Cinématique et dynamique des machines

University of Brussels, Belgium

Gaëtan Kerschen

University of Liège

Belgium

L03 **1DOF**

Course objectives

Given input → Mechanical system → Predict the output

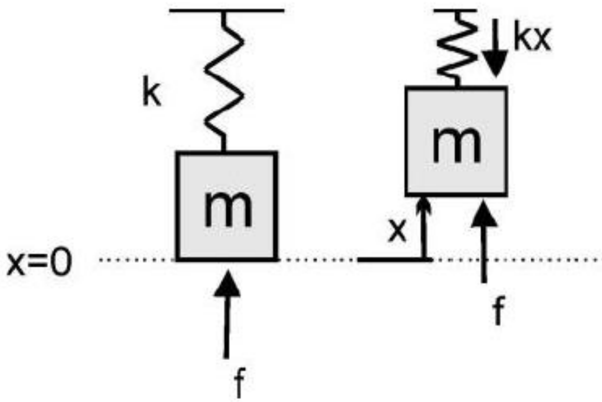
→ Write down the equations of motion $m\ddot{x} + c\dot{x} + kx = f(t)$

→ Calculate the response analytically or numerically $x = 0.19 \sin 2.3t$

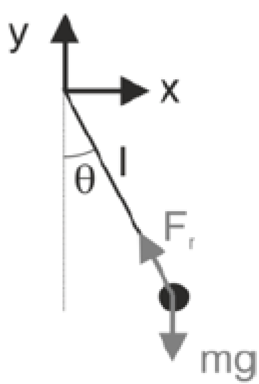
→ Is the structure safe ?

Focus on systems with 1DOF

A 1DOF system is a system whose configuration at each time instant can be described by a single variable.



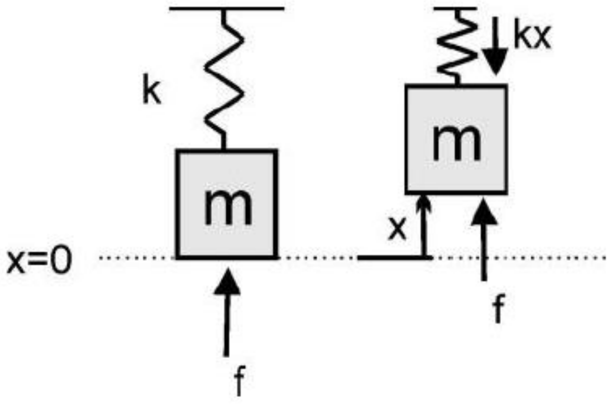
$$\boxed{m\ddot{x} + kx = f}$$



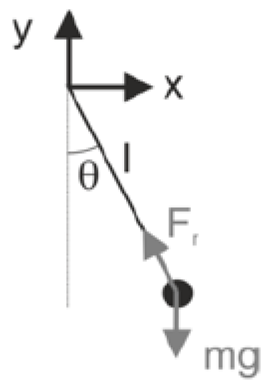
$$\boxed{ml\ddot{\theta} + mg\sin\theta = 0}$$

Focus on linear systems

Linear, second-order ordinary differential equation with constant coefficients



$$\boxed{m\ddot{x} + kx = f}$$ ✓

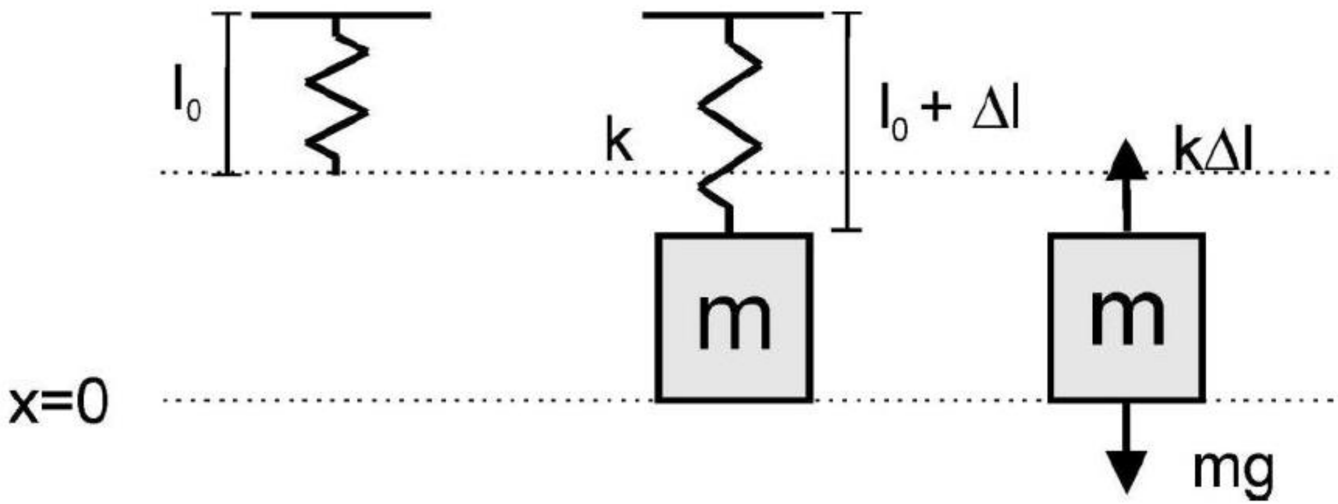


$$\boxed{ml\ddot{\theta} + mg\sin\theta = 0}$$ ✗

$$\boxed{ml\ddot{\theta} + mg\theta = 0}$$ ✓

Small displacements

Effect of gravity on spring systems ?



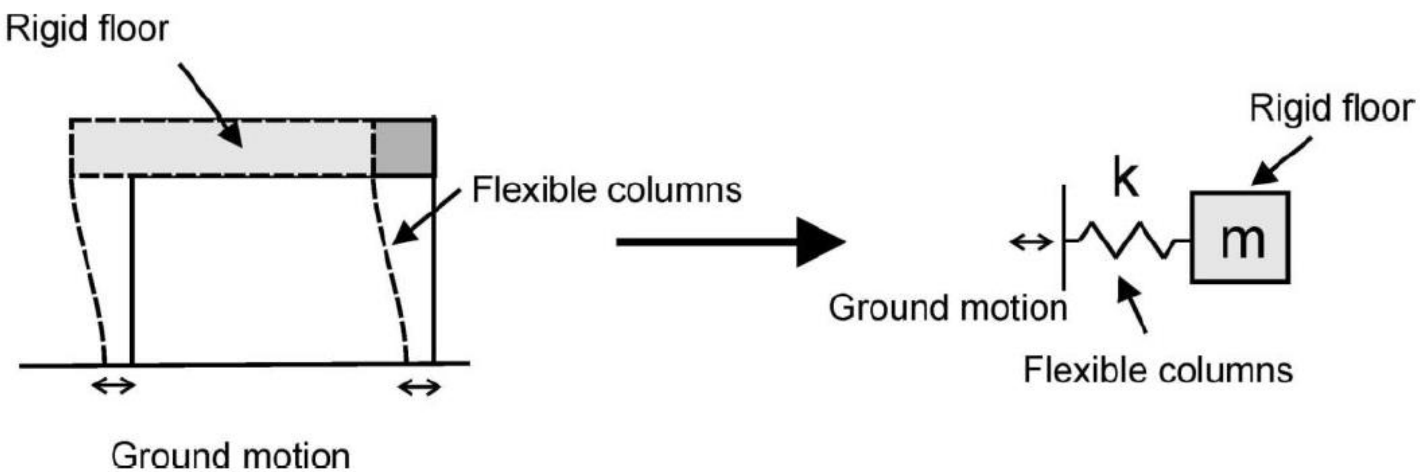
$mg = k\Delta l$ Static equilibrium

$m\ddot{x} = -k(\Delta l + x) + mg \;\rightarrow m\ddot{x} + kx = 0$ Dynamic equilibrium

No need to account for the effect of gravity in the equations of motion.

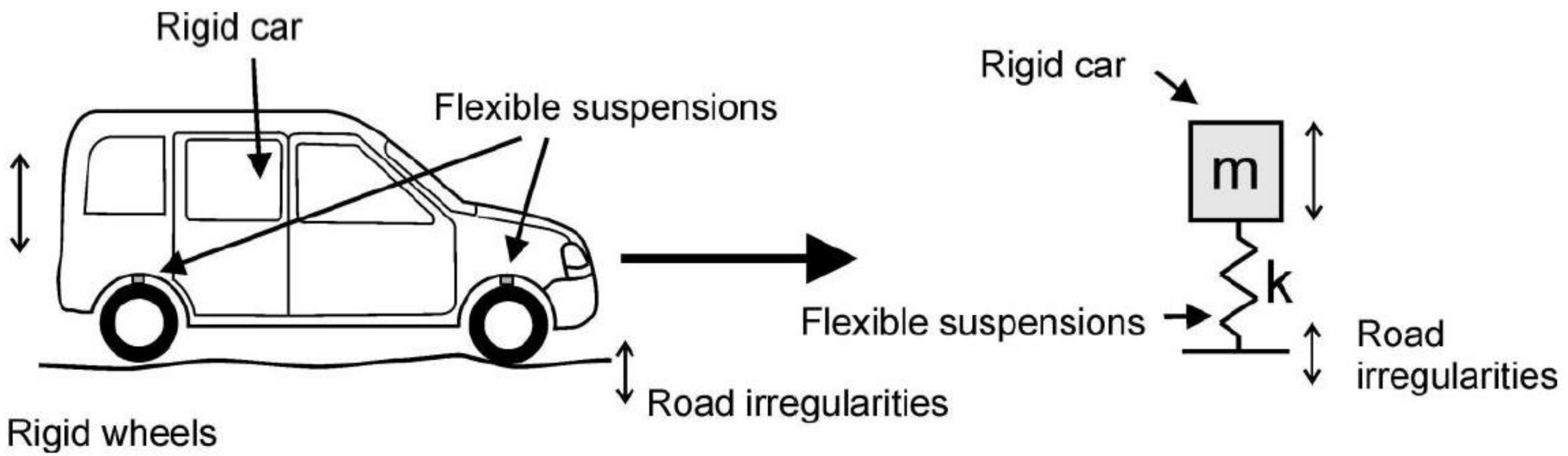
Motivation for studying 1DOF systems

An approximation of reality ! But it can sometimes be a very useful approximation…



