Cinématique et dynamique des machines

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L04 **MDOF**

Course objectives

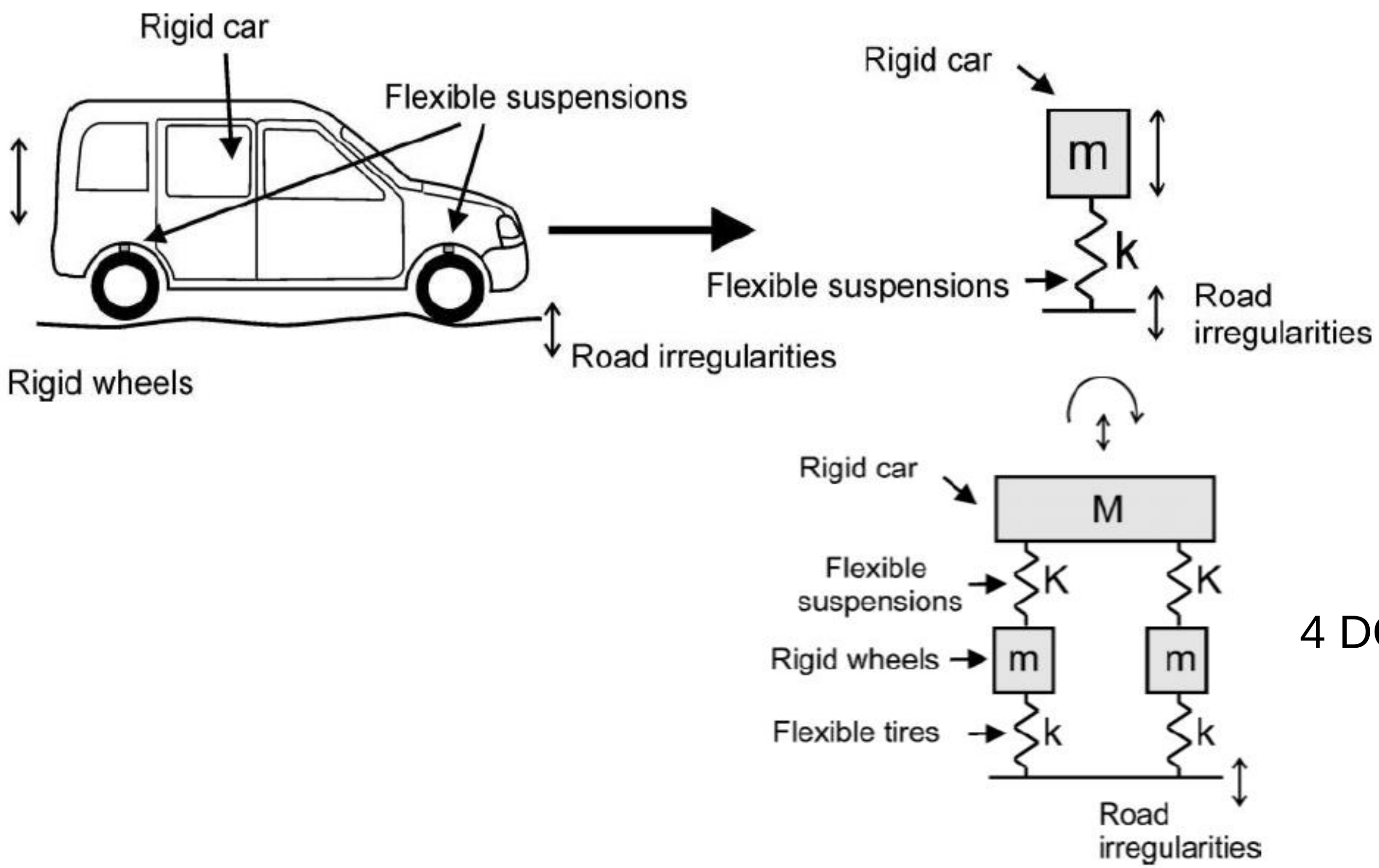
Given input → Mechanical system → Predict the output

→ Write down the equations of motion $m\ddot{x} + c\dot{x} + kx = f(t)$

→ Calculate the response analytically or numerically $x = 0.19\sin 2.3t$

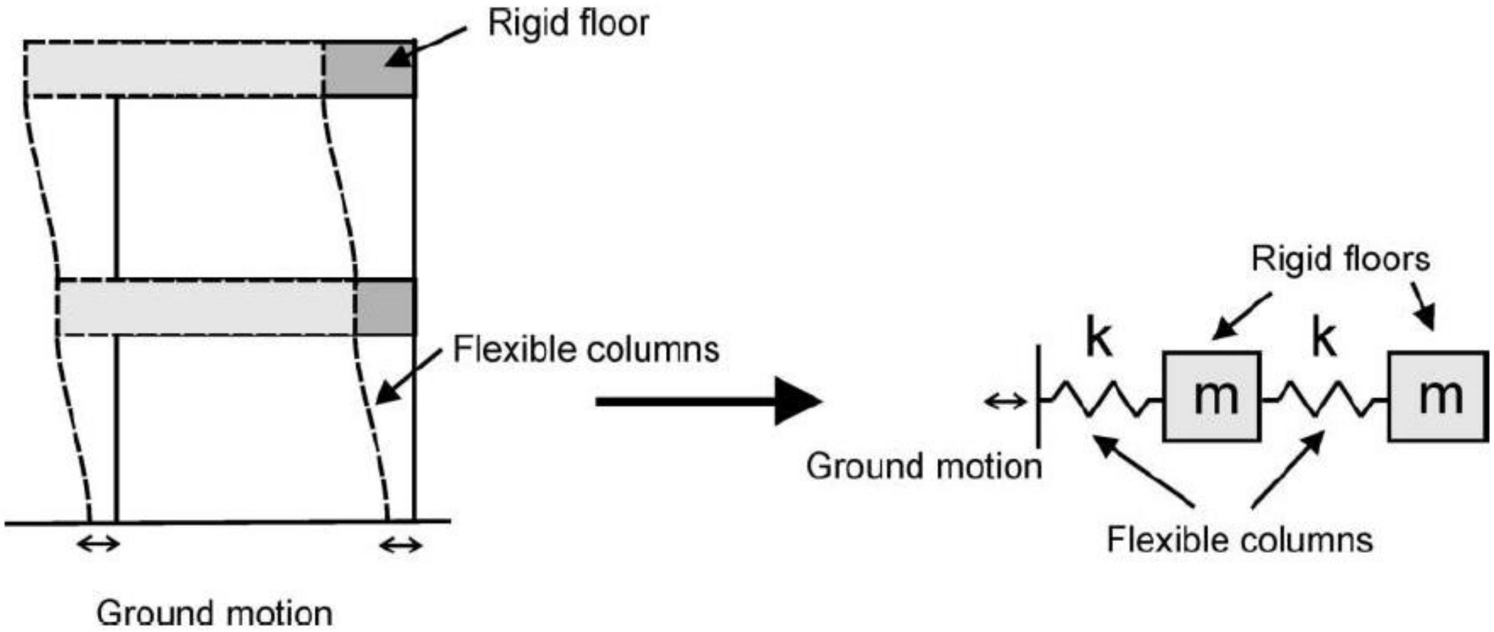
→ Is the structure safe ?

From 1DOF to MDOF systems



4 DOFs

From 1DOF to MDOF systems



Outline

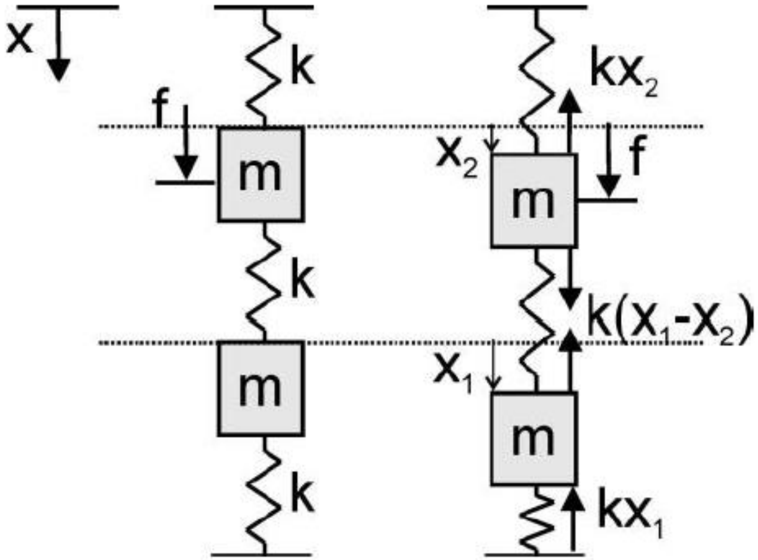
Undamped, unforced response

Undamped, harmonic response

Damped, harmonic response

Vibration absorbers

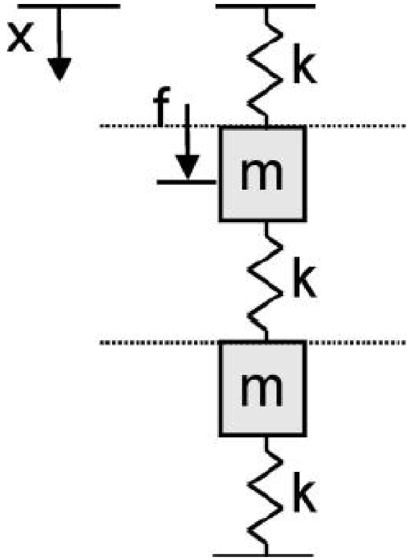
Equations of motion through Newton



$$m\ddot{x}_1 = -kx_1 - k(x_1 - x_2)$$
$$m\ddot{x}_2 = -kx_2 + k(x_1 - x_2) + f$$

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

Equations of motion through Lagrange



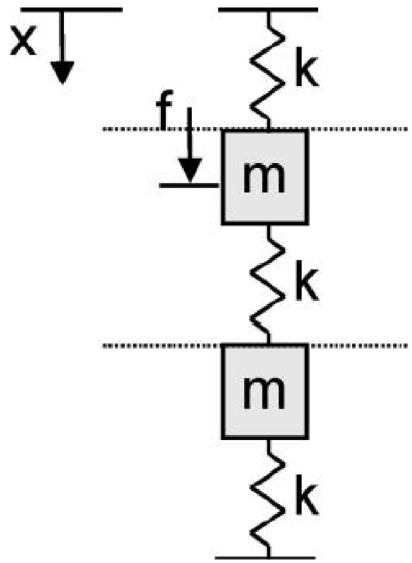
$$T = \frac{m\ddot{x}_1^2}{2} + \frac{m\ddot{x}_2^2}{2}$$

$$V = \frac{kx_1^2}{2} + \frac{kx_2^2}{2} + \frac{k(x_1 - x_2)^2}{2}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_k}\right) - \frac{\partial T}{\partial x_k} + \frac{\partial V}{\partial x_k} = Q_k, \qquad k = 1, \dots, n$$

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

Equations of motion through inspection



$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

Undamped, unforced problem

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = 0$$

Mass matrix **M**
(positive definite)

Stiffness matrix **K**
(positive definite)

Synchronous

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = 0 \Rightarrow \boldsymbol{x}(t) = \varphi(t)\boldsymbol{y} \Rightarrow \boldsymbol{K}\boldsymbol{y}\varphi(t) + \boldsymbol{M}\boldsymbol{y}\ddot{\varphi}(t) = \boldsymbol{0}$$

$\boldsymbol{y}^T \cdot$

>0 >0

$$\Rightarrow a\varphi(t) + b\ddot{\varphi}(t) = \boldsymbol{0} \Rightarrow \ddot{\varphi}(t) = -\omega^2\varphi(t)$$

$$\Rightarrow (\boldsymbol{K} - \omega^2\boldsymbol{M})\boldsymbol{y} = \boldsymbol{0} \Rightarrow \det(\boldsymbol{K} - \omega^2\boldsymbol{M}) = \boldsymbol{0}$$

Non trivial
solution

Eigenvalue problem of a 2DOF system

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 2k - \omega^2 m & -k \\ -k & 2k - \omega^2 m \end{bmatrix} \right) = \omega^4 m^2 - 4km\omega^2 + 3k^2 = 0$$

If $k = m = 1, \ \omega^4 - 4\omega^2 + 3 = 0$

$$\omega_1^2 = \frac{1}{2}\left\{4 - [4]^{1/2}\right\} = 1 \quad \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0 \qquad y_{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\omega_2^2 = \frac{1}{2}\left\{4 + [4]^{1/2}\right\} = 3 \quad \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0 \qquad y_{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Numerical solution of the eigenvalue problem

```
M=[1 0;0 1]; K=[2 -1;-1 2];
[Modes,Frequencies]=eig(K,M);
[Modes,Frequencies]=eig(inv(M)*K);


Modes=
    -0.7071 -0.7071
    -0.7071 0.7071
Frequencies=
     1   0
     0   3
```

