquad l = 1, ..., m \longrightarrow \text{n-m degrees of freedom}$$

$$\sum_k \left[\frac{\partial L}{\partial \dot{q}_k}\delta q_k\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_k \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} - Q_k\right] \delta q_k dt = 0$$

$\not\downarrow$ δq_k Not independent $\longrightarrow$ Lagrange multipliers

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k$$

Lagrange equations with constraints

Lagrange multipliers

$$\sum_{l=1}^{m} \lambda_l \left(\sum_{k=1}^{n} a_{lk} \delta q_k \right) = \sum_{k=1}^{n} \delta q_k \left(\sum_{l=1}^{m} \lambda_l a_{lk} \right) = 0$$

$$\longrightarrow \quad \int_{t_1}^{t_2} \sum_{k=1}^{n} \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} - Q_k - \sum_{l=1}^{m} \lambda_l a_{lk} \right] \delta q_k dt = 0$$

$$\boxed{\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = Q_k + \sum_{l=1}^{m} \underline{\lambda_l a_{lk}}} \qquad k = 1, ..., n$$

Generalized contraint forces

Lagrange equations with constraints

n+m unknowns $q_k, \quad \lambda_l$

n+m equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k + \sum_{l=1}^{m} \lambda_l a_{lk} \qquad k = 1, ..., n$$

$$\sum_{k=1}^{n} a_{lk} \delta q_k = 0 \qquad l = 1, ..., m$$

constraint equations

Lagrange equations with constraints

Holonomic system

$$f_l(q_1, ..., q_n, t) = 0 \longrightarrow \delta f_l = \sum_{k=1}^{n} \frac{\partial f_l}{\partial q_k} \delta q_k = 0 \longrightarrow a_{lk} = \frac{\partial f_l}{\partial q_k}$$

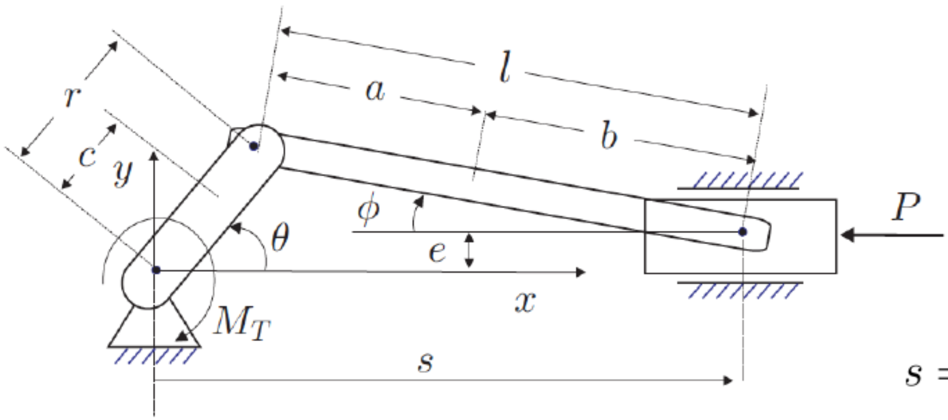
Non-holonomic system

$$\sum_{k=1}^{n} a_{lk} \delta q_k + a_{l0} dt = 0$$

Lagrange vs. Newton

- Newton
  - 6 equations per rigid body
  - Constraints appear as forces
- Lagrange
  - n coordinates
  - m constraints / Lagrange multipliers
  - m+n equations of motion

Piston engine



Coordinates

θ, ϕ, s n=3

Constraints

m=2

$$s = r\cos\theta + l\cos\phi$$
$$e = r\sin\theta - l\sin\phi$$

$$\lambda = \frac{r}{l} \longrightarrow \quad s = r\left(\cos\theta + \frac{1}{\lambda}\cos\phi\right)$$
$$\sin\phi = \left(\lambda\sin\theta - \frac{e}{l}\right)$$

No gravity so that the potential energy is zero.