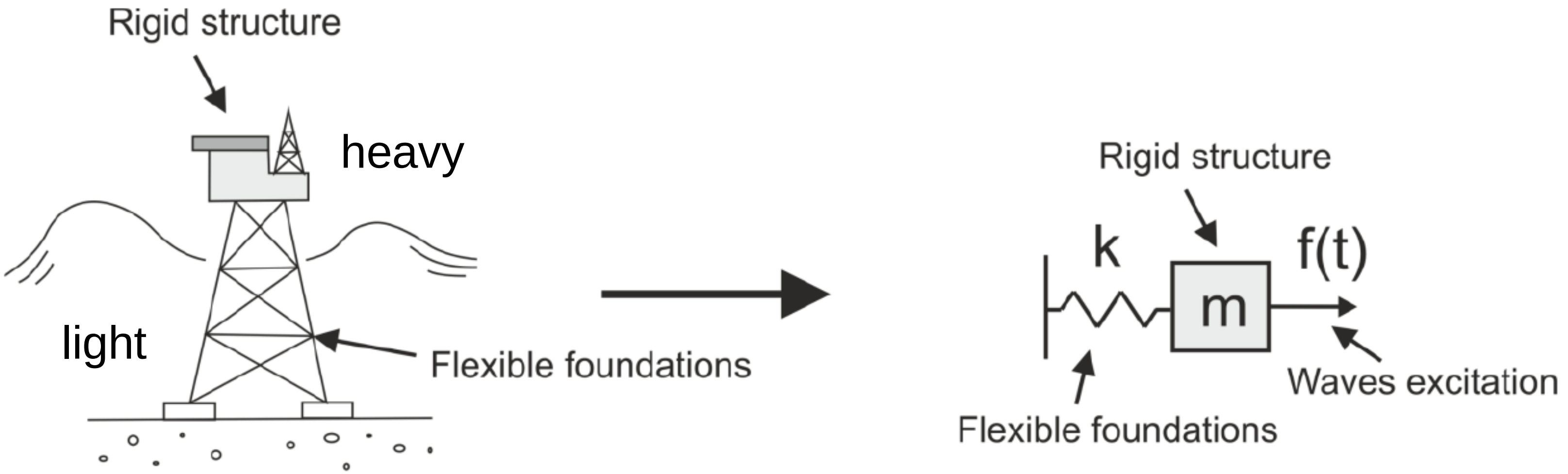
One-story building

Motivation for studying 1DOF systems



Vertical motion of a rigid car

Motivation for studying 1DOF systems



Offshore platform

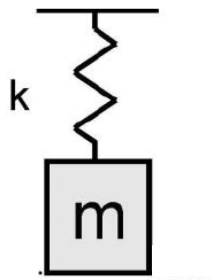
Outline

Undamped, unforced system

Damped, unforced system

Harmonically-forced system

Undamped, unforced system



$$m\ddot{x} + kx = 0$$

What is the solution to this equation ? $x = x(t)$?

Assumption of harmonic motion

$$m\ddot{x} + kx = 0$$

$$x = \boxed{A}\cos\boxed{\omega}t$$

? ?

$$-mA\omega^2\cos\omega t + kA\cos\omega t = 0$$

$$\boxed{\omega_n = \sqrt{\frac{k}{m}}}$$

Natural frequency in rad/s

But how do we determine A?

Initial conditions

$$m\ddot{x} + kx = 0 \quad \text{with} \quad x(0) = x_0$$

$$\Downarrow$$

$$x = A\cos\omega t$$

$$\Downarrow$$

$$x = x_0 \cos\omega t$$

What if $\dot{x}(0) = \dot{x}_0$ *?*

$$\dot{x} = -\omega A \sin\omega t \Rightarrow \dot{x}(0) = 0$$

More general solution

$$m\ddot{x} + kx = 0 \quad \text{with} \quad x(0) = x_0, \dot{x}(0) = \dot{x}_0$$

$$\boxed{x = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t}$$

$$\boxed{\omega_n = \sqrt{\frac{k}{m}}}$$

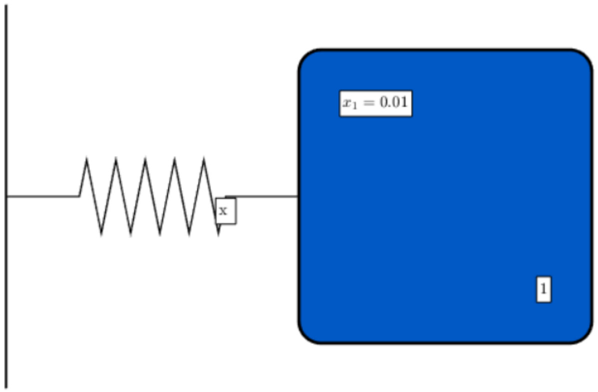
$$x = X \cos(\omega_n t - \varphi) = X \cos \varphi \cos \omega_n t + X \sin \varphi \sin \omega_n t$$

$$x_0 = X \cos \varphi$$

$$\dot{x}_0/\omega_n = X \sin \varphi$$

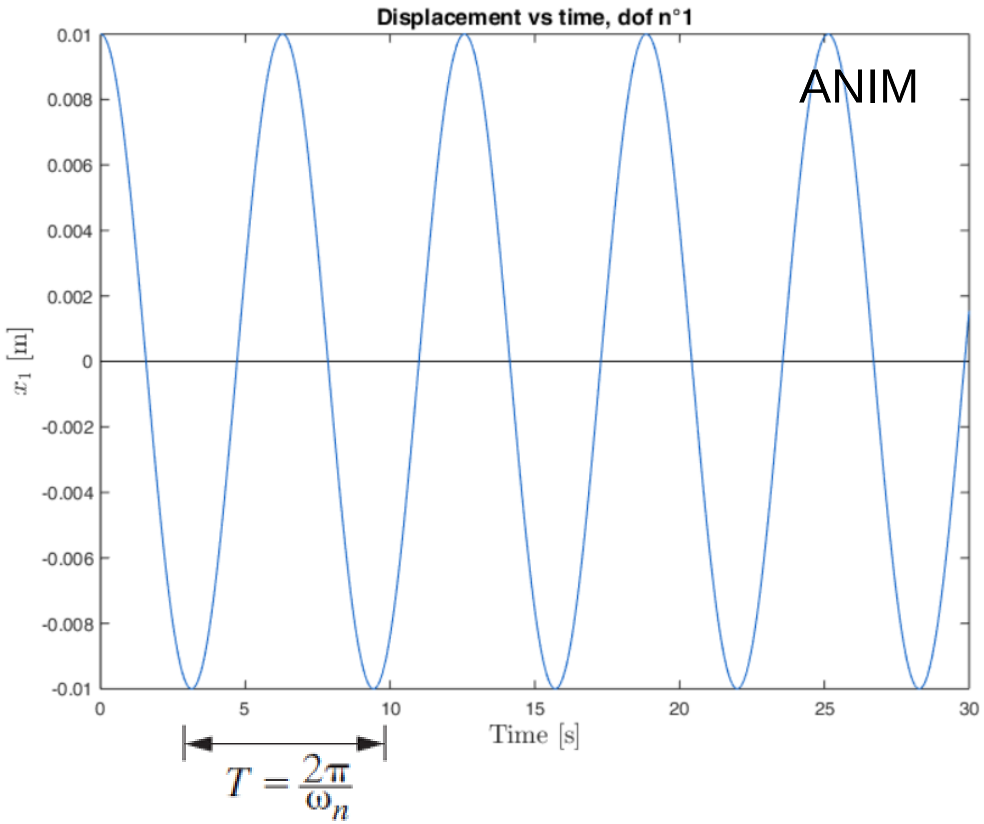
$$X = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2} \qquad \tan \varphi = \frac{\dot{x}_0}{\omega_n x_0}$$