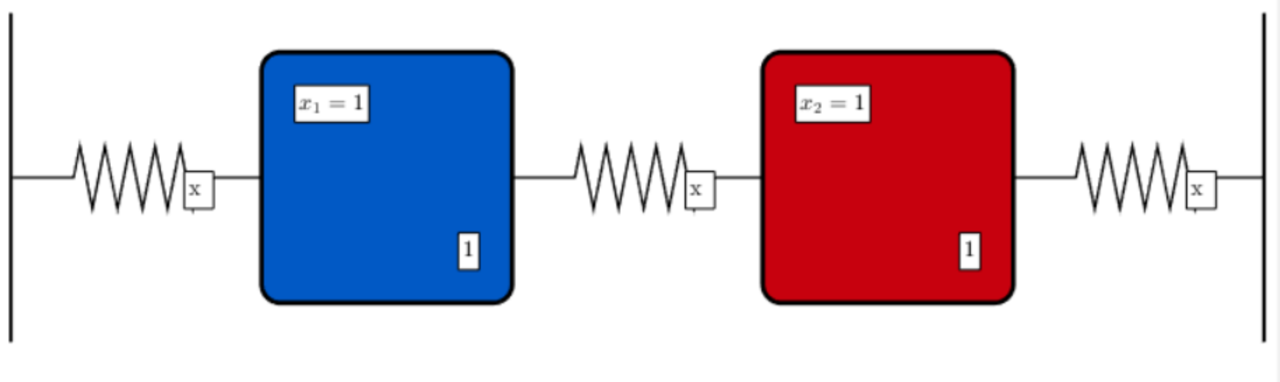
Response through modal superposition

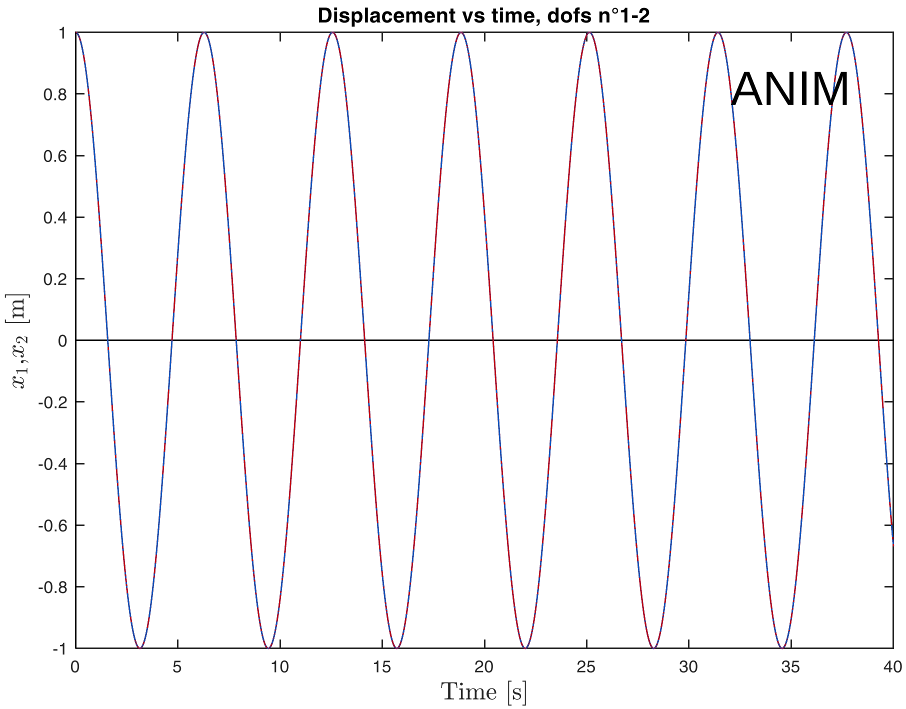
$$\ddot{\varphi}(t) = -\omega^2 \varphi(t) \qquad \varphi(t) = A\cos\omega t + B\sin\omega t$$

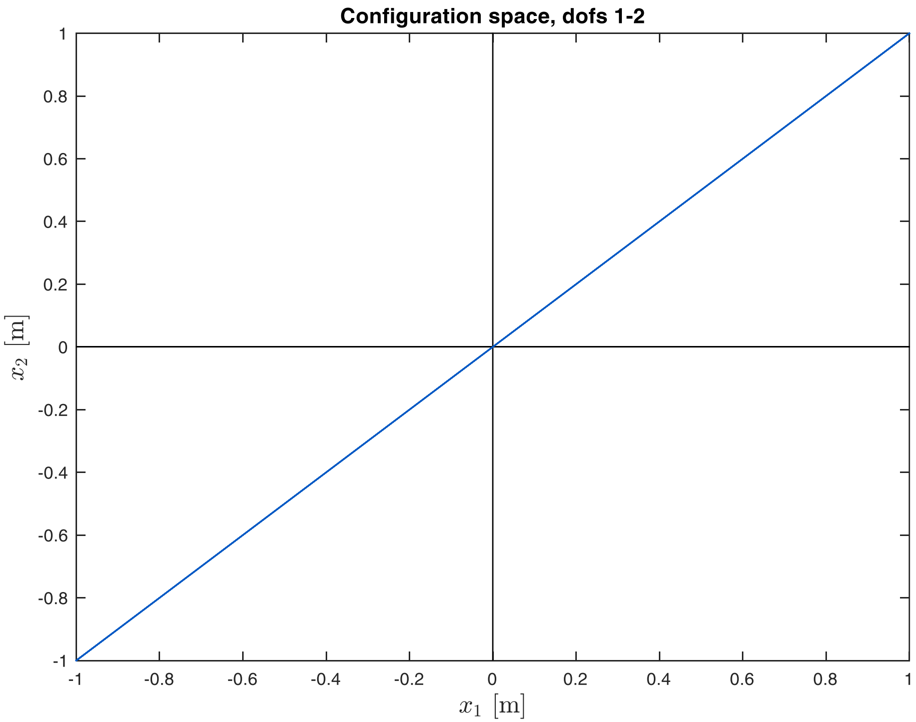
$$\boldsymbol{x}(t) = (A_1\cos\omega t + B_1\sin\omega t)\boldsymbol{y}_{(1)} + (A_2\cos\omega t + B_2\sin\omega t)\boldsymbol{y}_{(2)}$$

This solution can be directly generalized to *n* DOFs
(*n* eigenmodes and eigenfrequencies)

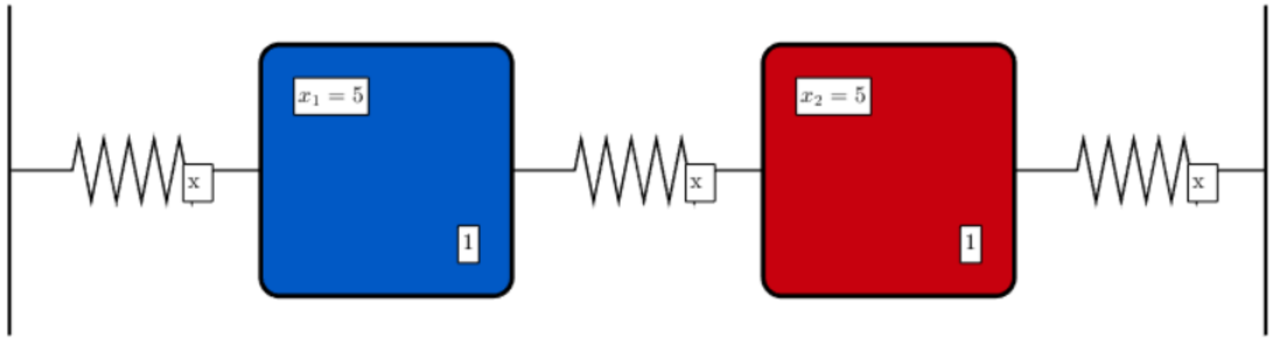
Excitation of the first mode

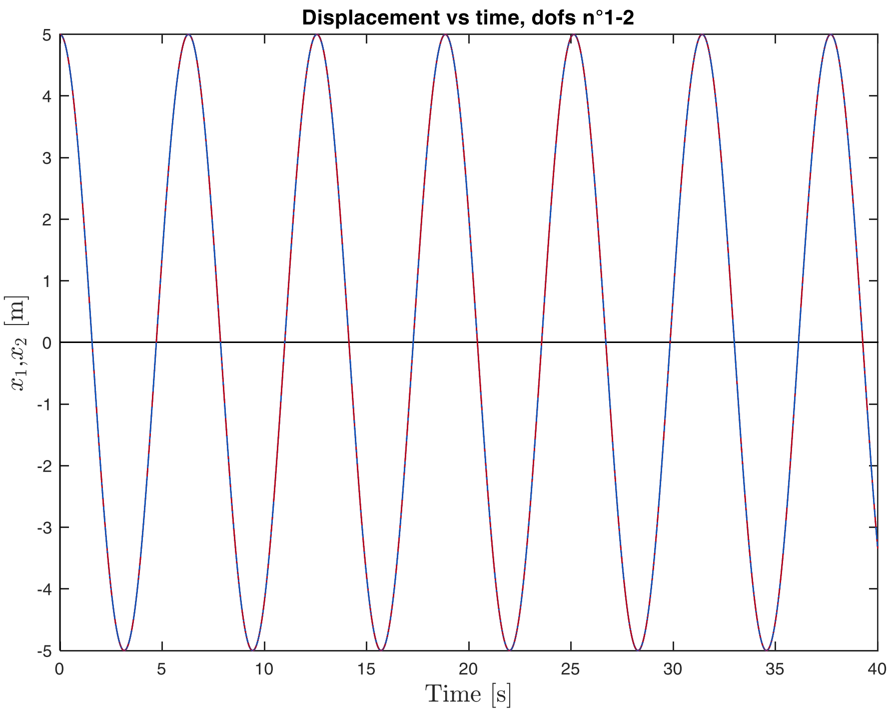




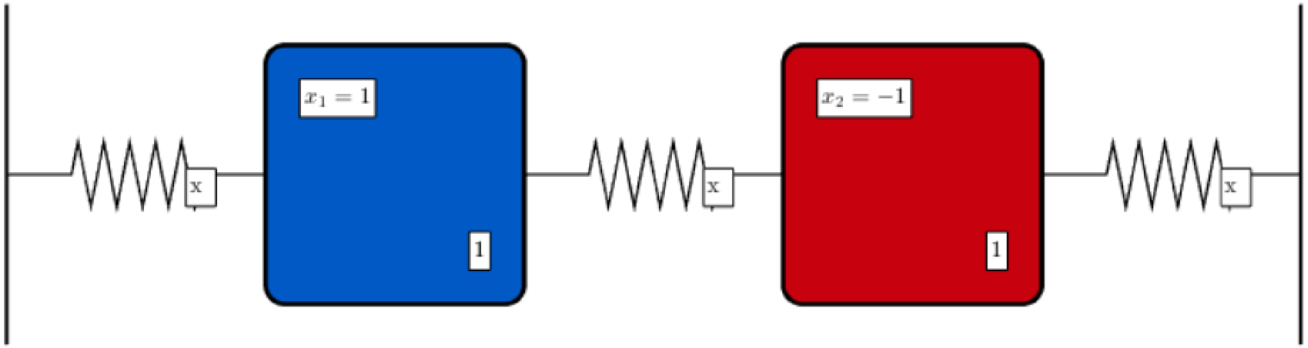


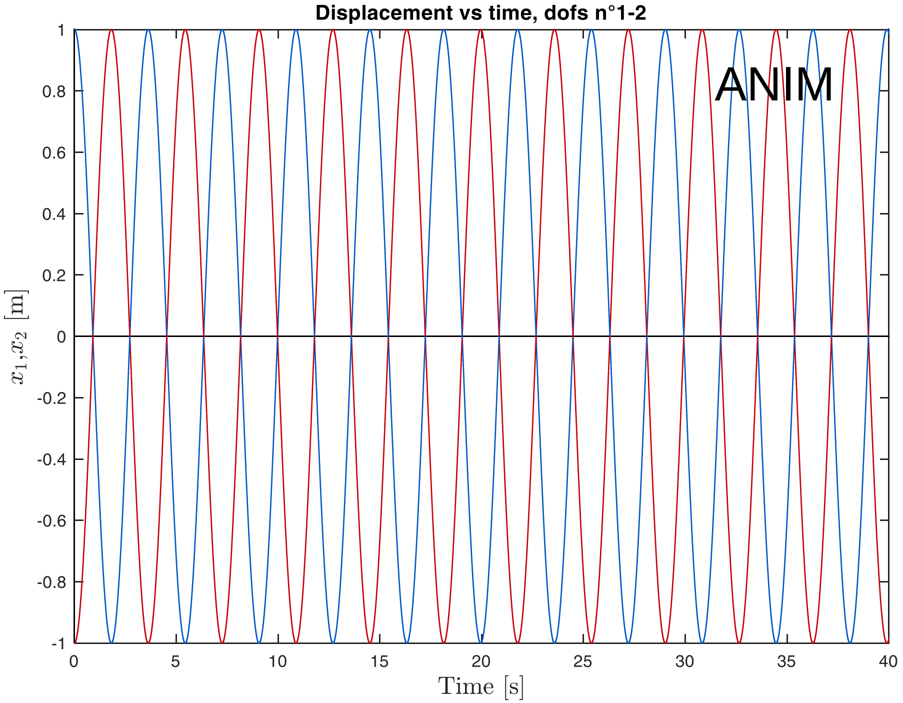
Excitation of the first mode (greater ICs)