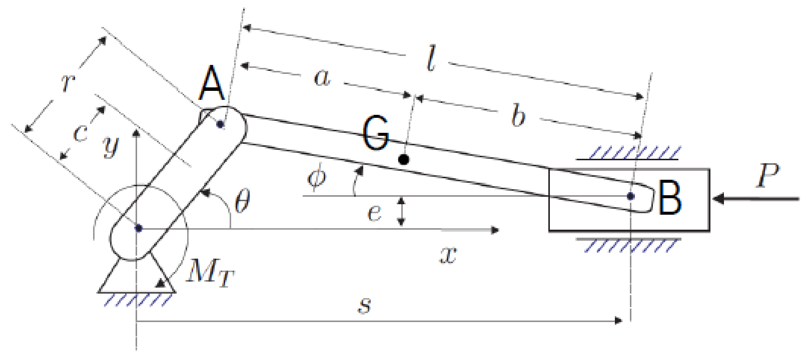
Piston engine: kinetic energy

Objective: write the kinetic energy as a function of θ only:

$$T_1 = \frac{1}{2}\left(J_1 + m_1 c^2\right)\dot{\theta}^2 = \frac{1}{2} J_0 \dot{\theta}^2$$

$$T_2 = \frac{1}{2} m_2 v_G^2 + \frac{1}{2} J_2 \dot{\phi}^2$$

$$T_3 = \frac{1}{2} m_3 \dot{s}^2$$



Distribute mass at points A and B

$$m_A = \frac{m_2 b}{l} \qquad m_B = \frac{m_2 a}{l}$$

$$T_2 = \frac{1}{2} m_A \left(r\dot{\theta}\right)^2 + \frac{1}{2} m_B \dot{s}^2 + J_{AB}\dot{\phi}^2$$

$$J_{AB} = J_2 - m_2 ab$$

Piston engine: kinetic energy

Define $k_\phi = \dfrac{\dot\phi}{\dot\theta}$ $\qquad k_s = \dfrac{\dot s}{\dot\theta}$

$$\sin\phi = \left(\lambda\sin\theta - \frac{e}{l}\right) \longrightarrow \cos\phi\dot\phi = \lambda\cos\theta\dot\theta$$

$$\longrightarrow \frac{\dot\phi}{\dot\theta} = \frac{\lambda\cos\theta}{\cos\phi} = \frac{\lambda\cos\theta}{\sqrt{1-\left(\lambda\sin\theta - \frac{e}{l}\right)^2}}$$

$$s = r\left(\cos\theta + \frac{1}{\lambda}\cos\phi\right)$$

$$\longrightarrow \dot s = -r\left(\sin\theta\dot\theta + \frac{1}{\lambda}\sin\phi\dot\phi\right)$$

$$\longrightarrow \frac{\dot s}{\dot\theta} = -r\left(\sin\theta + \frac{k_\phi}{\lambda}\sin\phi\right)$$

$$\longrightarrow \left|\begin{aligned} T_1 &= \frac{1}{2}\left(J_1 + m_1c^2\right)\dot\theta^2 = \frac{1}{2}J_0\dot\theta^2 \\ T_2 &= \frac{1}{2}m_A\left(r\dot\theta\right)^2 + \frac{1}{2}m_B\dot s^2 + J_{AB}\dot\phi^2 = \left(\frac{1}{2}m_Ar^2 + \frac{1}{2}m_Bk_s^2 + J_{AB}k_\phi^2\right)\dot\theta^2 \\ T_3 &= \frac{1}{2}m_3\dot s^2 = \frac{1}{2}m_3k_s^2\dot\theta^2 \end{aligned}\right.$$

The kinetic energy is a function of θ only ✓

Piston engine: simplify the inertia term

$$T = T_1 + T_2 + T_3 = \frac{1}{2} I(\theta)\dot{\theta}^2$$

$$\boxed{I(\theta) = J_0 + m_A r^2 + (m_3 + m_B)\, k_s^2 + J_{AB} k_\phi^2}$$

Assume e=0, limit to first order terms

$$I(\theta) \simeq J_0 + m_A r^2 + (m_3 + m_B)\, r^2 \sin^2\theta + J_{AB}\lambda^2 \cos^2\theta \quad = \boxed{A - B\cos 2\theta}$$

$$A = J_0 + m_A r^2 + \frac{1}{2}\left[(m_3 + m_B)\, r^2 + J_{AB}\lambda^2\right]$$

$$B = \frac{1}{2}\left[(m_3 + m_B)\, r^2 - J_{AB}\lambda^2\right]$$

Piston engine: Lagrange equations

External forces:

$$\delta W_{nc} = Q_\theta \delta_\theta = -P\delta s - M_T \delta\theta = -\left(Pk_s + M_T\right)\delta\theta \qquad k_s = \frac{\dot{s}}{\dot{\theta}} = \frac{\delta s}{\delta\theta}$$