Numerical example

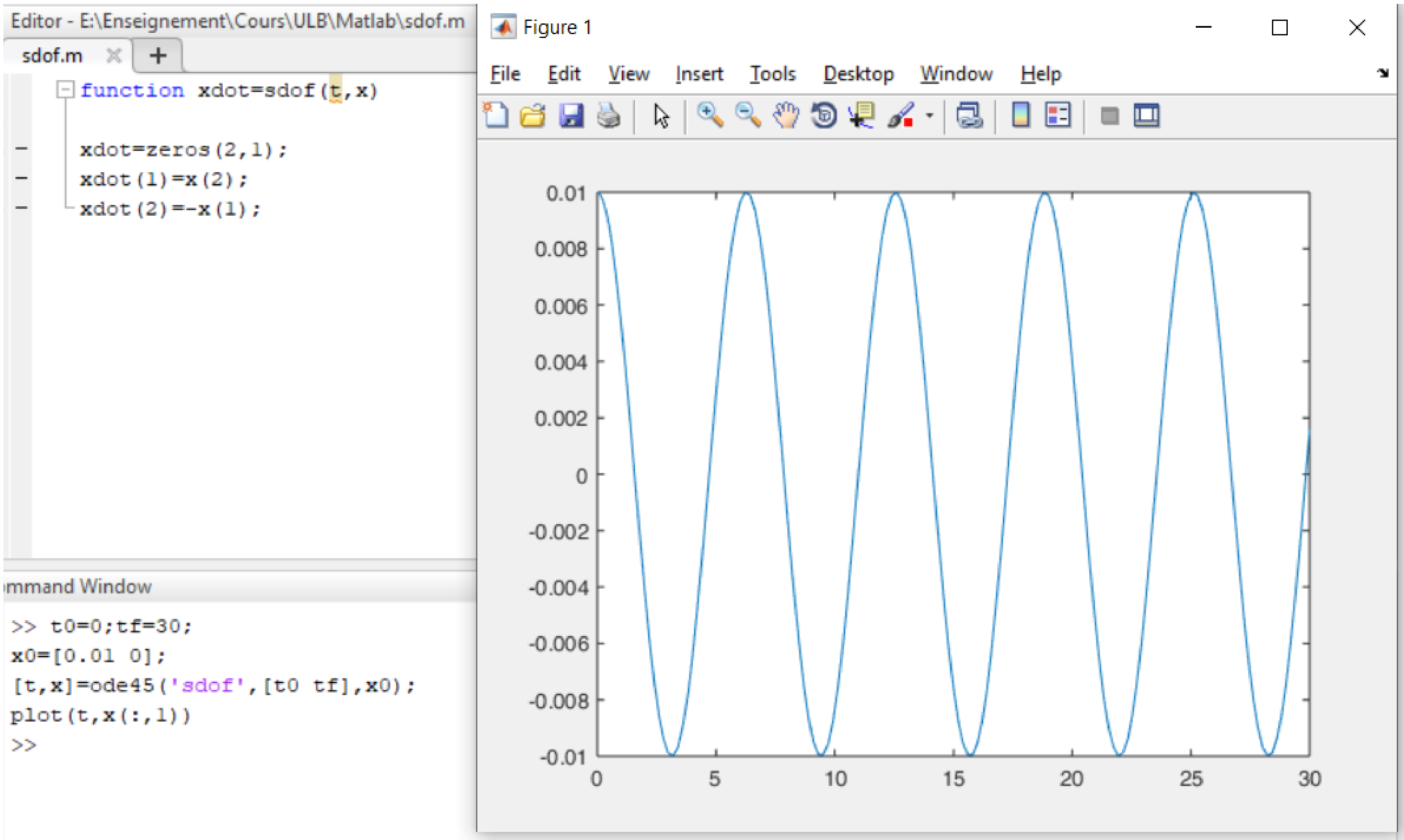


$$\omega_n = \sqrt{\frac{k}{m}} = 1\,rad/s = 0.16\,Hz$$

$$x = 0.01\cos t$$



Your own numerical propagation



Outline

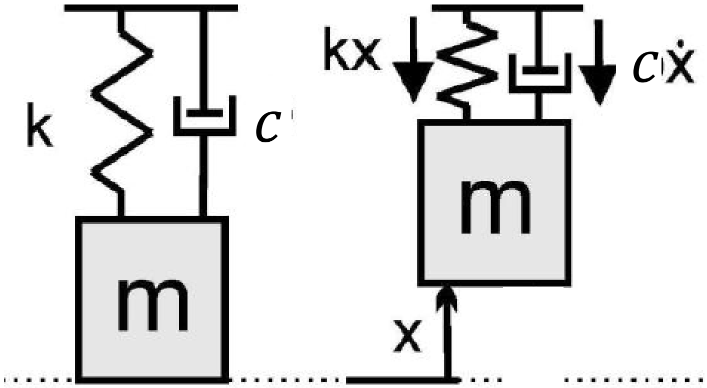
Undamped, unforced system

Damped, unforced system

Harmonically-forced system

All real-life systems possess damping

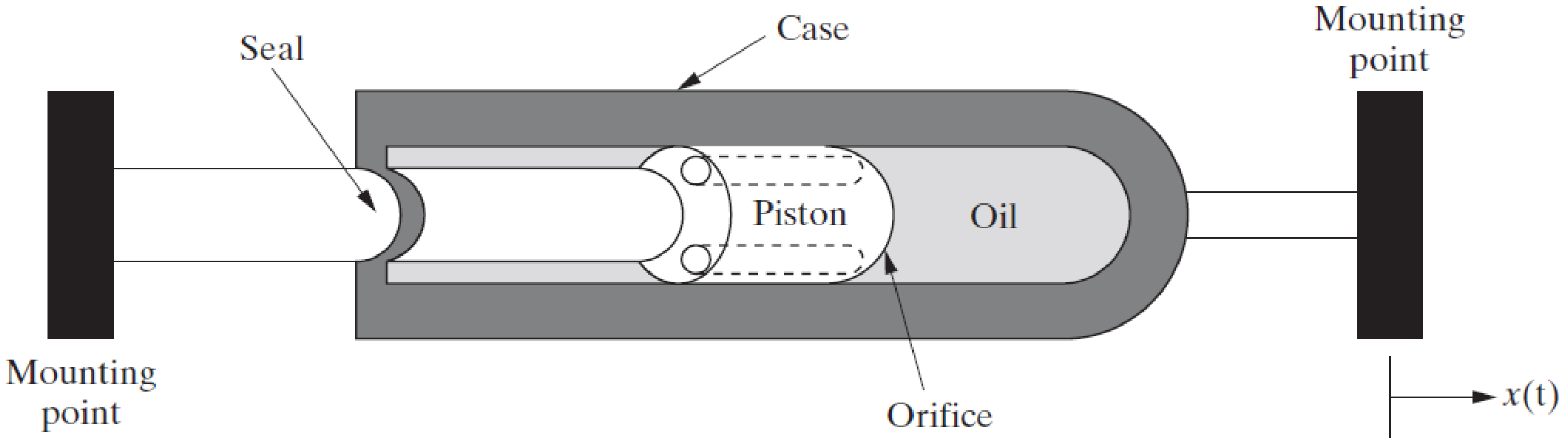
Undamped, unforced system oscillates indefinitely, which is not in accordance with reality.



$$m\ddot{x} + c\dot{x} + kx = 0$$

What is the solution to this equation ? $x = x(t)$?

Assumption of viscous damping



Dashpot that produces viscous damping

General solution

$$m\ddot{x} + c\dot{x} + kx = 0 \qquad x(t) = Ae^{\lambda t}$$

$$\Downarrow$$

$$(m\lambda^2 + c\lambda + k)Ae^{\lambda t} = 0 \;\rightarrow m\lambda^2 + c\lambda + k = 0$$

$$\Downarrow$$

$$\lambda_{1,2} = -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4km} \qquad c_{cr} = 2\sqrt{km}$$

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \qquad \zeta = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{km}}$$

$$\lambda_{1,2} = -\zeta\omega_n \pm i\omega_d \qquad \omega_d = \sqrt{1 - \zeta^2}\omega_n$$

Underdamped motion

$$c_{cr} < 2\sqrt{km}, \zeta < 1$$

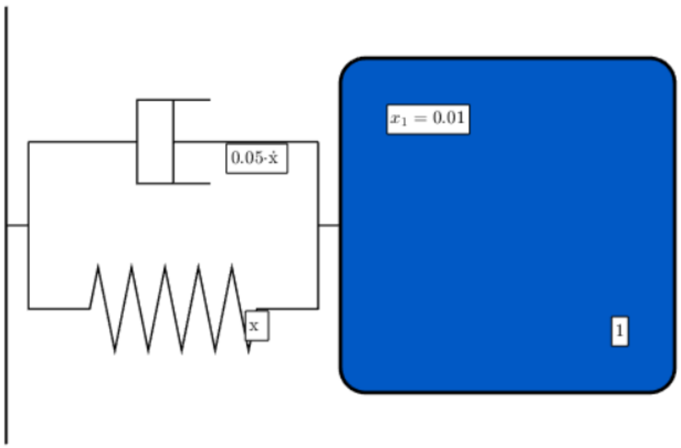
Imaginary roots,
damped oscillatory motion

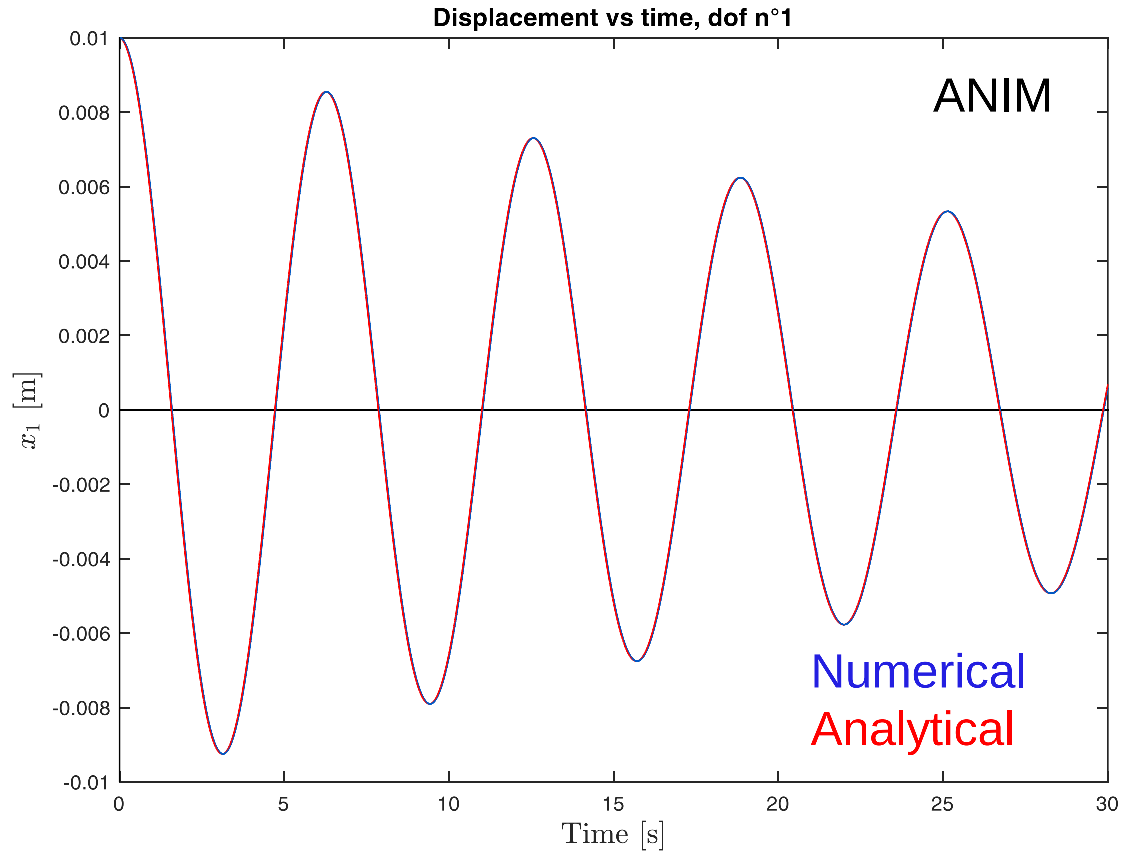
$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$ with $$\lambda_{1,2} = -\zeta\omega_n \pm i\omega_d$$

$$x(t) = e^{-\zeta\omega_n t}\left(Ae^{-i\omega_d t} + Be^{i\omega_d t}\right)$$

$$\boxed{x(t) = e^{-\zeta\omega_n t}\left(x_0 \cos\omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\sin\omega_d t\right)}$$