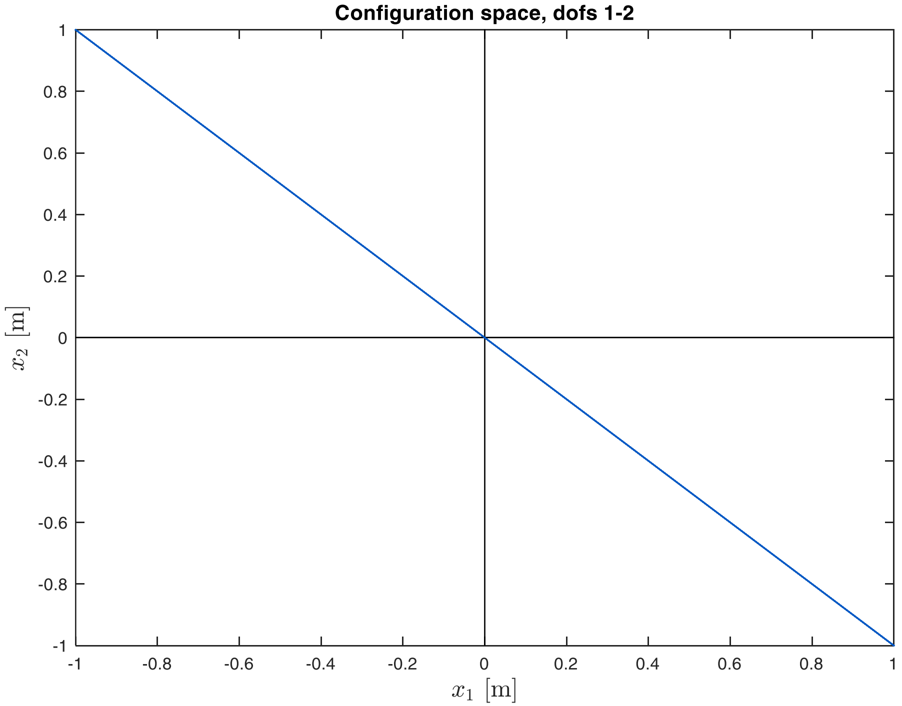




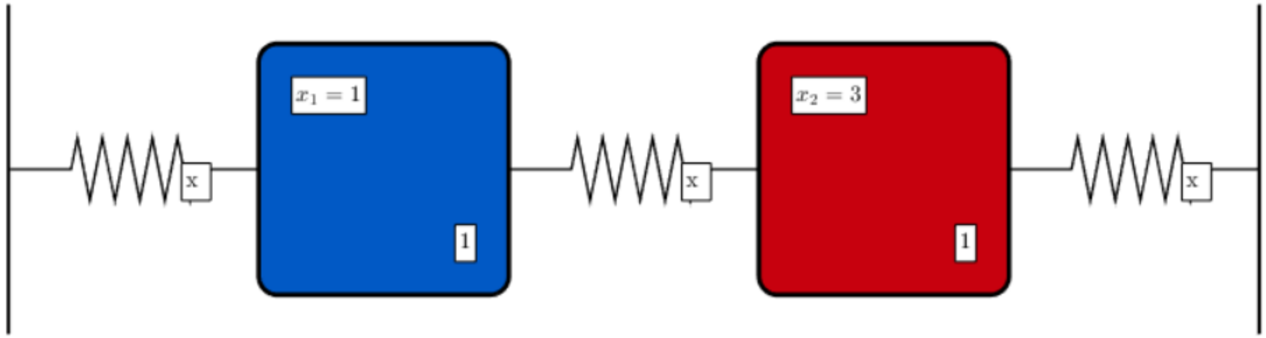
Excitation of the second mode



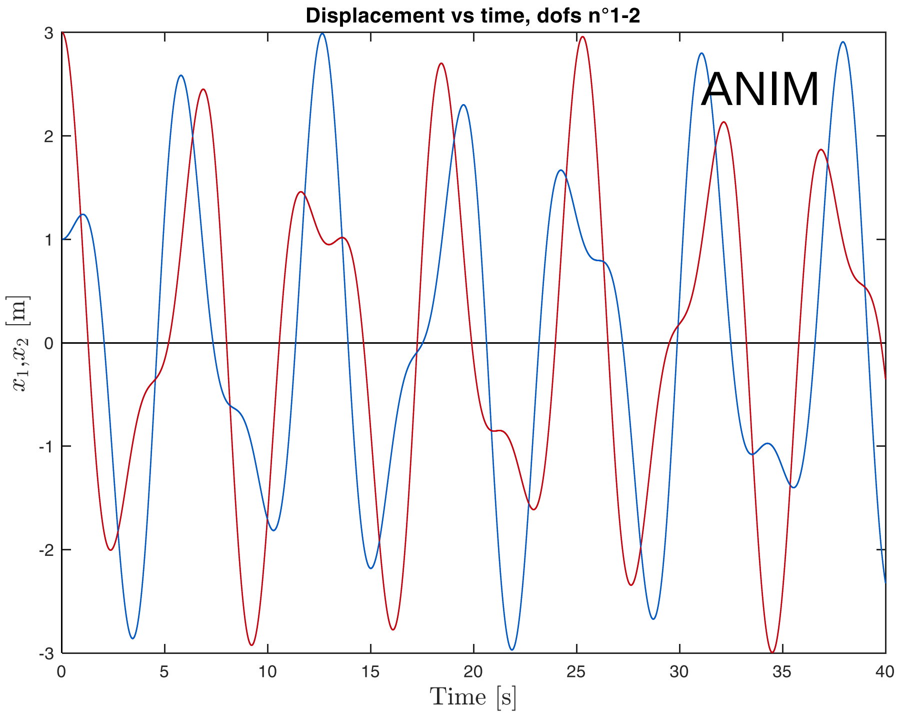




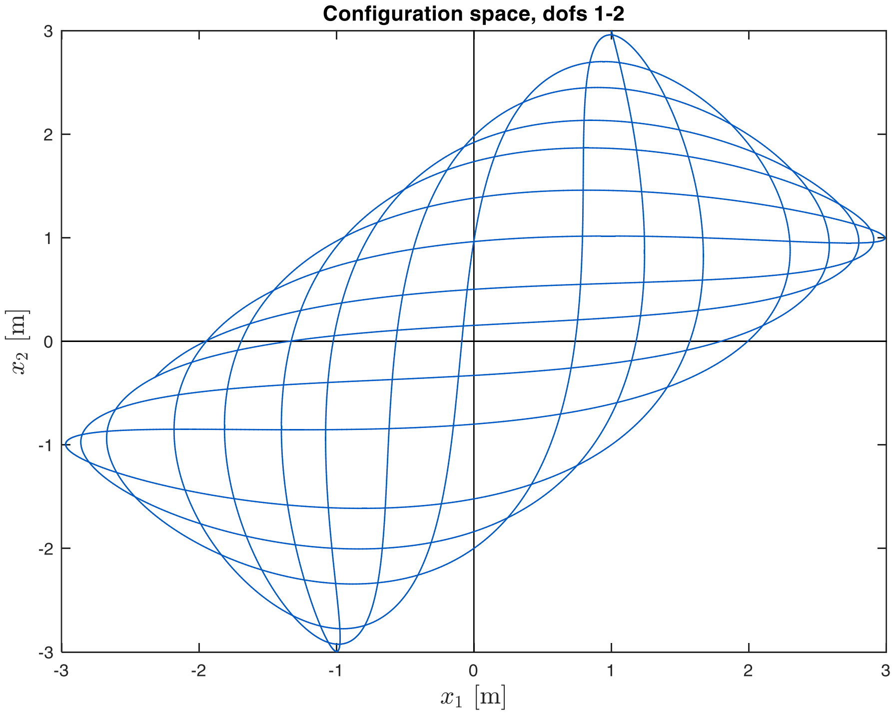
General ICs



$x_1 = 1$
1
$x_2 = 3$
1
x
x
x

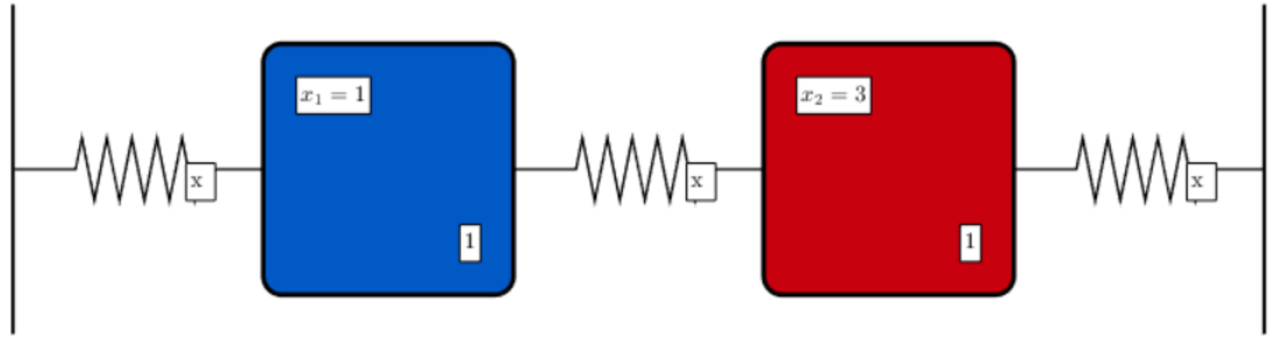


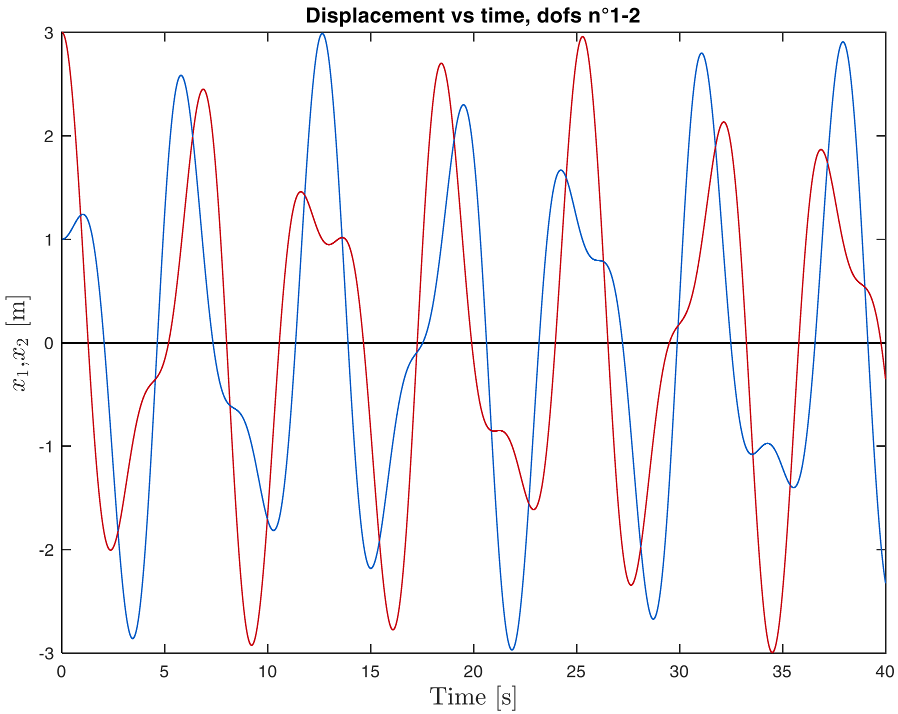
Displacement vs time, dofs n°1-2
ANIM
x_1,x_2 [m]
Time [s]
0
5
10
15
20
25
30
35
40
3
2
1
0
-1
-2
-3

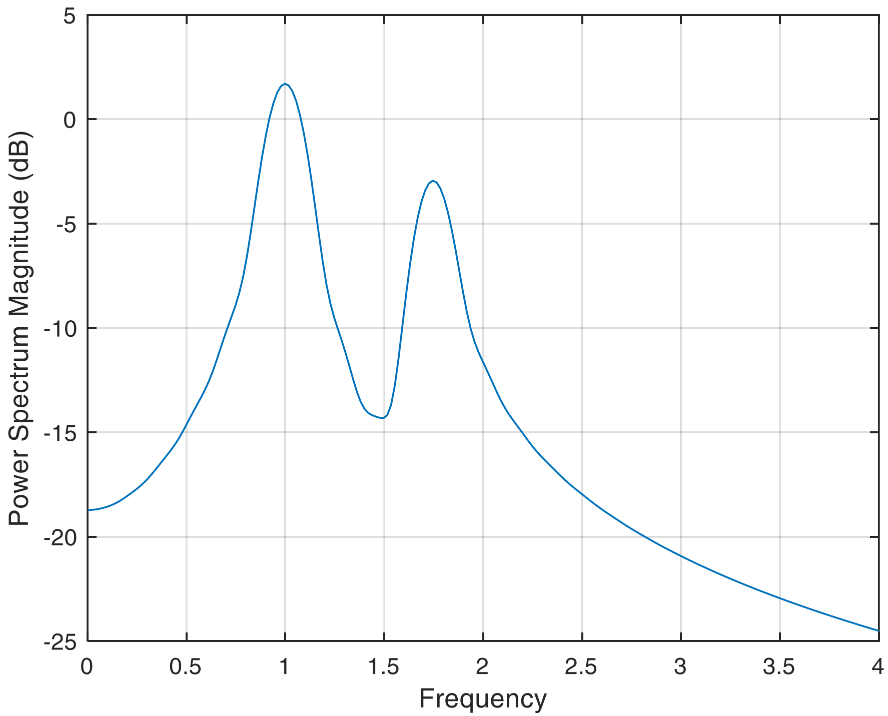


Configuration space, dofs 1-2
x_2 [m]
x_1 [m]
-3
-2
-1
0
1
2
3

General ICs







in rad/s

Vibration modes / normal modes / mode shapes

Clear physical meaning:

- Structural deformation at resonance
- Synchronous vibration of the structure

Important mathematical properties:

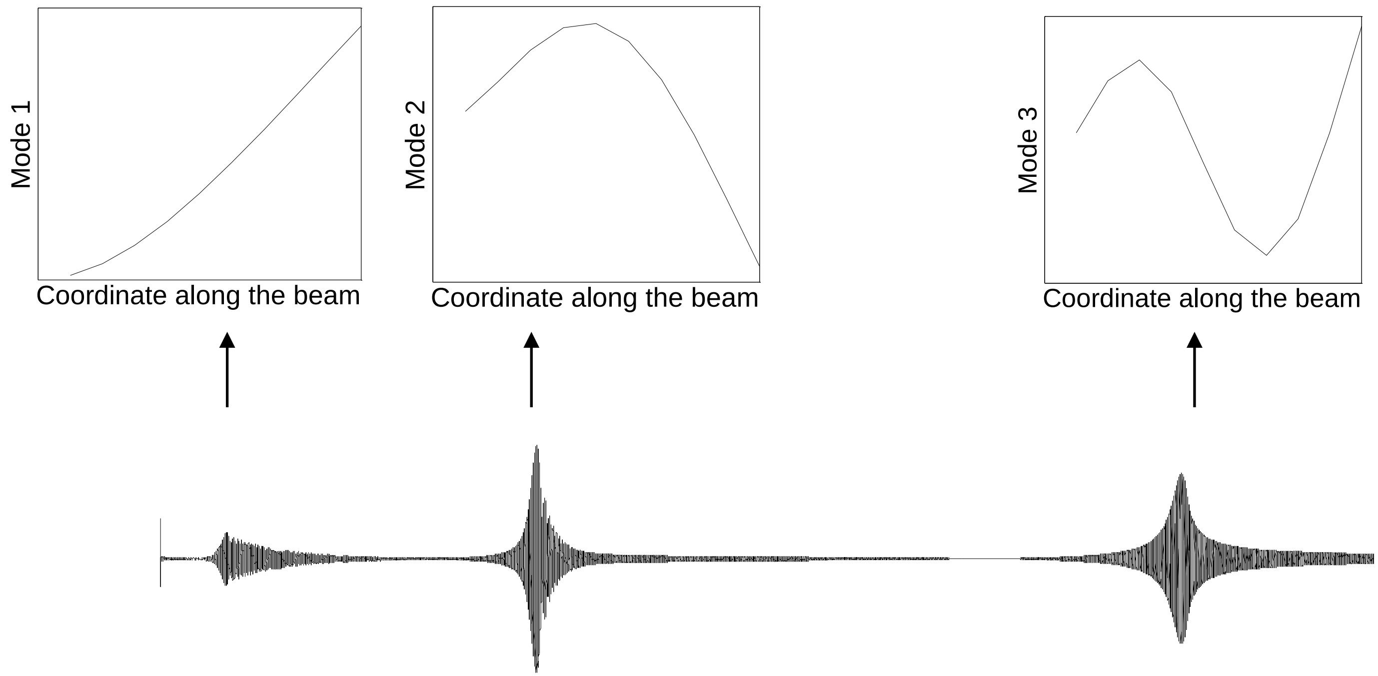
- Orthogonality
- Decoupling of the equations of motion (modal superposition)

Property 1: deformation at resonance

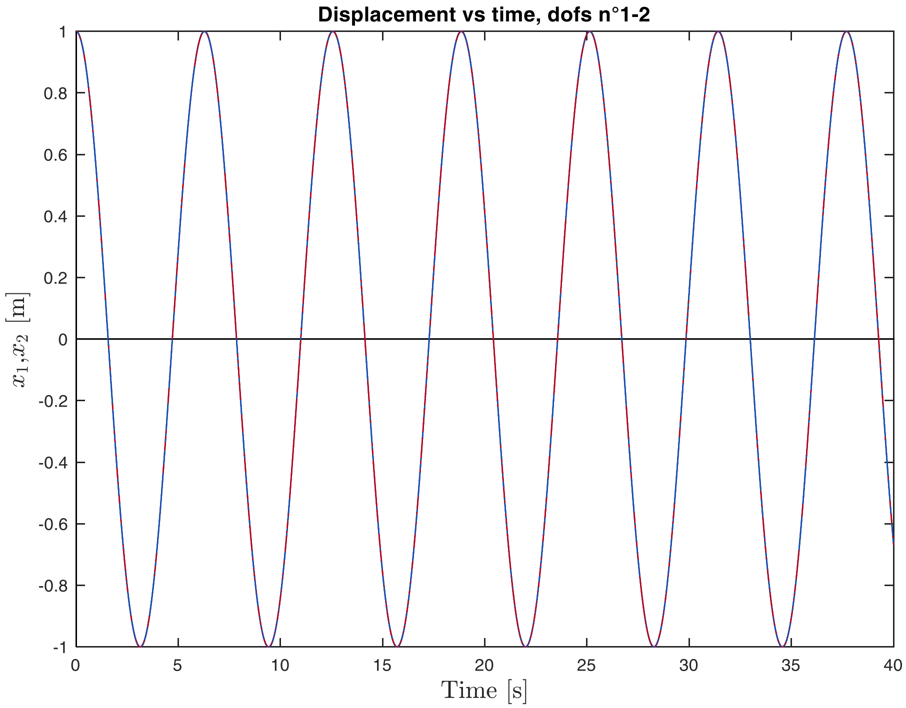
Property 1: deformation at resonance

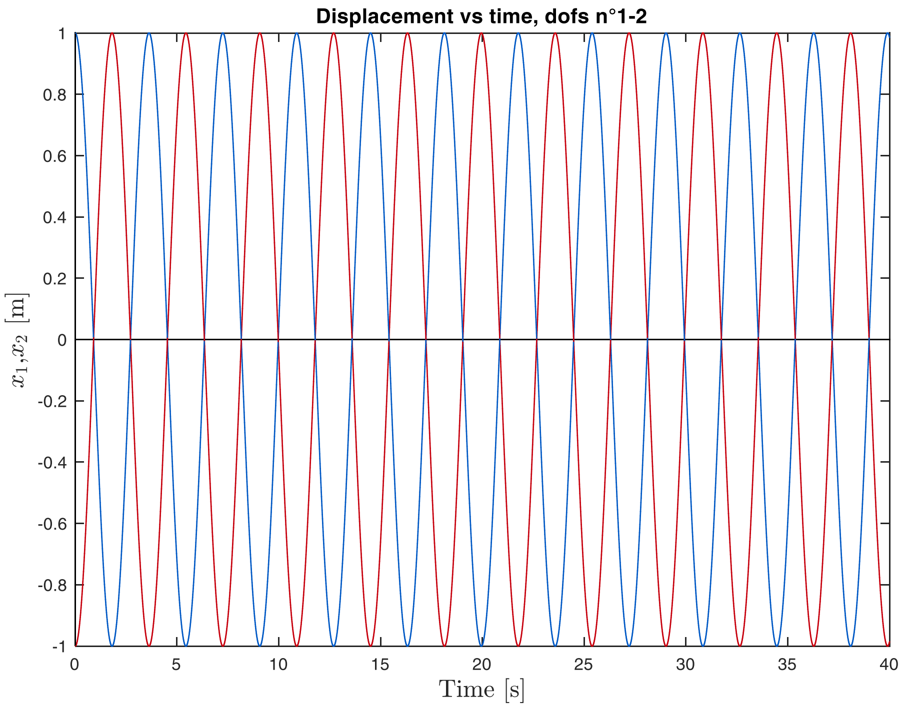
Modes

Time series



Property 2: synchronous motion





Property 3: orthogonality

$$\boldsymbol{K}\boldsymbol{y}_{(s)} - \omega_s^2 \boldsymbol{M}\boldsymbol{y}_{(s)} = \boldsymbol{0}$$

⇩

$$\boldsymbol{y}_{(r)}^T \boldsymbol{K}\boldsymbol{y}_{(s)} = \omega_s^2 \boldsymbol{y}_{(r)}^T \boldsymbol{M}\boldsymbol{y}_{(s)}$$

-

$$\boldsymbol{y}_{(s)}^T \boldsymbol{K}\boldsymbol{y}_{(r)} = \omega_r^2 \boldsymbol{y}_{(s)}^T \boldsymbol{M}\boldsymbol{y}_{(r)}$$

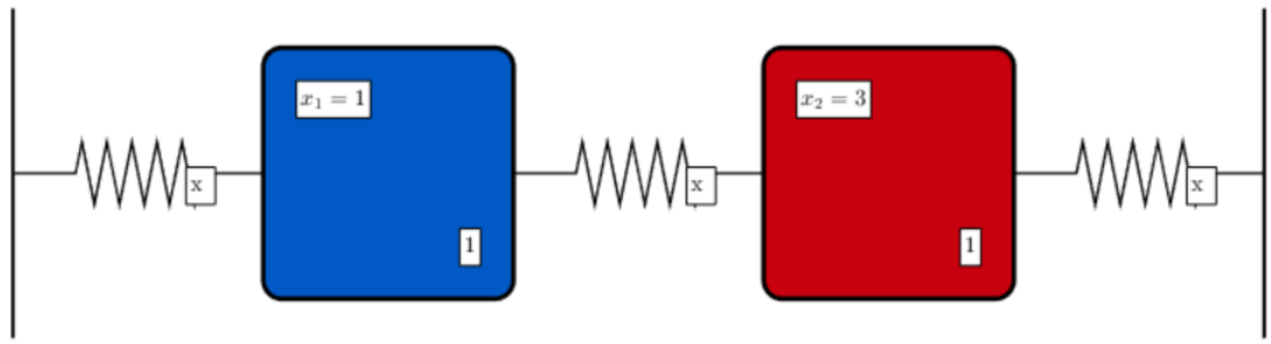
$$\boldsymbol{0} = (\omega_s^2 - \omega_r^2) \boldsymbol{y}_{(r)}^T \boldsymbol{M}\boldsymbol{y}_{(s)}$$

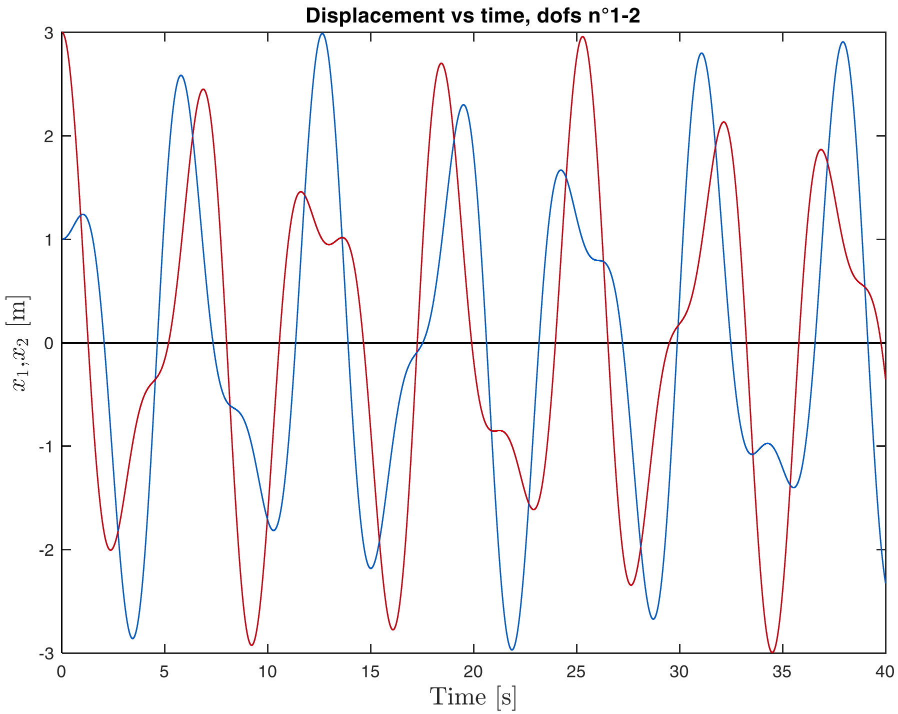
$r \neq s$

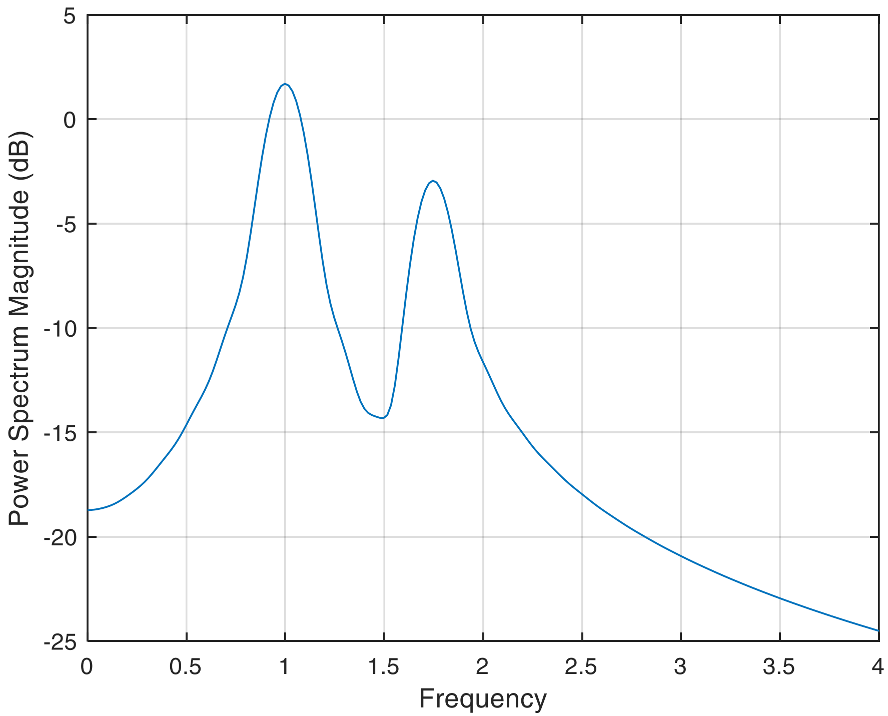
⇨ $$\boldsymbol{0} = \boldsymbol{y}_{(r)}^T \boldsymbol{M}\boldsymbol{y}_{(s)}$$