Lagrange equation $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\theta}}\right) - \dfrac{\partial L}{\partial \theta} = Q_\theta \qquad L = T = \dfrac{1}{2} I(\theta)\dot{\theta}^2$

$$I(\theta)\ddot{\theta} + \frac{dI}{dt}\dot{\theta} - \frac{1}{2}\frac{dI}{d\theta}\dot{\theta}^2 = -Pk_s - M_T \qquad \frac{dI}{dt} = \frac{dI}{d\theta}\frac{d\theta}{dt}$$

$$\boxed{I(\theta)\ddot{\theta} + \frac{1}{2}\frac{dI}{d\theta}\dot{\theta}^2 = -Pk_s - M_T}$$

Piston engine: constant speed

Constant speed $\dot{\theta} = \omega_0$ and P= 0

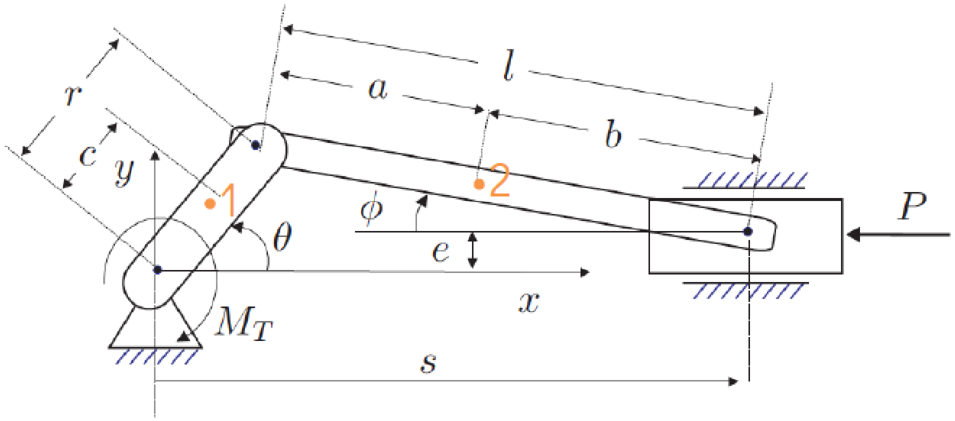
$$I(\theta)\ddot{\theta} + \frac{1}{2}\frac{dI}{d\theta}\dot{\theta}^2 = -Pk_s - M_T \longrightarrow \frac{1}{2}\frac{dI}{d\theta}\omega_0^2 = -M_T$$

$$I(\theta) = A - B\cos 2\theta$$

$$\boxed{B\sin 2\theta\omega_0^2 = -M_T}$$

It gives the moment to apply so that the system moves at constant speed.

Piston engine: calculate reaction forces



Generalized coordinates:

$$x_1, y_1, x_2, y_2, \theta, \phi, s$$

$$\begin{aligned}
s - r\cos\theta - l\cos\phi &= 0 = f_1 \\
e - r\sin\theta + l\sin\phi &= 0 = f_2 \\
x_1 - c\cos\theta &= 0 = f_3 \\
y_1 - c\sin\theta &= 0 = f_4 \\
x_2 - r\cos\theta - a\cos\phi &= 0 = f_5 \\
y_2 - b\sin\phi - e &= 0 = f_6
\end{aligned}$$