Underdamped motion





Consider several values of *c*

t=[0:0.01:30];k=1;m=1;c=0.05;x0=0.01;kssi=c/(2*sqrt(k*m));wn=sqrt(k/m);wd=sqrt(1-kssi^2)*wn;plot(t,exp(-kssi*t).*(x0*cos(wd*t)),'r--')

Logarithmic decrement

$$x(t) = e^{-\zeta\omega_n t}\left(x_0 \cos \omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d} \sin \omega_d t\right)$$

$$X = \sqrt{x_0^2 + \left(\frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\right)^2}$$

$$x(t) = Xe^{-\zeta\omega_n t}(\cos \omega_d t - \varphi)$$

$$\tan \varphi = \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d x_0}$$

Two successive maxima

$$\Delta = \ln \frac{x_n}{x_{n+1}} = \ln \frac{Xe^{-\zeta\omega_n t_n}(\cos \omega_d t_n - \varphi)}{Xe^{-\zeta\omega_n (t_n+T)}(\cos \omega_d (t_n + T) - \varphi)} = \zeta\omega_n T$$

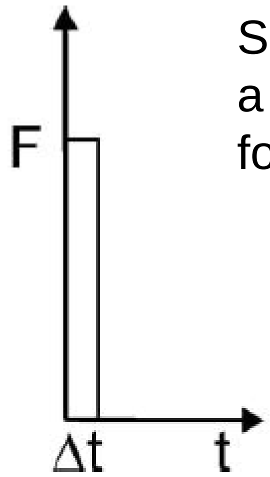
$$\Delta = \zeta\omega_n T = 2\pi \frac{\zeta}{\sqrt{1-\zeta^2}} \rightarrow \zeta = \frac{\Delta}{\sqrt{4\pi^2 + \Delta^2}}$$

Typical values of ζ

Civil structures ~5%

Aerospace structures <1%

Response to an impulsive force



Sudden application of a very short-duration force: shock/impulse

$$m\ddot{x} + c\dot{x} + kx = F\delta t$$

$$x(0^-) = 0,\, \dot{x}(0^-) = 0$$

An impulse imparts a change in momentum

$$x(0^+) = 0,\, \dot{x}(0^+) \neq 0$$

Initial velocity is impulse-like

$$lim_{\Delta t \to 0} \int_0^{\Delta t} (m\ddot{x} + c\dot{x} + kx)dt = F\ lim_{\Delta t \to 0} \int_0^{\Delta t} \delta t\ dt$$

$$lim_{\Delta t \to 0} \int_0^{\Delta t} m\ddot{x}\ dt = lim_{\Delta t \to 0} [m\dot{x}]_0^{\Delta t} = m\dot{x}(0^+)$$

$$lim_{\Delta t \to 0} \int_0^{\Delta t} c\dot{x} dt = lim_{\Delta t \to 0} [cx]_0^{\Delta t} = 0$$

$$lim_{\Delta t \to 0} \int_0^{\Delta t} kx\ dt = 0$$

$$\boxed{\dot{x}(0^+) = \frac{F}{m}}$$

Impulse response function

$$x(t) = e^{-\zeta\omega_n t}\left(\frac{\dot{x}_0}{\omega_d}\sin\omega_d t\right)$$

$$\dot{x}(0) = \frac{F}{m}$$

For a unit impulse, $F\delta t = 1$:

$$\boxed{h(t) = \frac{e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t}$$

Overdamped motion

$$c_{cr} > 2\sqrt{km}$$

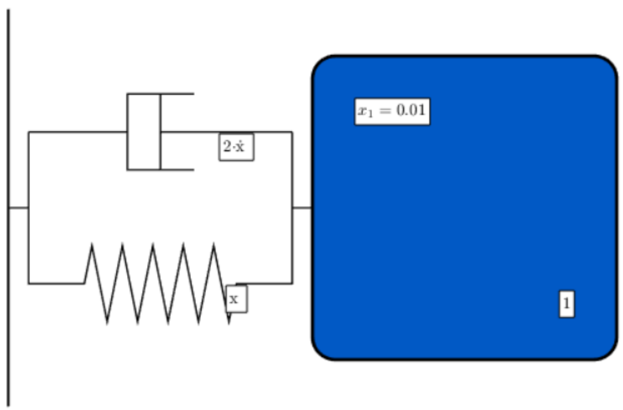
Real roots,
no oscillatory motion

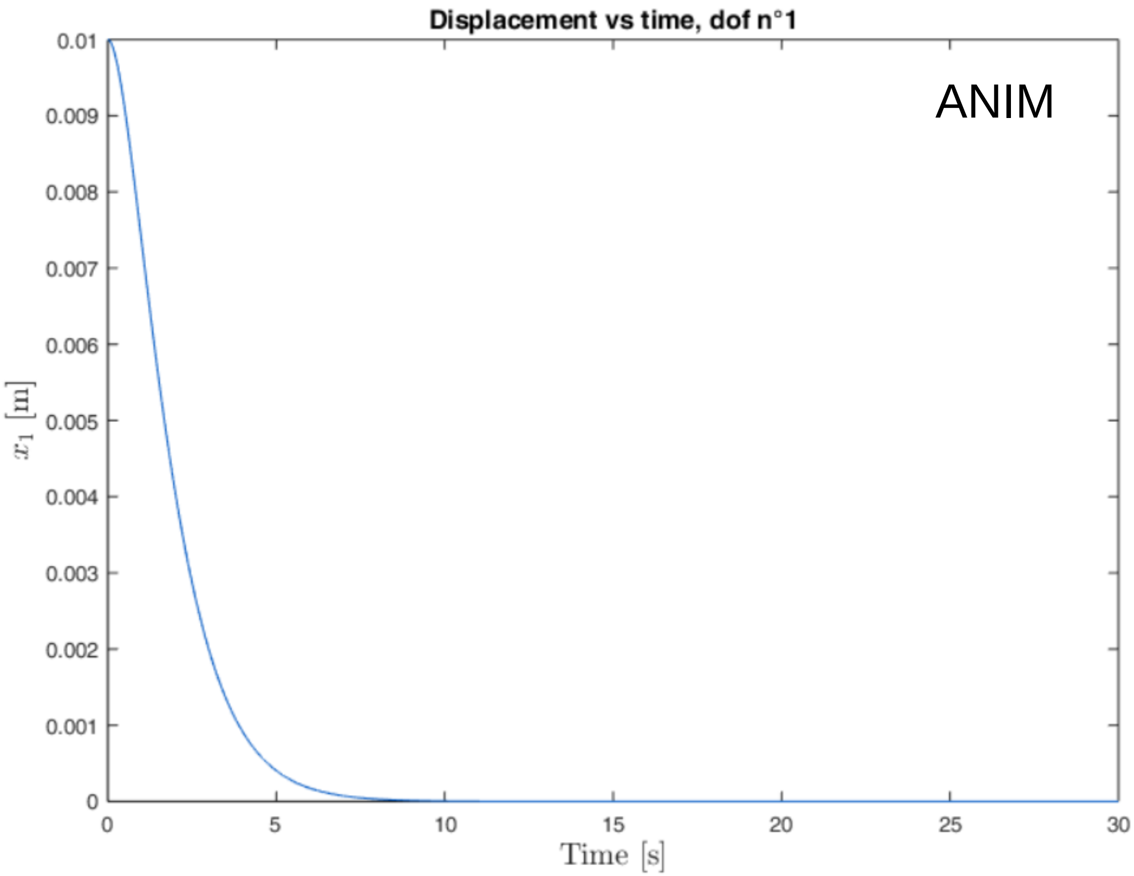
$$x(t) = e^{-\zeta\omega_n t}\left(Ae^{\omega_n\sqrt{\zeta^2-1}t} + BAe^{-\omega_n\sqrt{\zeta^2-1}t}\right)$$

$$A = \frac{\dot{x}_0 + (\zeta + \sqrt{\zeta^2 - 1})\omega_n x_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$

$$B = \frac{-\dot{x}_0 - (\zeta - \sqrt{\zeta^2 - 1})\omega_n x_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$

Overdamped motion





A design problem example

$$m = [2 - 3]\ kg$$

$$k > 200\ N/m$$

$$\dot{x}_0 < 0.3\ m/s$$

$$x_0 = 0$$

Choose the damping coefficient *c* such that

$$\boxed{\text{Amplitude} < 25mm}$$

Calculate the time of the maximum amplitude

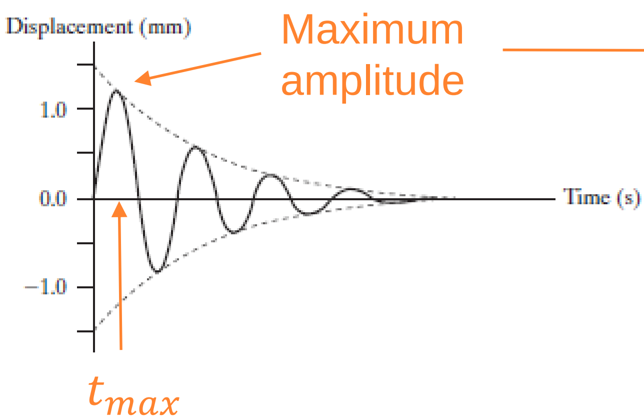
$$m = [2-3]\,kg, k > 200\,N/m \quad \rightarrow \quad 8.16\,rad/s < \omega_n < 10\,rad/s$$

$$\dot{x}_0 < 0.3\frac{m}{s}, x_0 = 0 \quad \rightarrow \quad x(t) = \frac{\dot{x}_0}{\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t$$

$$\downarrow$$

Worst case when

$\dot{x}_0 = 0.3\,m/s,\ \omega_d = 8.16\,rad/s$



$$\dot{x}(t) = 0$$

$$\omega_d e^{-\zeta\omega_n t} \cos \omega_d t - \zeta\omega_n e^{-\zeta\omega_n t} \sin \omega_d t = 0$$

$$\tan \omega_d t_{max} = \frac{\omega_d}{\zeta\omega_n} \rightarrow t_{max} = \frac{1}{\omega_d} \tan^{-1} \frac{\omega_d}{\zeta\omega_n}$$

Calculate the damping coefficient

$$t_{max} = \frac{1}{\omega_d} \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \qquad x(t) = \frac{\dot{x}_0}{\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t$$

$$Amplitude = \frac{\dot{x}_0}{\sqrt{1-\zeta^2}\omega_n} e^{\frac{-\zeta}{\sqrt{1-\zeta^2}} \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)} < 25mm$$

$$Numerical\ solution \rightarrow \zeta = 0.281$$

$$\boxed{c = 2m\omega_n\zeta = 2 \times 3 \times 8.16 \times 0.281 = 13.76kg/s}$$

$$c = 0 \rightarrow Amplitude = 37mm$$

Outline

Undamped, unforced system

Damped, unforced system

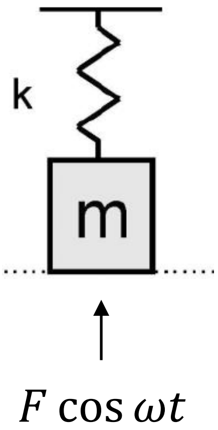
Harmonically-forced system

Importance of harmonic excitation

Fourier theorem: other forcing functions can be represented as a series of harmonic terms.