⇨ $$\boldsymbol{0} = \boldsymbol{y}_{(r)}^T \boldsymbol{K}\boldsymbol{y}_{(s)}$$

Property 4: modal superposition







in rad/s

Property 4: modal superposition

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = 0$$

MODAL EXPANSION

$$\boldsymbol{x} = \sum_{s=1}^{n} \boldsymbol{y}_{(s)} \varphi_{(s)}(t) = \boldsymbol{Y}\boldsymbol{\varphi}(t) \Rightarrow \ddot{\boldsymbol{\varphi}} + \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & \omega_n^2 \end{bmatrix} \boldsymbol{\varphi} = \boldsymbol{0}$$

eigenmodes ($\boldsymbol{y}_{(s)}$)

normal coordinates ($\varphi_{(s)}$)

Uncoupled normal equations $\Rightarrow$ *the energy initially imparted to one mode remains in this mode for ever*

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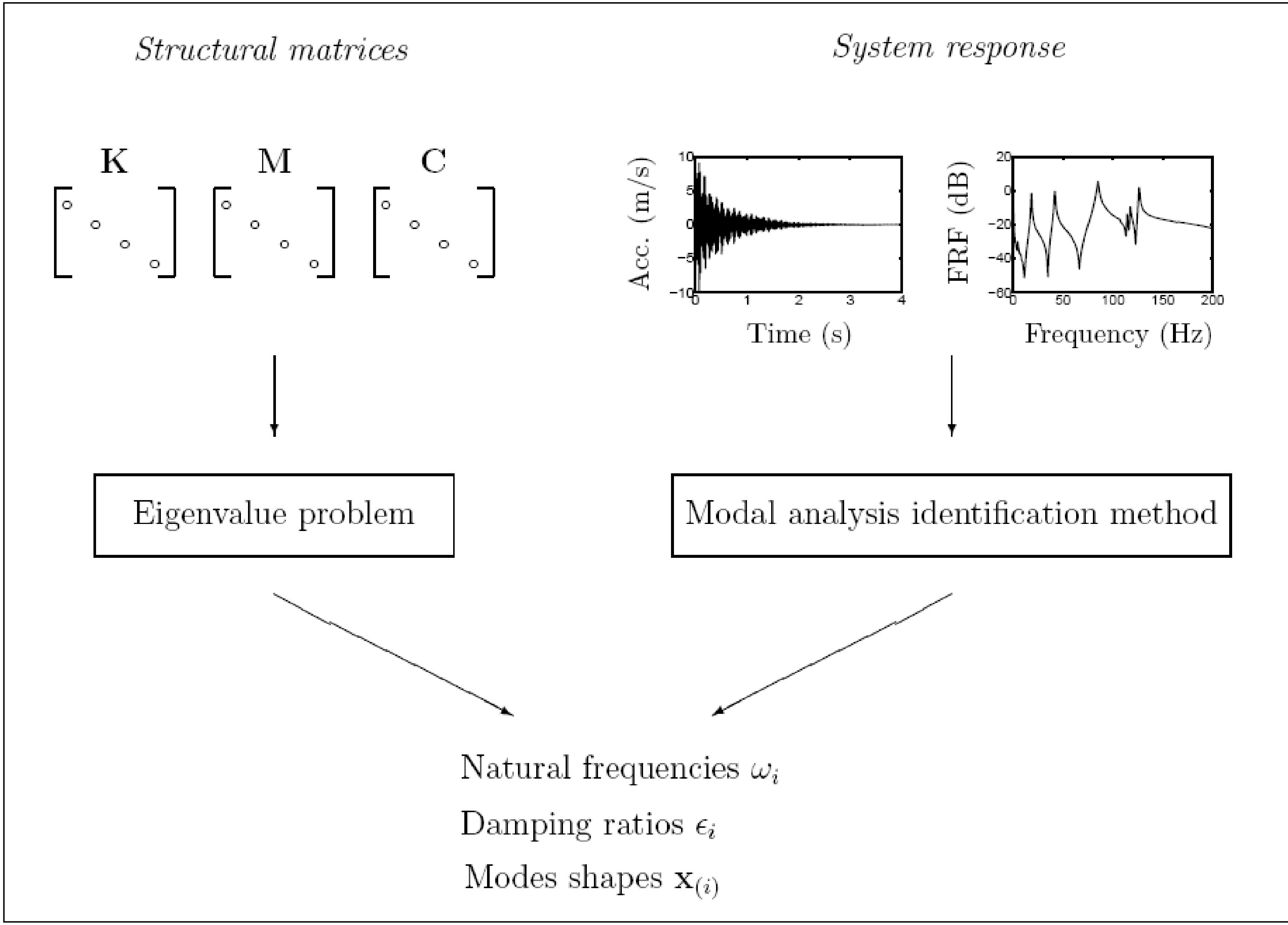
Property 4: modal superposition

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = 0$$

⇩

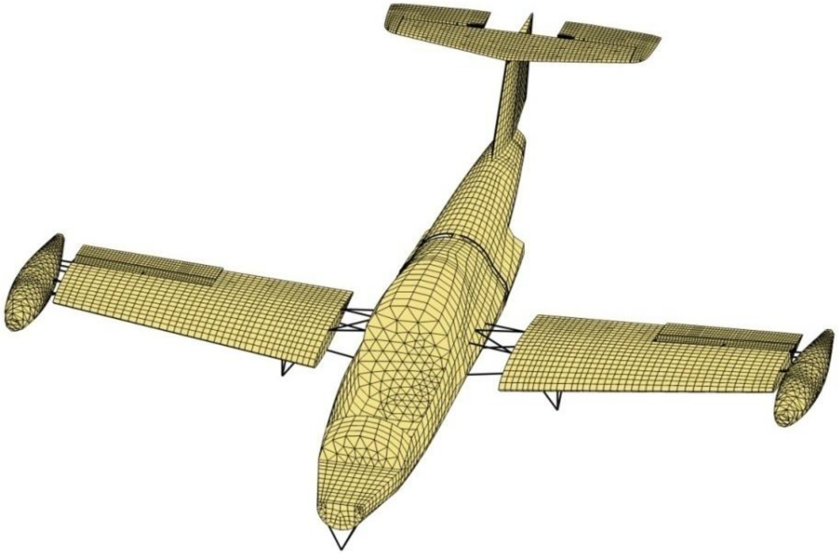
$$\boldsymbol{x}(t) = \left(\sum_{s=1}^{n} \frac{\boldsymbol{y}_{(s)}\boldsymbol{y}_{(s)}^T\boldsymbol{M}}{\mu_s} \cos\omega_s t\right)\boldsymbol{x}_0 + \left(\sum_{s=1}^{n} \frac{\boldsymbol{y}_{(s)}\boldsymbol{y}_{(s)}^T\boldsymbol{M}}{\mu_s\omega_s} \sin\omega_s t\right)\dot{\boldsymbol{x}}_0$$

$$\mu_s = \boldsymbol{y}_{(s)}^T\boldsymbol{M}\boldsymbol{y}_{(s)}$$



Numerical and experimental modal analysis (TMA and EMA)

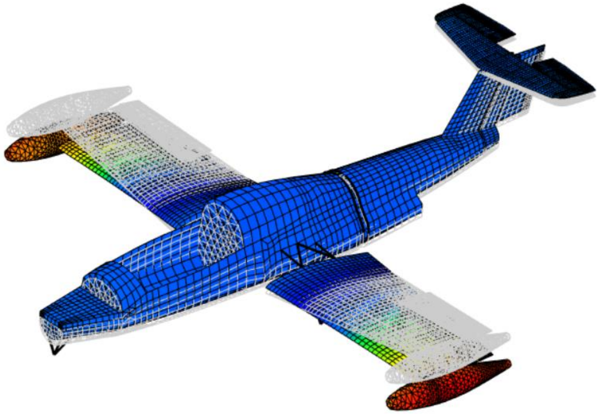
Aircraft example



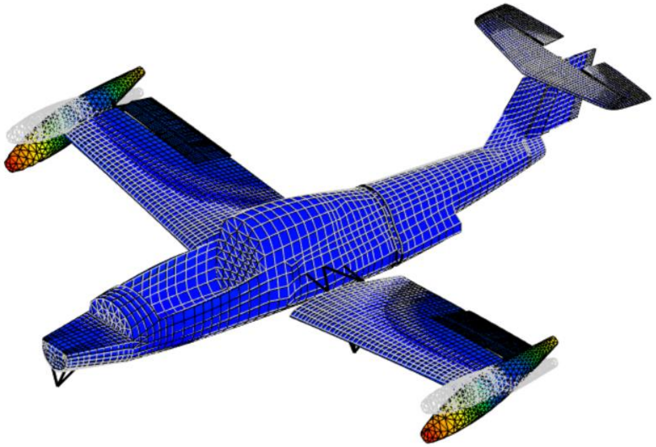
Finite element model
(2D shells and beams, 85000 DOFs)

Numerical modal analysis

Mode	Freq. (Hz)	Mode	Freq. (Hz)
1	0.0936	13	21.2193
2	0.7260	14	22.7619
3	0.9606	15	23.6525
4	1.2118	16	25.8667
5	1.2153	17	28.2679
6	1.7951	18	29.3309
7	2.1072	19	31.0847
8	2.5157	20	34.9151
9	3.5736	21	39.5169
10	8.1913	22	40.8516
11	9.8644	23	47.3547
12	16.1790	24	52.1404



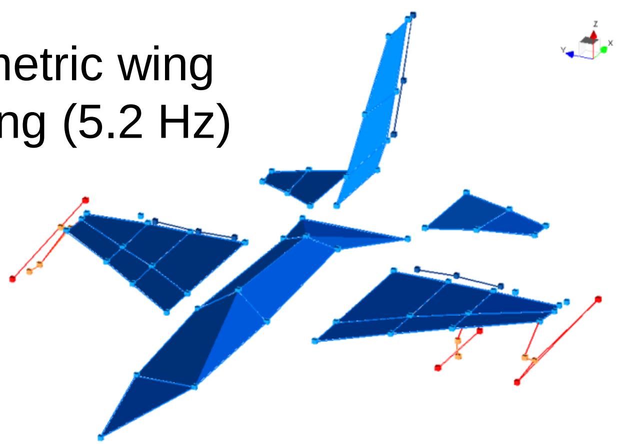
Wing bending



Wing torsion (symmetric)

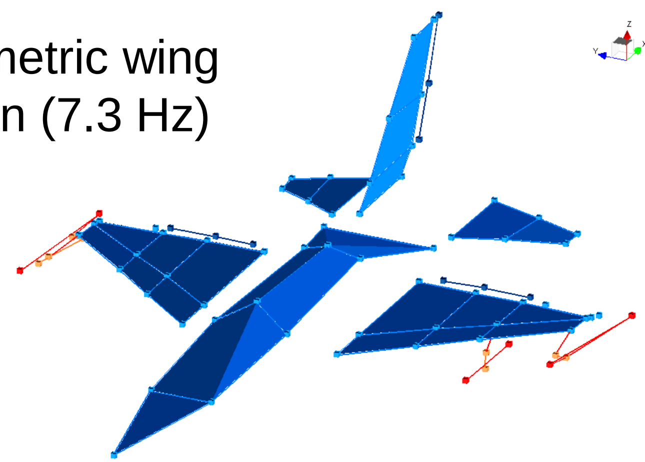
Experimental modal analysis

Symmetric wing
bending (5.2 Hz)



Mode 3 : 4.8668 Hz, 0.66 % Low level (8N)

Symmetric wing
torsion (7.3 Hz)



Mode 5 : 7.0303 Hz, 0.58 % Low level (8N)

Outline

Undamped, unforced response

Undamped, harmonic response

Damped, harmonic response

Vibration absorbers

Undamped system with harmonic excitation

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{f}$$

$$\boldsymbol{x} = \boldsymbol{X}e^{i\omega t} \qquad \boldsymbol{f} = \boldsymbol{F}e^{i\omega t}$$

$$\Downarrow$$

$$(\boldsymbol{K} - \omega^2\boldsymbol{M})\boldsymbol{X} = \boldsymbol{F}$$

2DOF system

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} e^{i\omega t} \qquad f = Fe^{i\omega t}$$