Piston engine: equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k + \sum_{l=1}^{m} \lambda_l a_{lk} \Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = Q_k + \sum_{l=1}^{m} \lambda_l a_{lk}}$$

$$\begin{aligned} T = &\frac{1}{2} m_1 \left(\dot{x_1}^2 + \dot{y_1}^2\right) + \frac{1}{2} J_1 \dot{\theta}^2 \\ &+ \frac{1}{2} m_2 \left(\dot{x_2}^2 + \dot{y_2}^2\right) + \frac{1}{2} J_2 \dot{\phi}^2 \\ &+ \frac{1}{2} m_3 \dot{s}^2 \end{aligned}$$

$$\begin{aligned} s - r\cos\theta - l\cos\phi &= 0 = f_1 \\ e - r\sin\theta + l\sin\phi &= 0 = f_2 \\ x_1 - c\cos\theta &= 0 = f_3 \\ y_1 - c\sin\theta &= 0 = f_4 \\ x_2 - r\cos\theta - a\cos\phi &= 0 = f_5 \\ y_2 - b\sin\phi - e &= 0 = f_6 \end{aligned}$$

$$x_1 (k = 1): \quad \frac{d}{dt}(m_1 \dot{x_1}) = \sum_{l=1}^{6} a_{l1} \lambda_l$$

$$a_{l1} = \frac{\partial f_l}{\partial x_1} \longrightarrow$$ All = 0 except $a_{31} = 1$

$$\longrightarrow \boxed{m_1 \ddot{x_1} = \lambda_3}$$

Piston engine: equation of motion

$$\begin{aligned} m_1\ddot{x}_1 &= \lambda_3 \\ m_1\ddot{y}_1 &= \lambda_4 \\ m_2\ddot{x}_2 &= \lambda_5 \\ m_2\ddot{y}_2 &= \lambda_6 \end{aligned}$$

$$\begin{aligned} s - r\cos\theta - l\cos\phi &= 0 = f_1 \\ e - r\sin\theta + l\sin\phi &= 0 = f_2 \\ x_1 - c\cos\theta &= 0 = f_3 \\ y_1 - c\sin\theta &= 0 = f_4 \\ x_2 - r\cos\theta - a\cos\phi &= 0 = f_5 \\ y_2 - b\sin\phi - e &= 0 = f_6 \end{aligned}$$

$$J_1\ddot{\theta} = -M_T + (\lambda_1 + \lambda_5)\, r\sin\theta - \lambda_2 r\cos\theta + c\,(\lambda_3\sin\theta - \lambda_4\cos\theta)$$

$$J_2\ddot{\phi} = l\,(\lambda_1\sin\phi + \lambda_2\cos\phi) + \lambda_5 a\sin\phi - \lambda_6 b\cos\phi$$

$$m_3\ddot{s}_3 = \lambda_1 - P$$

Piston engine: physical interpretation of multipliers

Newton's second law :

$$m_1\ddot{x}_1 = X_{01} + X_{12}$$

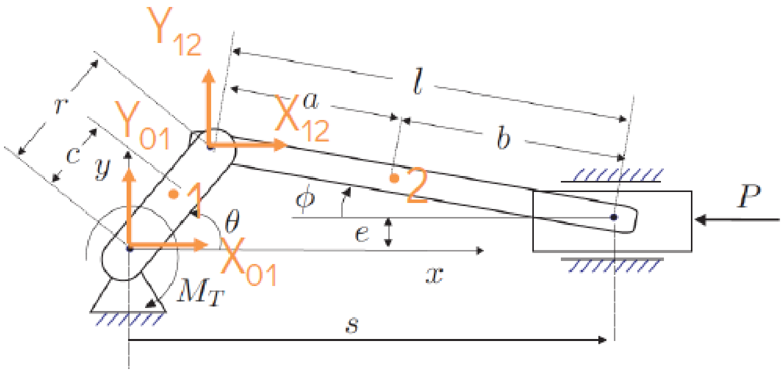
$$m_1\ddot{y}_1 = Y_{01} + Y_{12}$$

Lagrange:

$$\begin{aligned} m_1\ddot{x}_1 &= \lambda_3 \\ m_1\ddot{y}_1 &= \lambda_4 \end{aligned}$$