Since we assume linearity, the reponse can be calculated by knowing the response to the indivudal terms in the series.

Harmonically-forced undamped system



$$m\ddot{x} + kx = F\cos\omega t$$

What is the solution to this equation ? $x = x(t)$?

Homogeneous and particular solutions

$$m\ddot{x} + kx = F\cos\omega t$$

$$x = x_H + x_P$$

$$x_H = A\cos\omega_n t + B\sin\omega_n t$$

$$x_P = X\cos\omega t$$

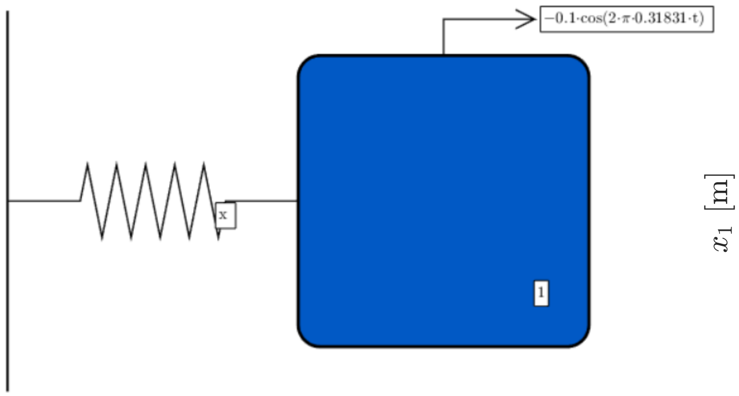
$$-m\omega^2 X + kX = F$$

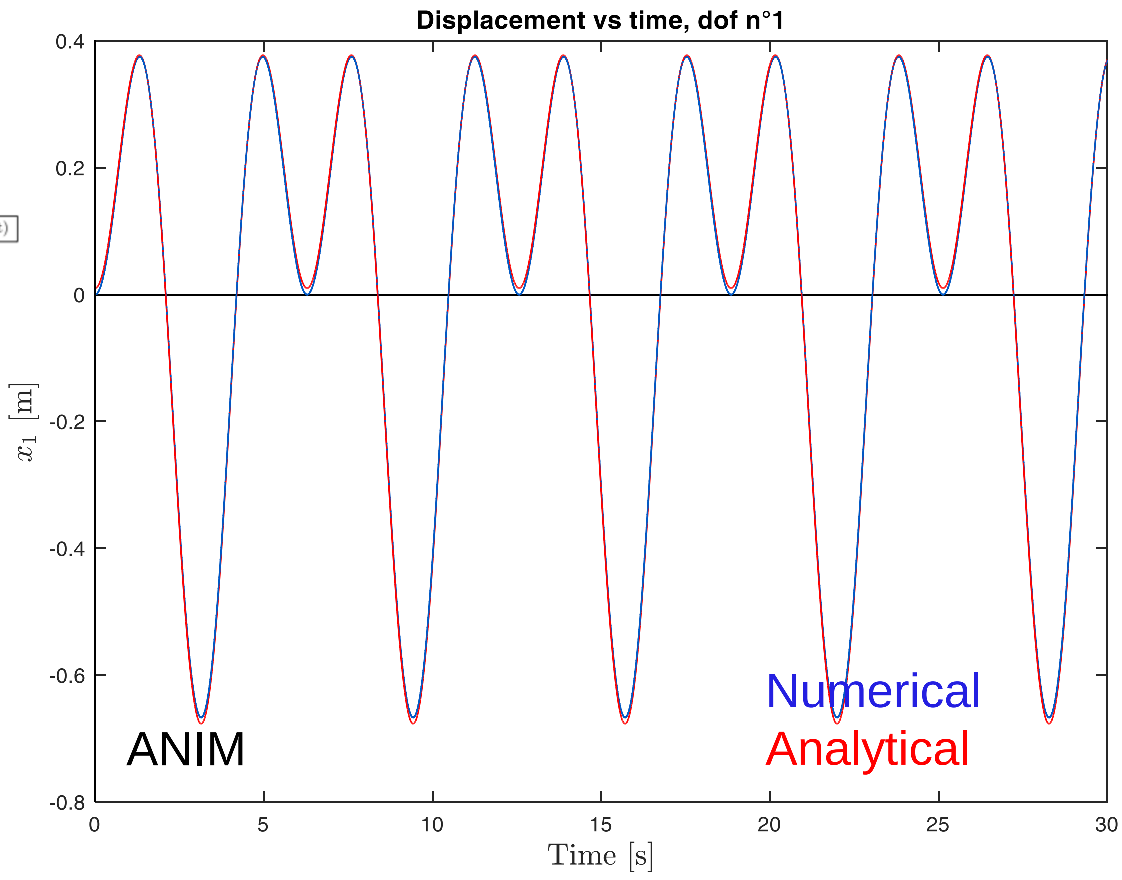
$$X = \frac{F}{k - m\omega^2}$$

$$\Rightarrow \quad x = A\cos\omega_n t + B\sin\omega_n t + \frac{F}{k - m\omega^2}\cos\omega t$$

$$\boxed{x = \left(x_0 - \frac{F}{k - m\omega^2}\right)\cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t + \frac{F}{k - m\omega^2}\cos\omega t}$$

Harmonic motion






```
x0=0.01;k=1;m=1;wn=1;w=2;F=1;t=[0:0.01:30];hold on;plot(t,(x0-F/(k-
w^2*m))*cos(wn*t)+(F/(k-w^2*m))*cos(w*t),'r--')
```

Dynamic amplification factor

$$x = \left(x_0 - \frac{F}{k - m\omega^2}\right)\cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t + \frac{F}{k - m\omega^2}\cos\omega t$$

The static response is $X_0 = \dfrac{F}{k}$

The dynamic response is $\dfrac{F}{k - m\omega^2}\cos\omega t = X\cos\omega t$

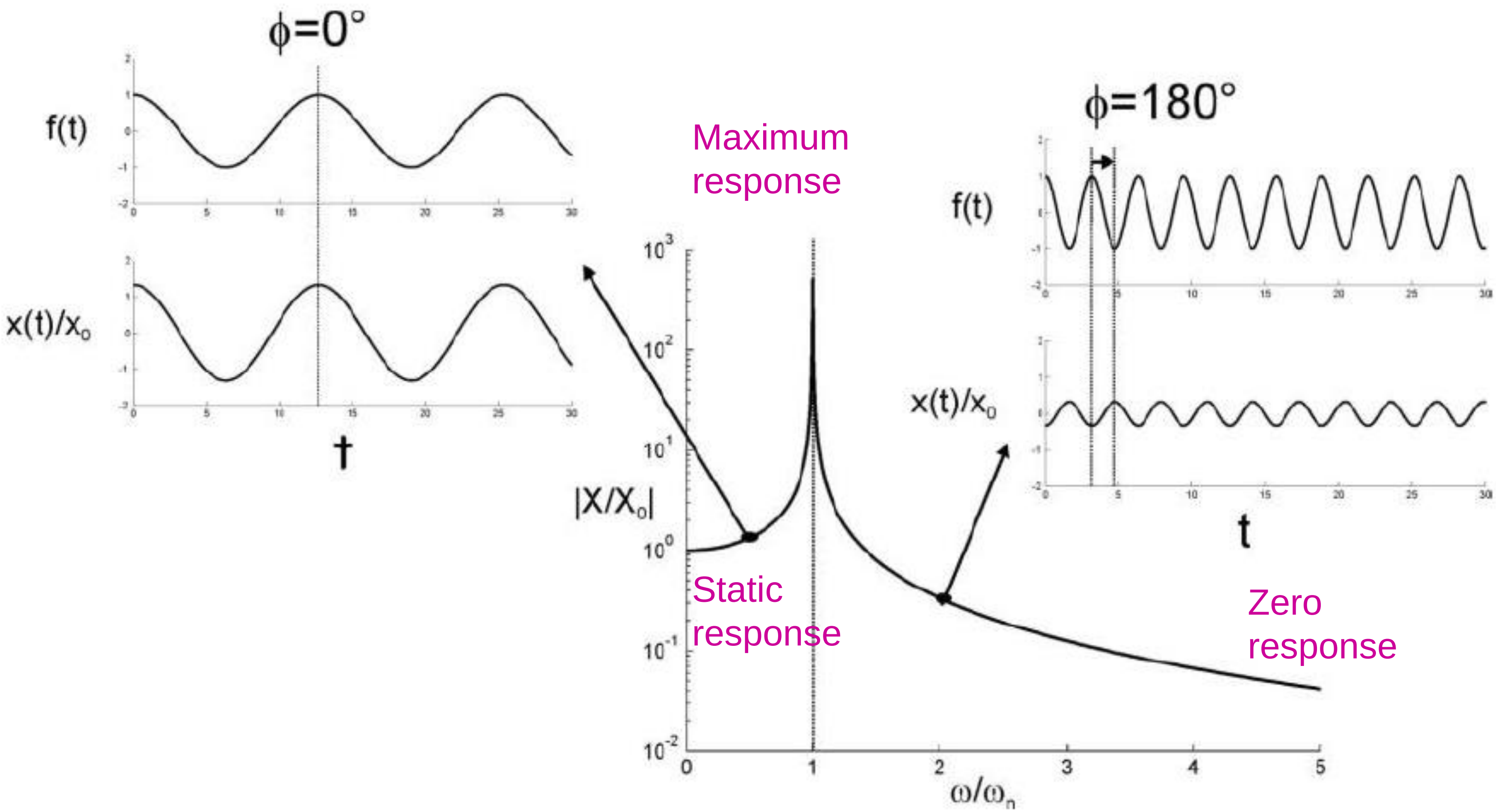
$$\boxed{\frac{X}{X_0} = \frac{1}{1 - \dfrac{\omega^2}{\omega_n^2}}}$$