⇩

$$\begin{bmatrix} 2k - \omega^2 m & -k \\ -k & 2k - \omega^2 m \end{bmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ F \end{pmatrix}$$

$$\frac{X_1}{F} = \frac{-k}{(2k - \omega^2 m)^2 - k^2}$$

$$\omega_1^2 = k/m$$

$$\omega_2^2 = 3k/m$$

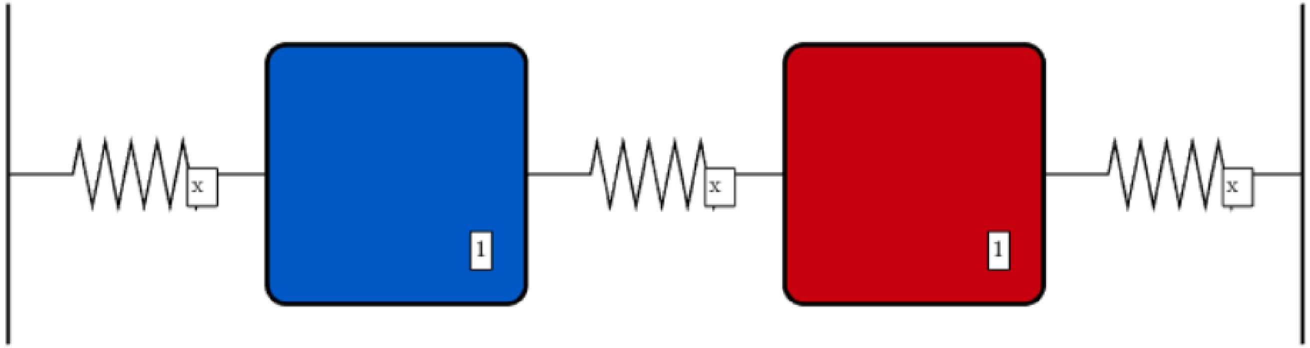
Resonance

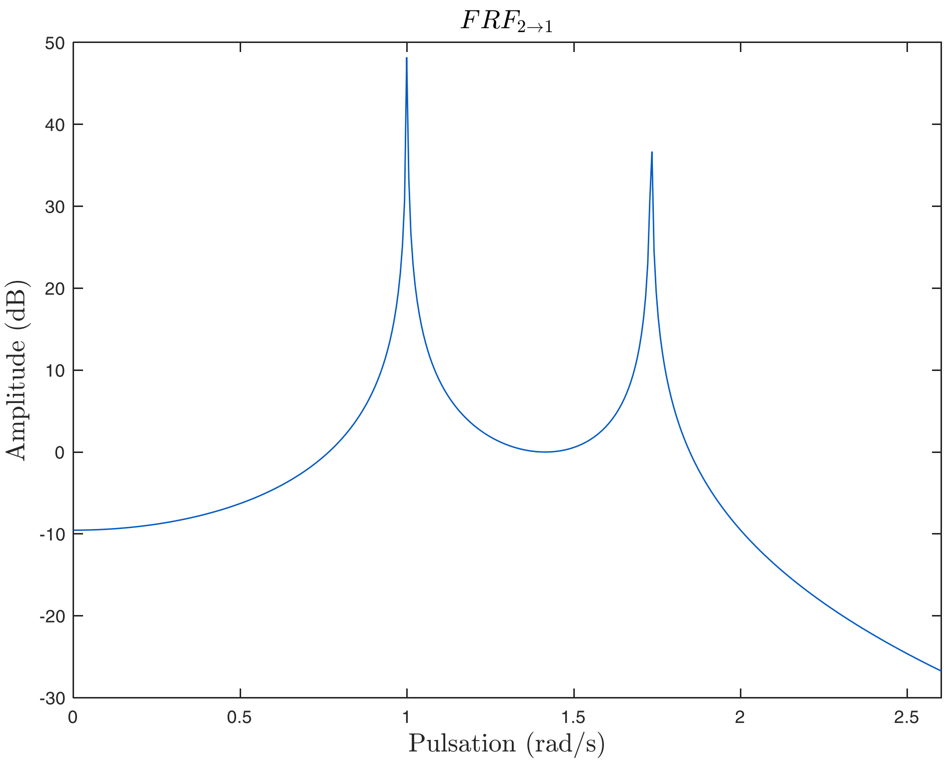
$$\frac{X_2}{F} = \frac{2k - \omega^2 m}{(2k - \omega^2 m)^2 - k^2}$$

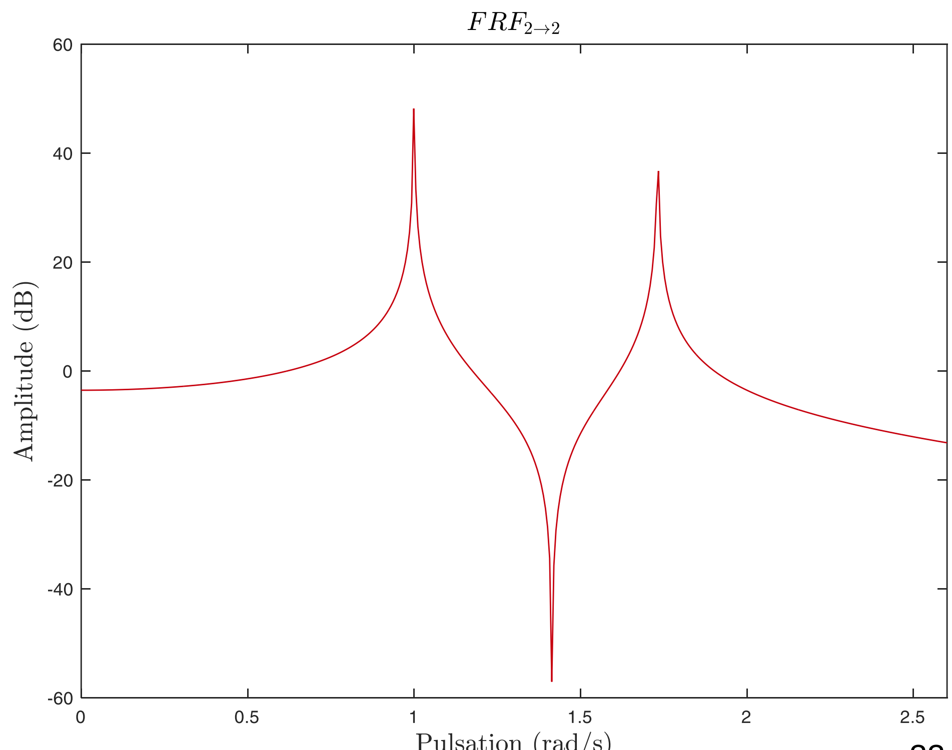
$$\omega_A^2 = 2k/m$$

Antiresonance

2DOF system


x
1
x
1
x


$FRF_{2\to 1}$
Amplitude (dB)
Pulsation (rad/s)
50
40
30
20
10
0
-10
-20
-30
0
0.5
1
1.5
2
2.5


$FRF_{2\to 2}$
Amplitude (dB)
Pulsation (rad/s)
60
40
20
0
-20
-40
-60
0
0.5
1
1.5
2
2.5

Outline

Undamped, unforced response

Undamped, harmonic response

Damped, harmonic response

Vibration absorbers

Damped systems with harmonic excitation

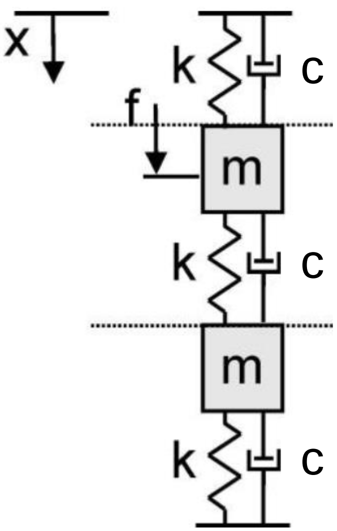
$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{f}$$

$$\boldsymbol{x} = \boldsymbol{X}e^{i\omega t} \qquad \boldsymbol{f} = \boldsymbol{F}e^{i\omega t}$$

⇩

$$(-\omega^2\boldsymbol{M} + i\omega\boldsymbol{C} + \boldsymbol{K}\,)\boldsymbol{X} = \boldsymbol{F}$$

2DOF system



$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2c & -c \\ -c & 2c \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} e^{i\omega t} \qquad f = Fe^{i\omega t}$$

$$\begin{bmatrix} 2k + 2i\omega c - \omega^2 m & -k - i\omega c \\ -k - i\omega c & 2k + 2i\omega c - \omega^2 m \end{bmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ F \end{pmatrix}$$

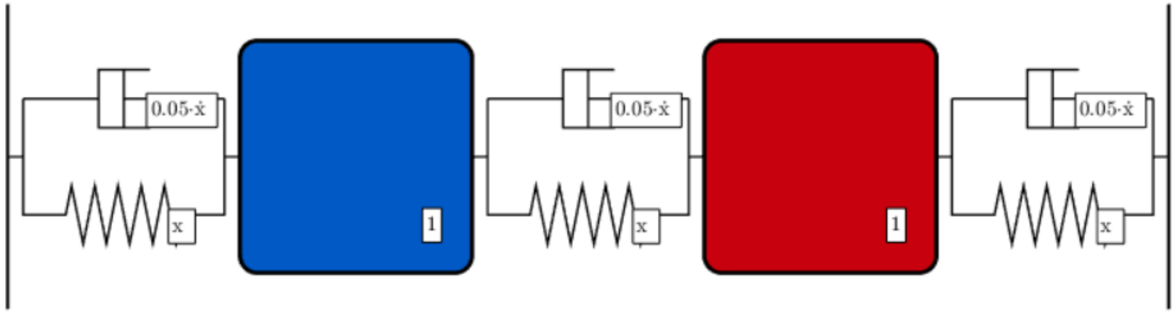
$$\frac{X_1}{F} = \frac{-k + i\omega c}{(2k + 2i\omega c - \omega^2 m)^2 - (k + i\omega c)^2}$$

Damped resonance

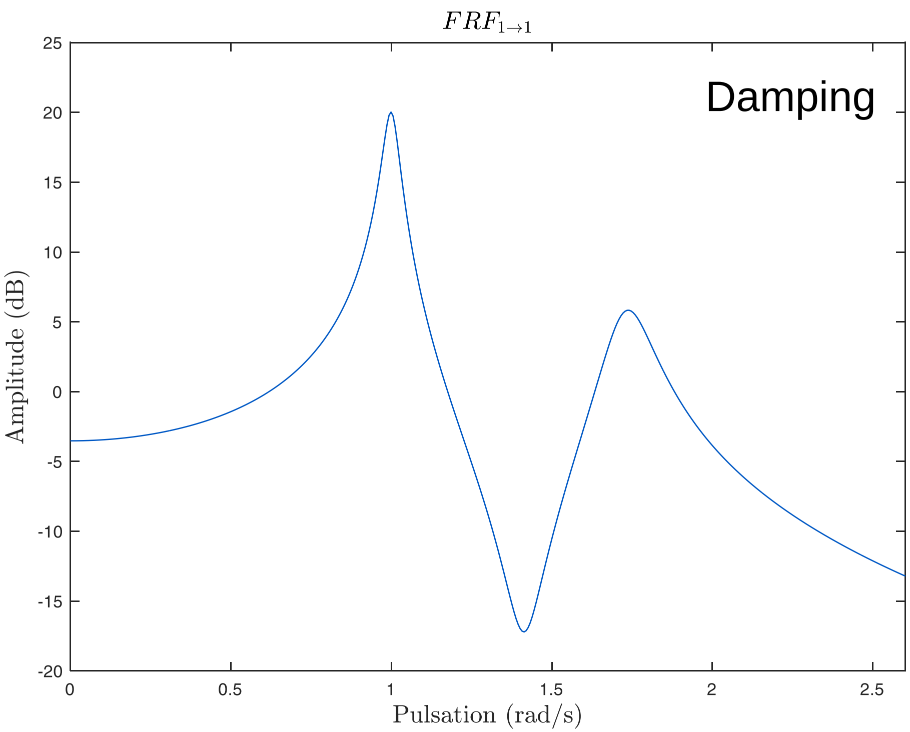
$$\frac{X_2}{F} = \frac{2k + 2i\omega c - \omega^2 m}{(2k + 2i\omega c - \omega^2 m)^2 - (k + i\omega c)^2}$$