$\longrightarrow$ $\lambda_3 = X_{01} + X_{12}$

$\longrightarrow$ $\lambda_4 = Y_{01} + Y_{12}$

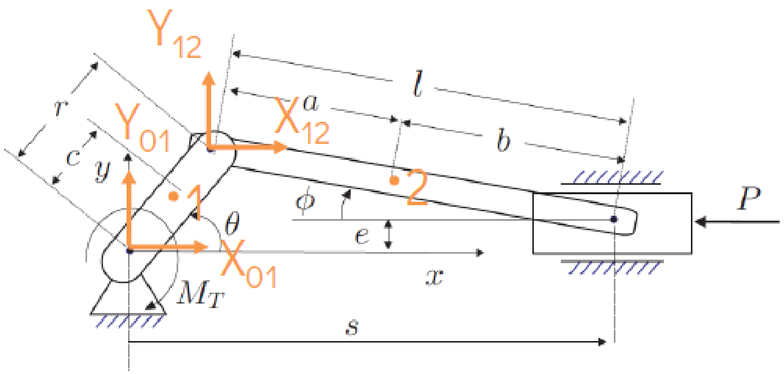


Piston engine: physical interpretation of multipliers

Theorème du moment cinétique vs. Lagrange

$$J_1\ddot{\theta} = -M_T + X_{01} c \sin\theta - Y_{01} c \cos\theta - X_{12}(r-c)\sin\theta + Y_{12}(r-c)\cos\theta$$

$$J_1\ddot{\theta} = -M_T + (\lambda_1 + \lambda_5)\, r \sin\theta - \lambda_2 r \cos\theta + c\,(\lambda_3 \sin\theta - \lambda_4 \cos\theta)$$



$$-X_{12} = \lambda_1 + \lambda_5$$

$$Y_{12} = -\lambda_2$$

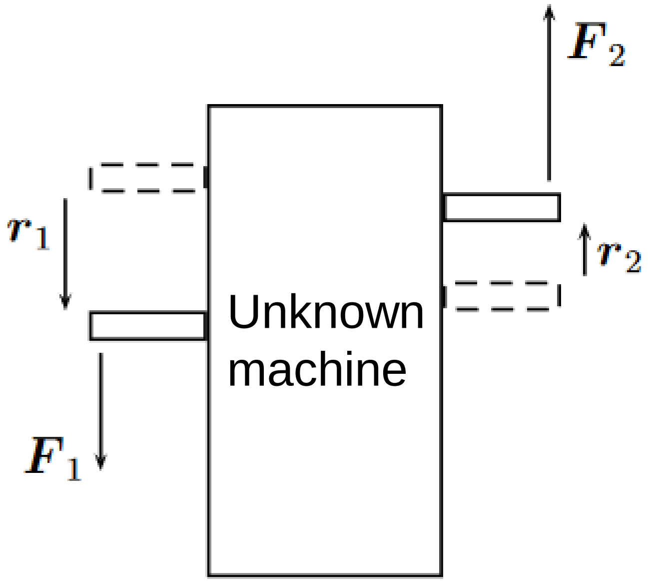
Lagrange multipliers are combinations of reaction forces.

Summary from Meirovitch

Newtonian mechanics formulates the equations of motion in terms of physical coordinates and forces, which are in general vector quantities. It requires one free-body diagram for each mass and it includes reaction forces and constraint forces in the equations of motion. These forces play the role of unknowns, which makes it necessary to work with more equations of motion than the number of degrees of freedom of the system. As a result, as the number of degrees of freedom increases, Newtonian mechanics rapidly loses its appeal as a way of deriving equations of motion.

Analytical mechanics, or Lagrangian mechanics, does not have the disadvantages cited above, and must be regarded as the method of choice for deriving equations of motion for multi-degree-of-freedom systems, as well as for distributed-parameter systems. It permits the derivation of all the equations of motion from three scalar quantities, namely, the kinetic energy, potential energy and virtual work of the nonconservative forces. It does not require free-body diagrams, and in fact it considers the system as a whole, rather than the individual components. As a result, reaction forces and constraint forces do not appear in the formulation, and the number of equations of motion coincides with the number of degrees of freedom. The process of deriving the equations of motion is rendered almost routine by the use of Lagrange's equations.

Summary from Meirovitch



Equilibrium conditions: $\boldsymbol{F}_1 \cdot \boldsymbol{r}_1 + \boldsymbol{F}_2 \cdot \boldsymbol{r}_2 = 0$

Not solvable by Newton !