Positive if $\omega < \omega_n$

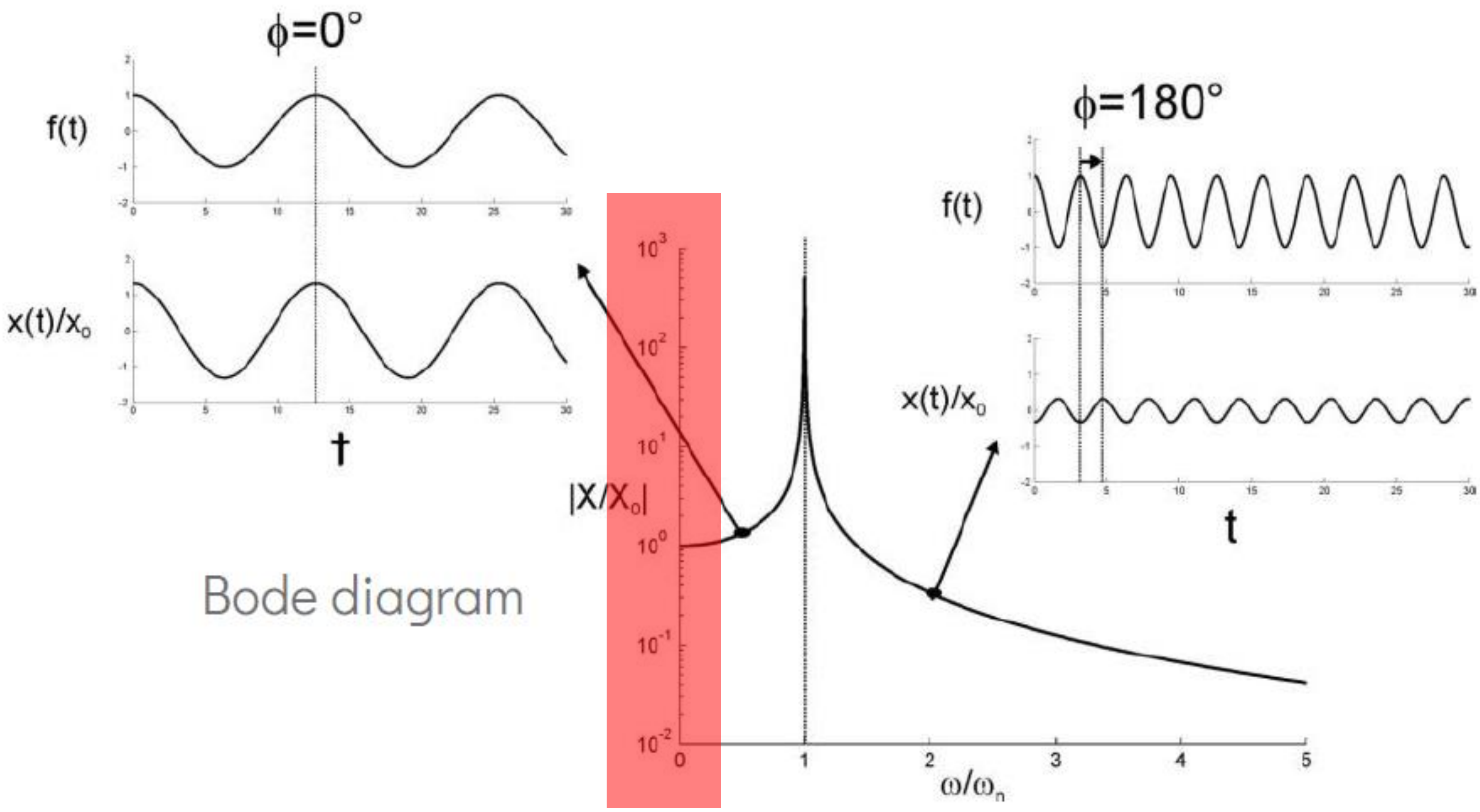
Infinite if $\omega = \omega_n$

Negative if $\omega > \omega_n$

Bode diagrams



Decibel dB



$$\mathrm{dB} \equiv 10 \log_{10}\left(\frac{x_1}{x_0}\right)^2 = 20 \log_{10} \frac{x_1}{x_0}$$

Illustration

https://www.youtube.com/watch?v=cfKwnTfNhog

Resonance, a key concept in vibration theory

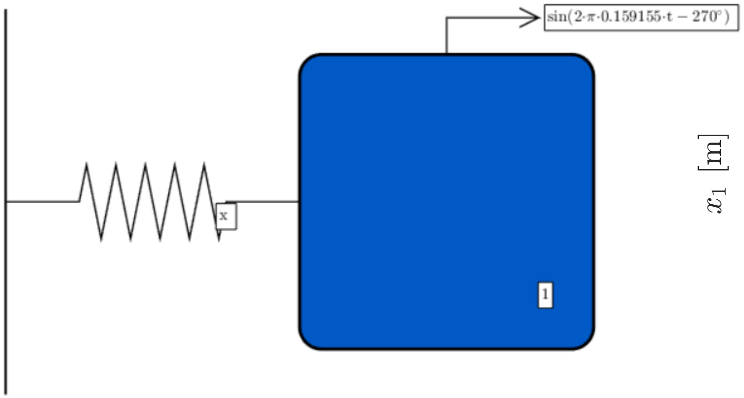
$$x = \left(x_0 - \frac{F}{k - m\omega^2}\right)\cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t + \frac{F}{k - m\omega^2}\cos\omega t$$

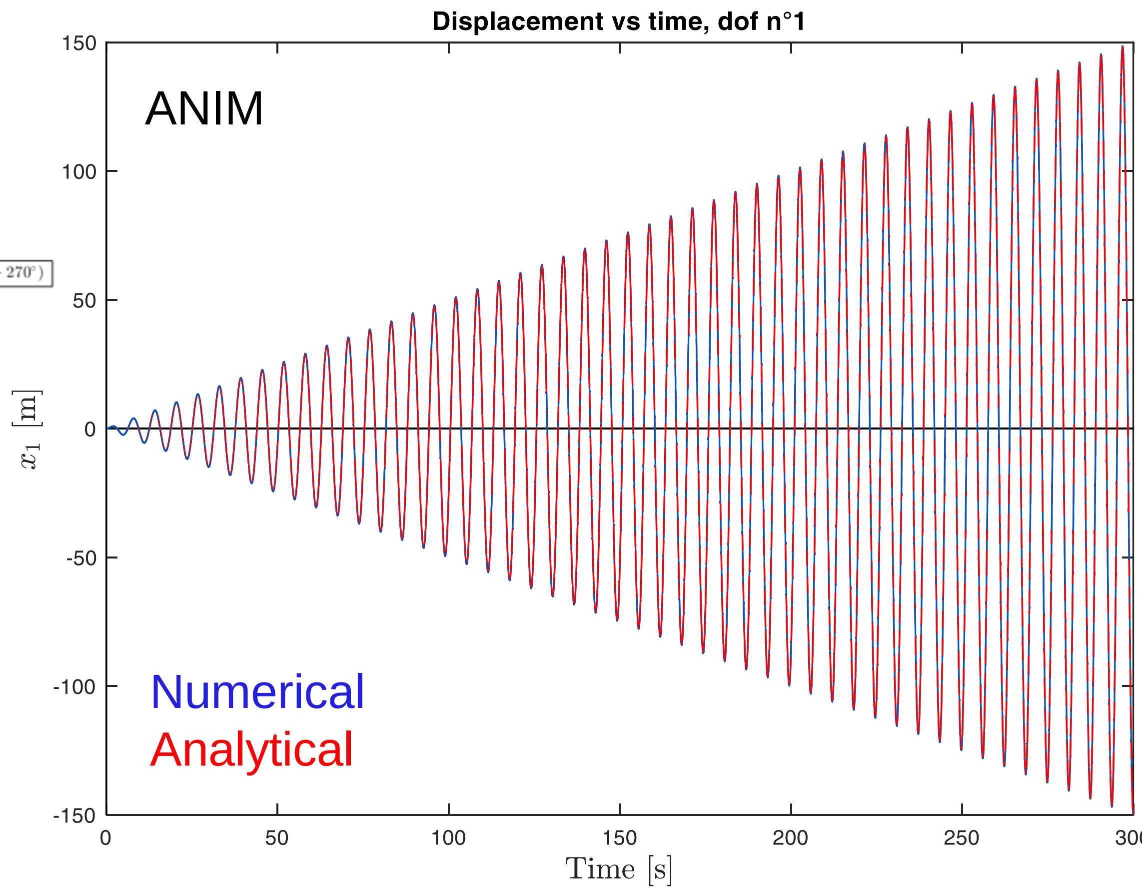
$$\omega = \sqrt{\frac{k}{m}}\ ?$$

$$x_P = tX\sin\omega t$$

$$\boxed{x = x_0\cos\omega t + \frac{\dot{x}_0}{\omega}\sin\omega t + \frac{Ft}{2m\omega}\sin\omega t}$$