No strict antiresonance

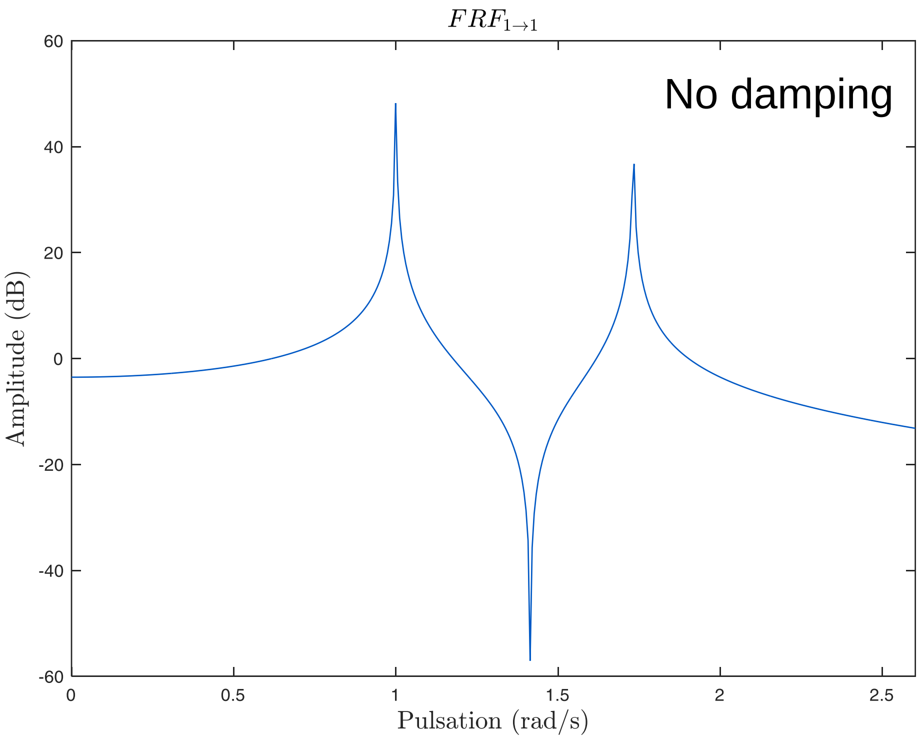
2DOF system



0.05·ẋ
x
1
0.05·ẋ
x
1
0.05·ẋ
x

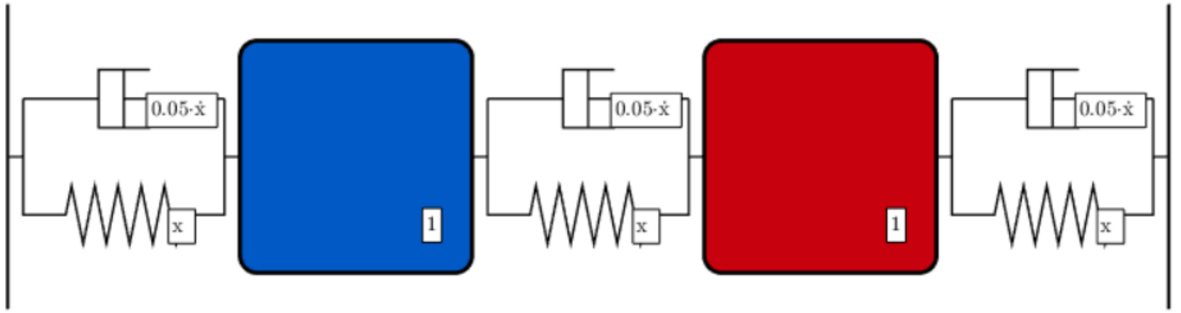


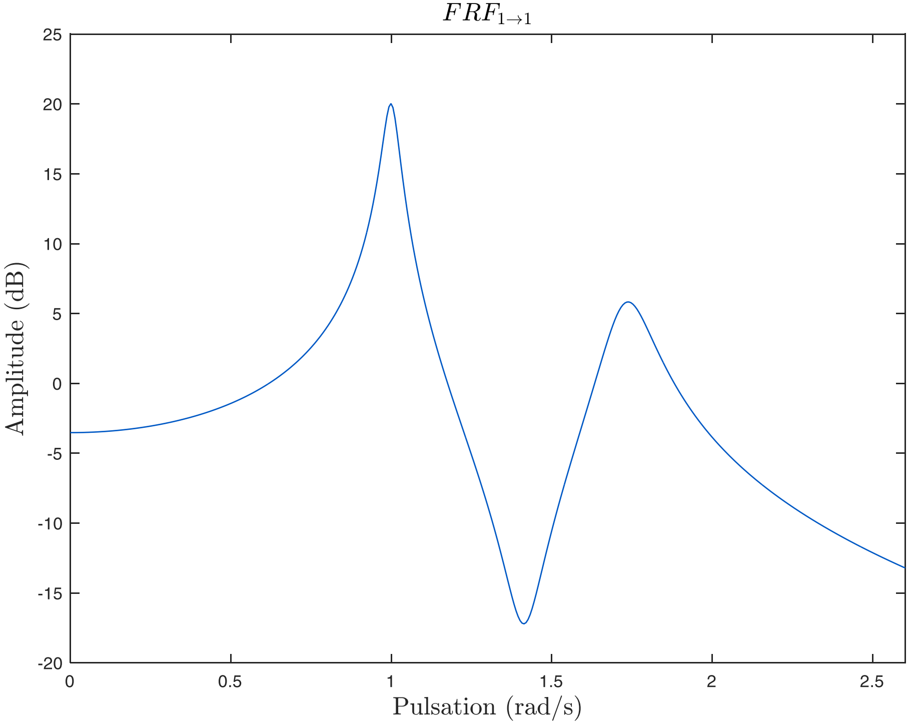
$FRF_{1\to 1}$
Damping
Amplitude (dB)
Pulsation (rad/s)

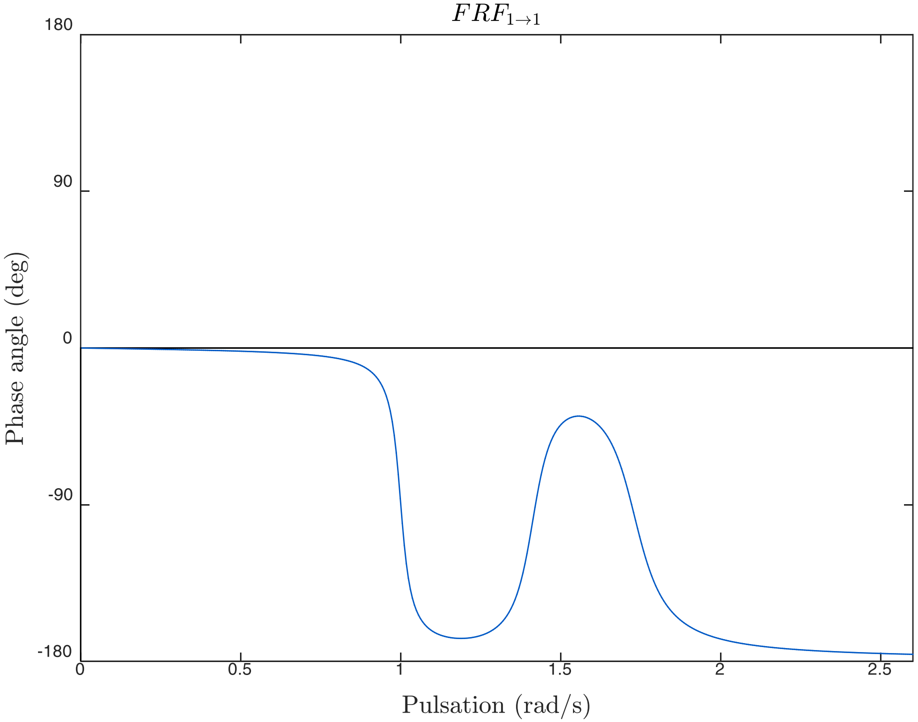


$FRF_{1\to 1}$
No damping
Amplitude (dB)
Pulsation (rad/s)

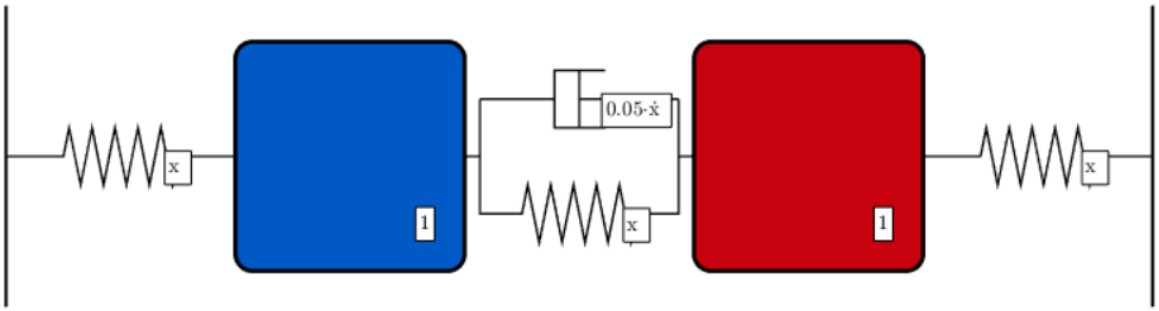
2DOF system

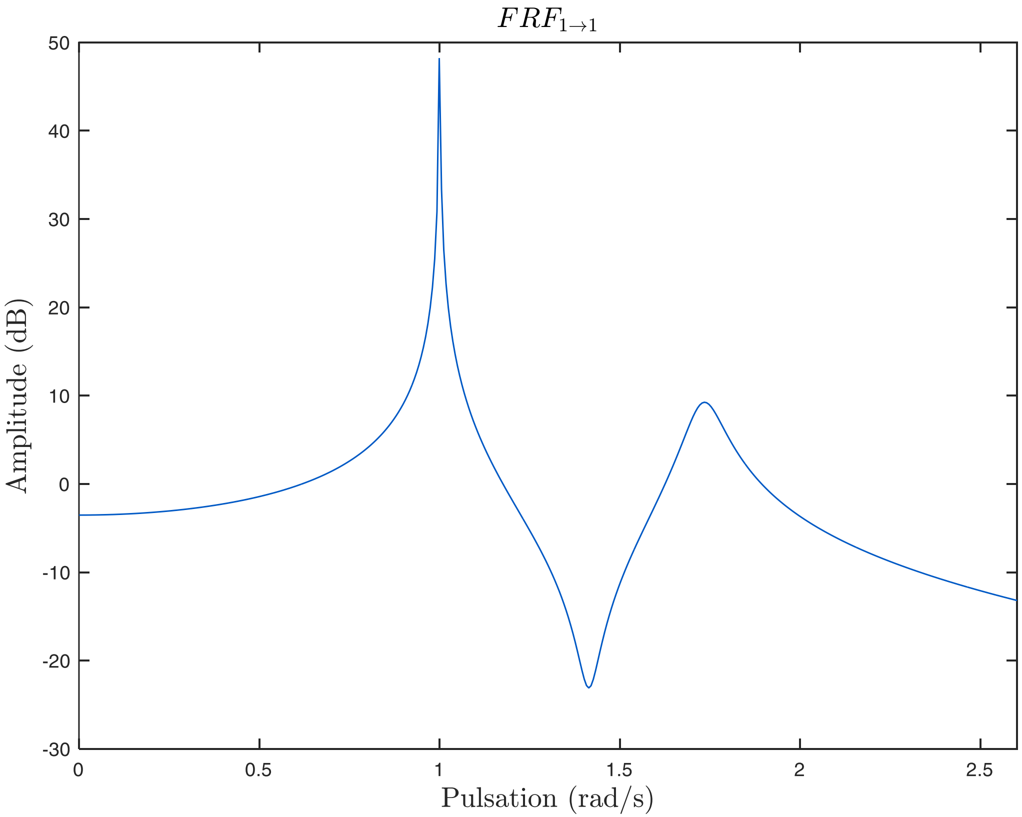

0.05·ẋ
0.05·ẋ
0.05·ẋ
x
1
x
1
x


$FRF_{1\to1}$
Amplitude (dB)
Pulsation (rad/s)
25
20
15
10
5
0
-5
-10
-15
-20
0
0.5
1
1.5
2
2.5

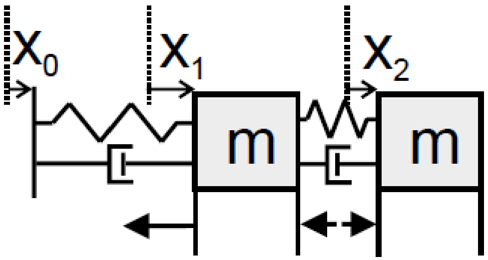

$FRF_{1\to1}$
Phase angle (deg)
Pulsation (rad/s)
180
90
0
-90
-180
0
0.5
1
1.5
2
2.5

Can you predict the damping of the modes ?





Base-excited systems



Equations of motion:

$$
\begin{aligned}
m\ddot{x}_1 &= -k(x_1 - x_0) - b(\dot{x}_1 - \dot{x}_0) - k(x_1 - x_2) - b(\dot{x}_1 - \dot{x}_2) \\
m\ddot{x}_2 &= k(x_1 - x_2) + b(\dot{x}_1 - \dot{x}_2)
\end{aligned}
$$

$$
\begin{aligned}
x_{1r} &= x_1 - x_0 \\
x_{2r} &= x_2 - x_0
\end{aligned}
\quad \longrightarrow \quad
\begin{aligned}
m\ddot{x}_{1r} + 2b\dot{x}_{1r} - b\dot{x}_{2r} + 2kx_{1r} - kx_{2r} &= -m\ddot{x}_0 \\
m\ddot{x}_{2r} + b\dot{x}_{2r} - b\dot{x}_{1r} + kx_{2r} - kx_{1r} &= -m\ddot{x}_0
\end{aligned}
$$

$$
\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_{1r} \\ \ddot{x}_{2r} \end{Bmatrix} + \begin{bmatrix} 2b & -b \\ -b & b \end{bmatrix} \begin{Bmatrix} \dot{x}_{1r} \\ \dot{x}_{2r} \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_{1r} \\ x_{2r} \end{Bmatrix} = \begin{Bmatrix} -m\ddot{x}_0 \\ -m\ddot{x}_0 \end{Bmatrix}
$$

Base-excited systems

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x_1}_r \\ \ddot{x_2}_r \end{Bmatrix} + \begin{bmatrix} 2b & -b \\ -b & b \end{bmatrix} \begin{Bmatrix} \dot{x_1}_r \\ \dot{x_2}_r \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_{1r} \\ x_{2r} \end{Bmatrix} = \begin{Bmatrix} -m\ddot{x}_0 \\ -m\ddot{x}_0 \end{Bmatrix}$$

Matrix notations:

$$M\ddot{x}_r + C\dot{x}_r + Kx_r = -M\ddot{x}_b$$

$$x_r = \begin{Bmatrix} x_1 - x_0 \\ x_2 - x_0 \end{Bmatrix} \qquad \ddot{x}_b = \begin{Bmatrix} \ddot{x}_0 \\ \ddot{x}_0 \end{Bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \ddot{x}_0 = T\,\ddot{x}_0$$