Resonance, a key concept in vibration theory



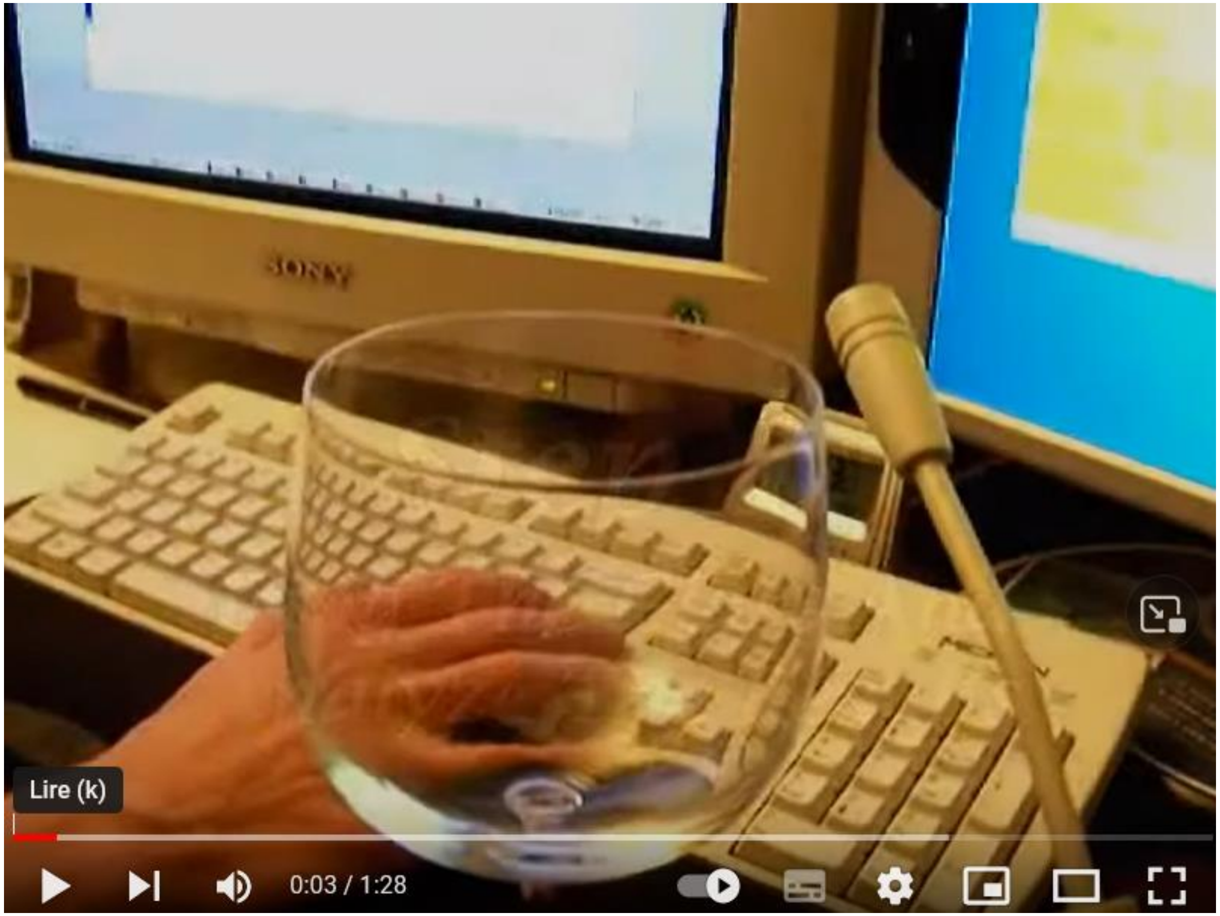


k=1;m=1;w=1;F=1;t=[0:0.01:300];hold on;plot(t,F*t/2/m/w.*sin(w*t),'r--')

Resonance, a key concept in vibration theory

https://www.youtube.com/watch?v=10lWpHyN0Ok

Resonance, a key concept in vibration theory



https://www.youtube.com/watch?v=JiM6AtNLXX4

Resonance, a key concept in vibration theory

https://www.youtube.com/watch?v=n9ULMIjvSIg

Resonance, a key concept in vibration theory

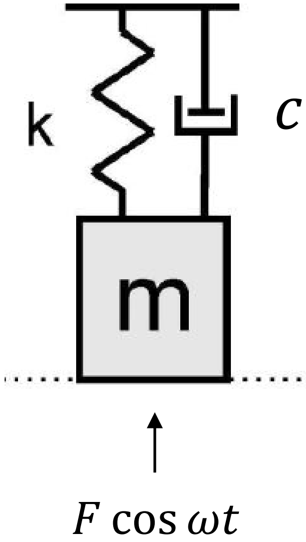
https://www.youtube.com/watch?v=LV_UuzEznHs

Beneficial effect of damping

https://www.youtube.com/watch?v=wqiSz6P5GtQ

https://www.youtube.com/watch?v=02rpUMdr7qo

Harmonically-forced damped system



$$m\ddot{x} + c\dot{x} + kx = F \cos \omega t$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$\zeta = \frac{c}{2\sqrt{km}}$$

$$f = \frac{F}{m}$$

$$\Downarrow$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f \cos \omega t$$

What is the solution to this equation ? $x = x(t)$?

We now disregard the transient response (homogeneous solution) and focus on the **steady-state response.**

Complex amplitude

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = f/m$$

$$x(t) = X\,e^{i\omega t}$$

$$f(t) = F\,e^{i\omega t}$$

$$(\omega_n^2 + 2i\xi\omega\omega_n - \omega^2)X = F/m$$

$$X = \frac{F}{m}\left(\frac{1}{\omega_n^2 + 2i\xi\omega\omega_n - \omega^2}\right) = \frac{F}{k}\left(\frac{1}{1 - \frac{\omega^2}{\omega_n^2} + 2i\xi\frac{\omega}{\omega_n}}\right)$$

$$= X_0\left(\frac{1}{1 - \frac{\omega^2}{\omega_n^2} + 2i\xi\frac{\omega}{\omega_n}}\right)$$

Amplitude and phase

$$X_r = X_0 \frac{1 - \frac{\omega^2}{\omega_n^2}}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}$$

$$X_i = X_0 \frac{-2\xi \frac{\omega}{\omega_n}}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}$$

$$|X/X_0| = \sqrt{\frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}$$

$$\tan\phi = \frac{-2\xi \frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}}$$

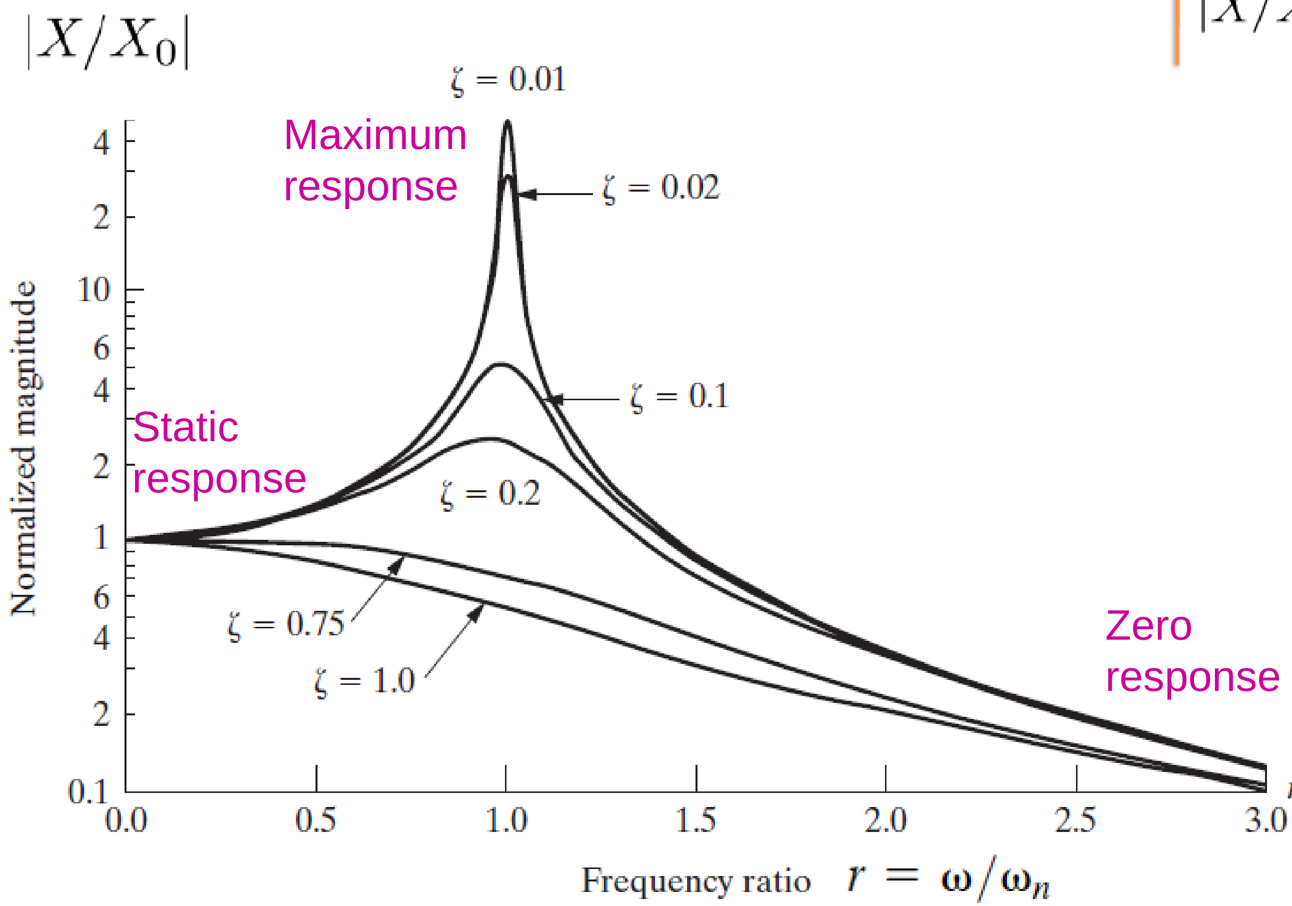
Bode diagrams

Maximum $\omega/\omega_n = \sqrt{1-2\xi^2}$

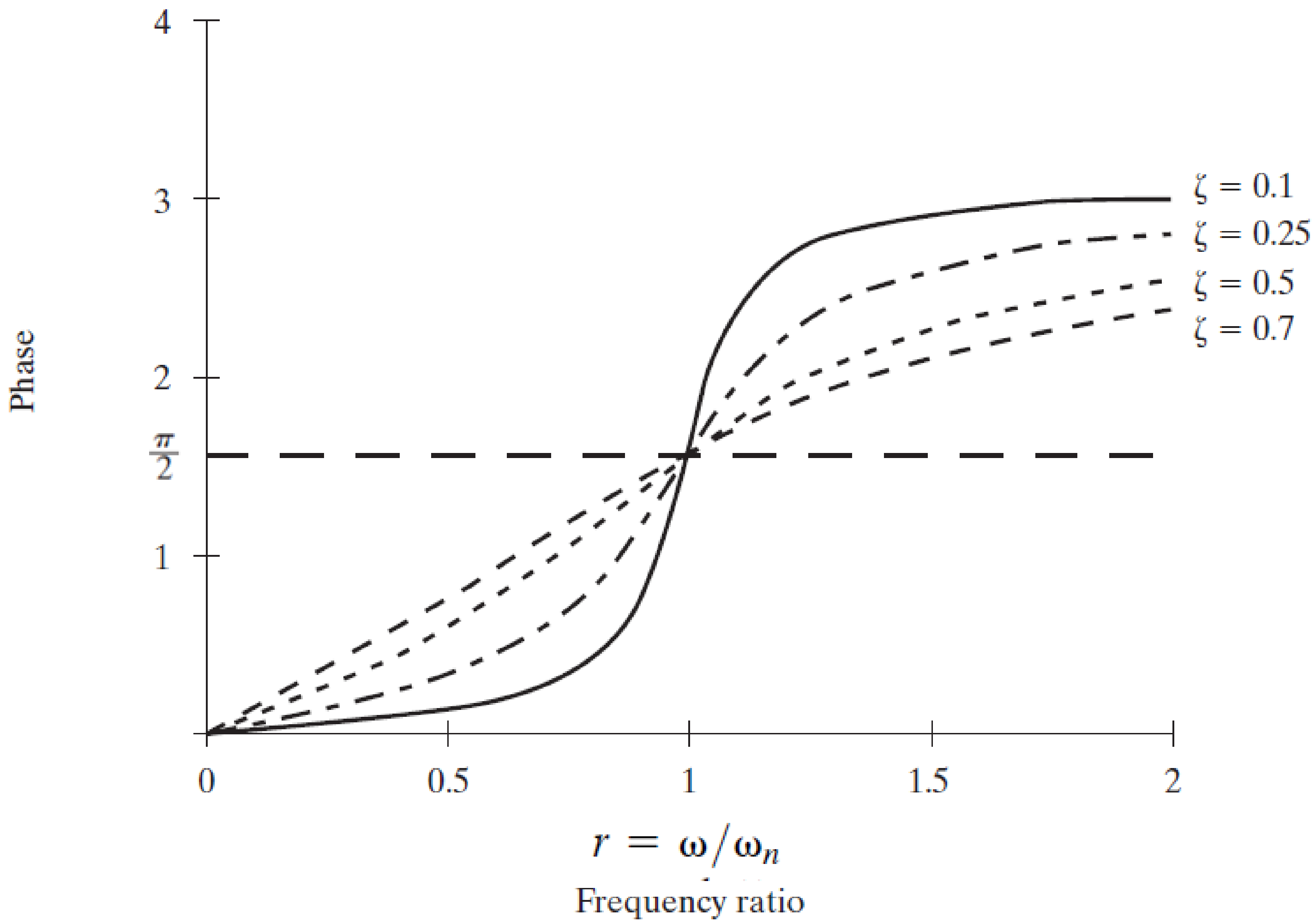
Max ampl. $|X/X_0| = \dfrac{1}{2\xi\sqrt{1-\xi^2}}$

For ξ small

$$\begin{aligned} \omega/\omega_n &= 1 \\ |X/X_0| &= \frac{1}{2\xi} \end{aligned}$$



Bode diagrams

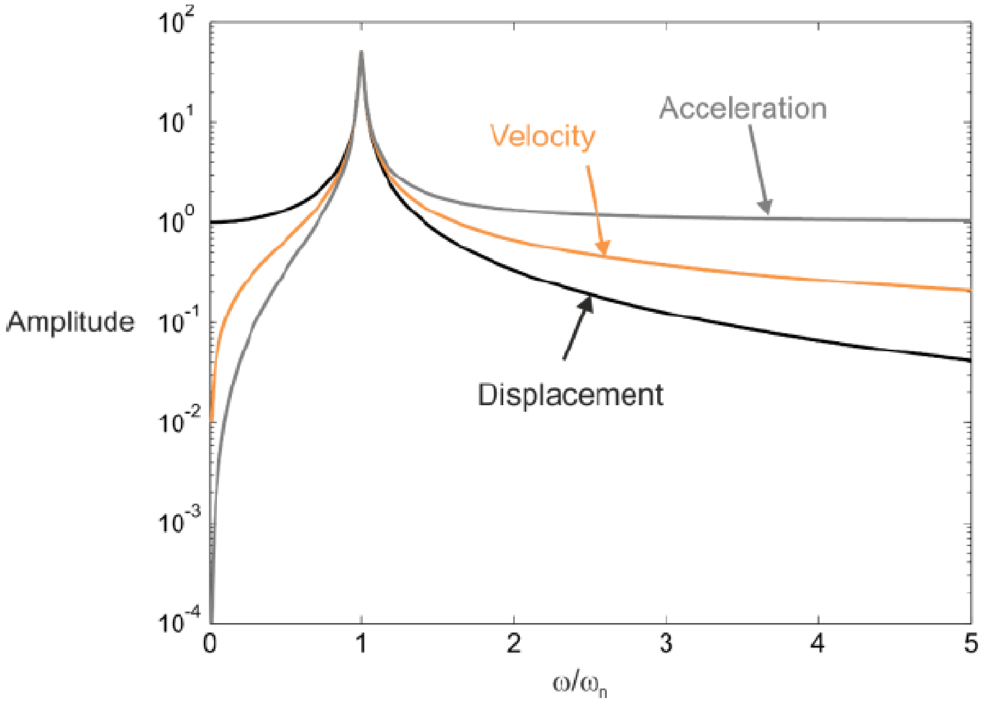


Physical explanation of resonance

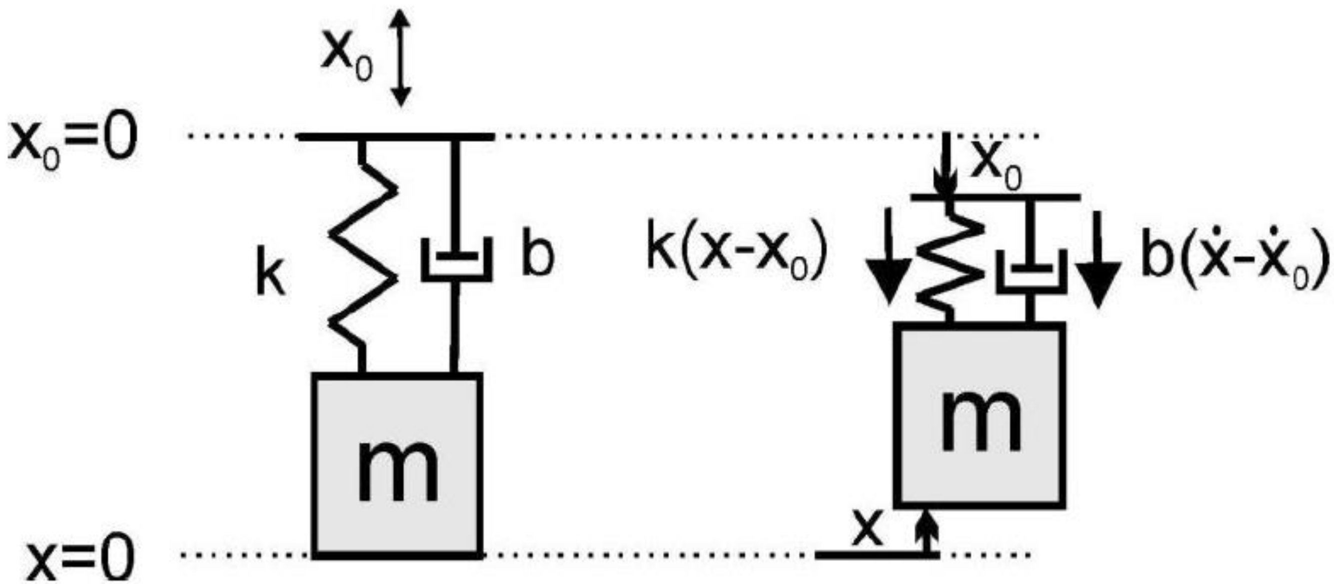
To explain physically the phenomena of resonance, consider the steady-state forced response of the system where the applied force is $F_0 \cos(\omega t)$, the displacement is $x_p(t) = X \cos(\omega t - \theta)$, and the velocity is $\dot{x}_p(t) = -\omega X \sin(\omega t - \theta)$. At resonance, $\theta = \pi/2$. Thus $\dot{x}_p(t)_{(\text{resonance})} = \omega X \cos(\omega t)$. This shows that at resonance the velocity and the force are exactly in phase but have different magnitudes. Physically, this means that the force is always pushing in the direction of the velocity and that the force changes magnitude and direction just as the velocity does. This condition will cause the vibration amplitude of the system to reach its maximum value because at resonance the external force never opposes the velocity.

Different signals

$$
\begin{aligned}
\text{Displacement} \quad u(t) &= Ae^{i\omega t} \\
\text{Velocity} \quad v(t) &= \frac{d\,u(t)}{dt} = i\omega Ae^{i\omega t} = i\omega u(t) \\
\text{Acceleration} \quad a(t) &= \frac{d\,v(t)}{dt} = -\omega^2 Ae^{i\omega t} = -\omega^2 u(t)
\end{aligned}
$$



Base-excited systems



$$m\ddot{x} = -k(x - x_0) - b(\dot{x} - \dot{x_0})$$

$$x_r = x - x_0$$

$$m\ddot{x}_r + b\dot{x}_r + kx_r = \underbrace{-m\ddot{x}_0}_{\text{Excitation}}$$

In summary

Thorough analysis of a 1DOF oscillator with/without damping and with/without forcing.

Important concepts of dynamic amplification and resonance.

Bode diagrams.