→ All developments for force excitation apply

Outline

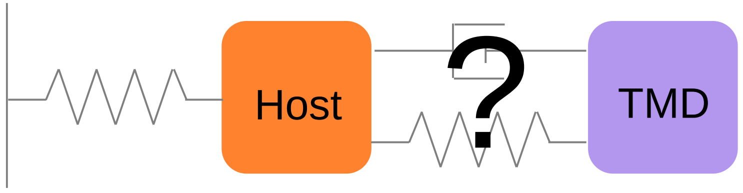
Undamped, unforced response

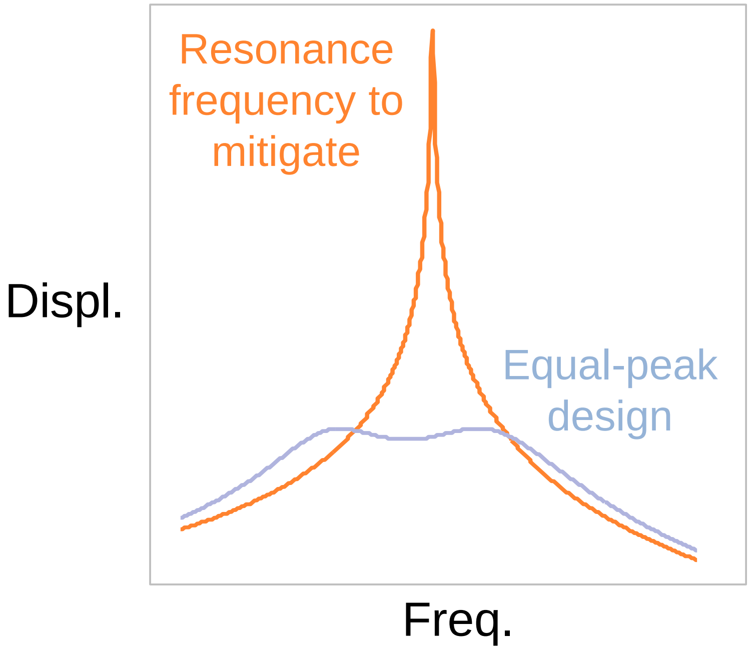
Undamped, harmonic response

Damped, harmonic response

Vibration absorbers

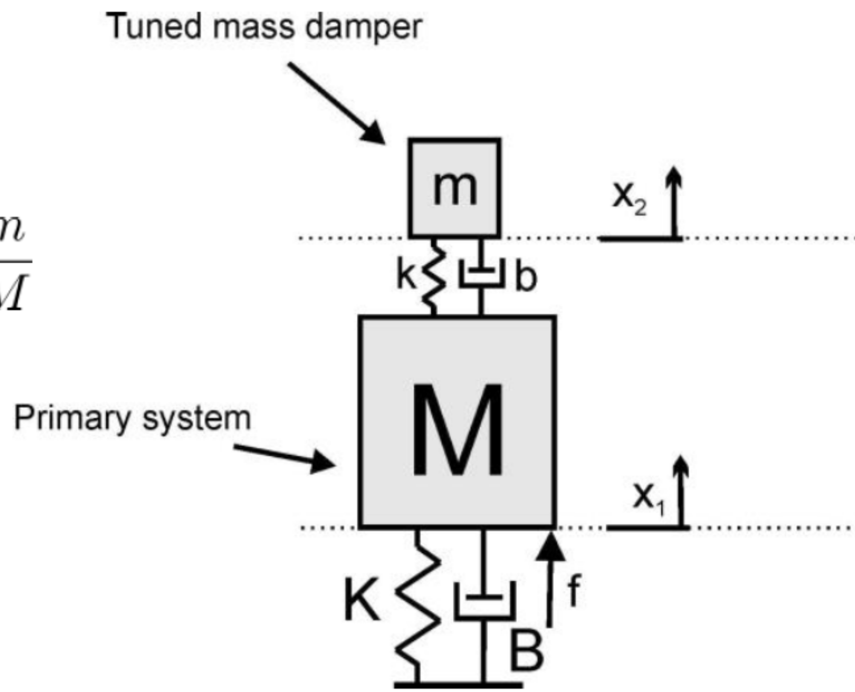
The Design Problem: Resonance Mitigation





Tuned mass damper (TMD) principle

$$\mu = \frac{m}{M}$$



$$\omega_n = \sqrt{\frac{k}{m}}$$

$$\Omega = \sqrt{\frac{K}{M}}$$

$$\nu = \frac{\omega_n}{\Omega}$$

Equations of motion:

$$\begin{bmatrix} M & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} B+b & -b \\ -b & b \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} \begin{bmatrix} K+k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f \\ 0 \end{Bmatrix}$$

Harmonic excitation

Harmonic excitation:

$$\begin{bmatrix} K+k+i\omega(B+b)-\omega^2 M & -(k+i\omega b) \\ -(k+i\omega b) & k-\omega^2 m+i\omega b \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \end{Bmatrix}$$

$$X_1/F = \frac{k-\omega^2 m+i\omega b}{(K+k+i\omega(B+b)-\omega^2 M)(k-\omega^2 m+i\omega b)-(k+i\omega b)^2}$$

Undamped vibration absorber (b=0)

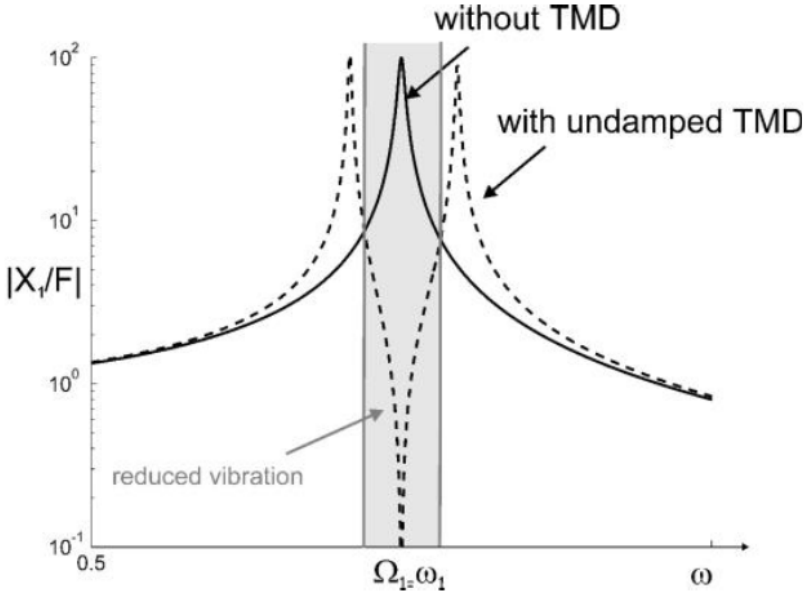
$$X_1/F = \frac{k-\omega^2 m}{(K+k+i\omega B-\omega^2 M)(k-\omega^2 m)-k^2}$$

$$X_1 = 0 \quad \text{for} \quad \omega = \sqrt{\frac{k}{m}} = \omega_n$$

Undamped TMD

If you choose $\omega_n = \Omega$ you can cancel the vibration of the primary system at its natural frequency

$$\mu = 0.03,\ \nu = 1$$

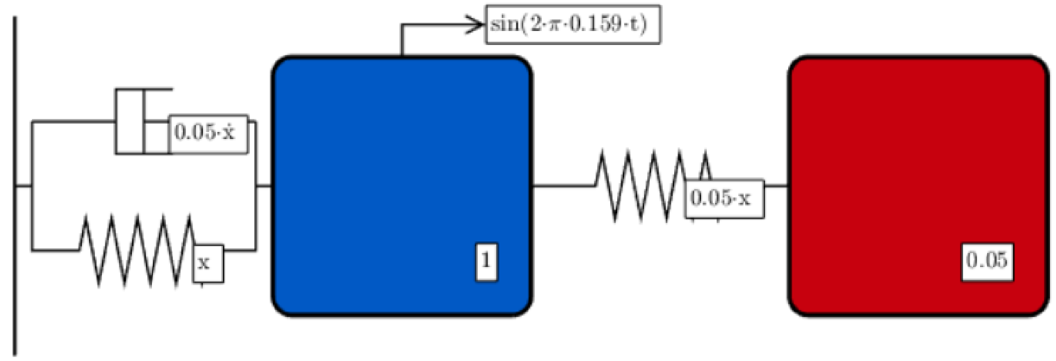


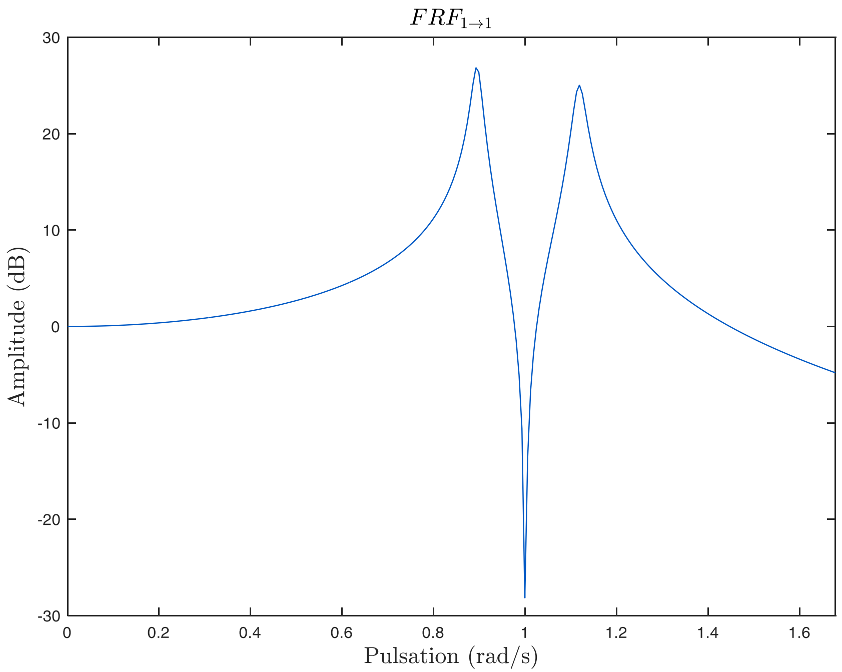
The damping device is tuned to the eigenfrequency of the primary system

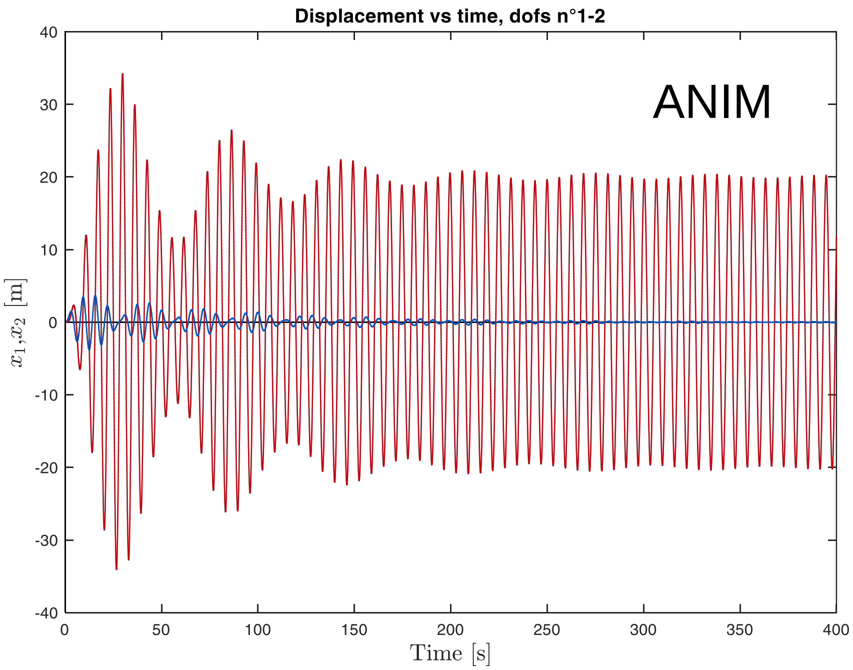
-> Reduces vibrations in a narrow band around eigenfrequency

-> Amplification outside of this narrow band

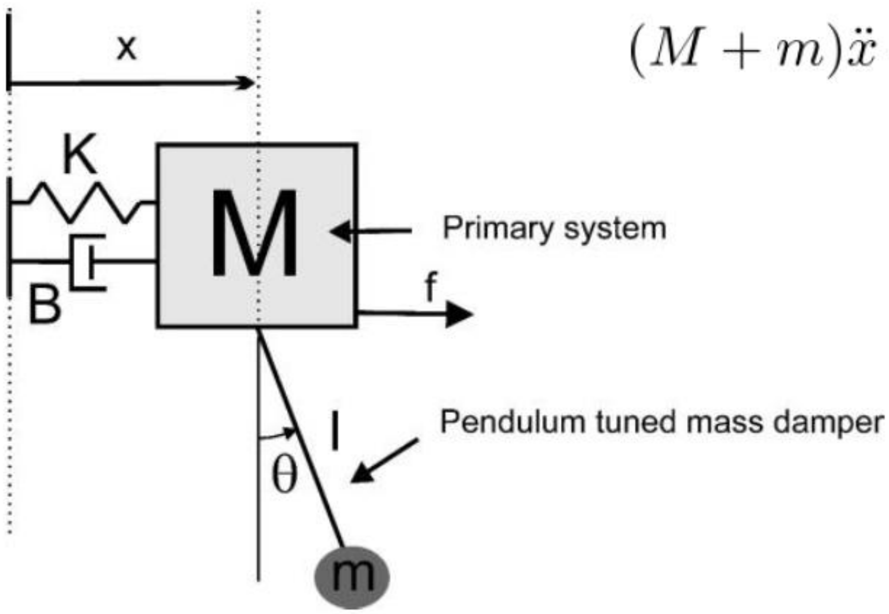
Peformance of an undamped TMD







Pendulum TMD



$$
\begin{aligned}
(M+m)\ddot{x} + ml\ddot{\theta} + Kx + B\dot{x} &= f \\
m(\ddot{x} + l\ddot{\theta}) + mg\theta &= 0
\end{aligned}
$$

θ small

$$\omega_n = \sqrt{\frac{g}{l}} \qquad \Omega = \sqrt{\frac{K}{M}}$$

$$\nu = \frac{\omega_n}{\Omega} \qquad \mu = \frac{m}{M}$$

$$\begin{bmatrix} M+m & ml \\ 1 & l \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} K & 0 \\ 0 & g \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} f \\ 0 \end{Bmatrix}$$

Inertial coupling of the two systems

Pendulum TMD

Harmonic excitation:

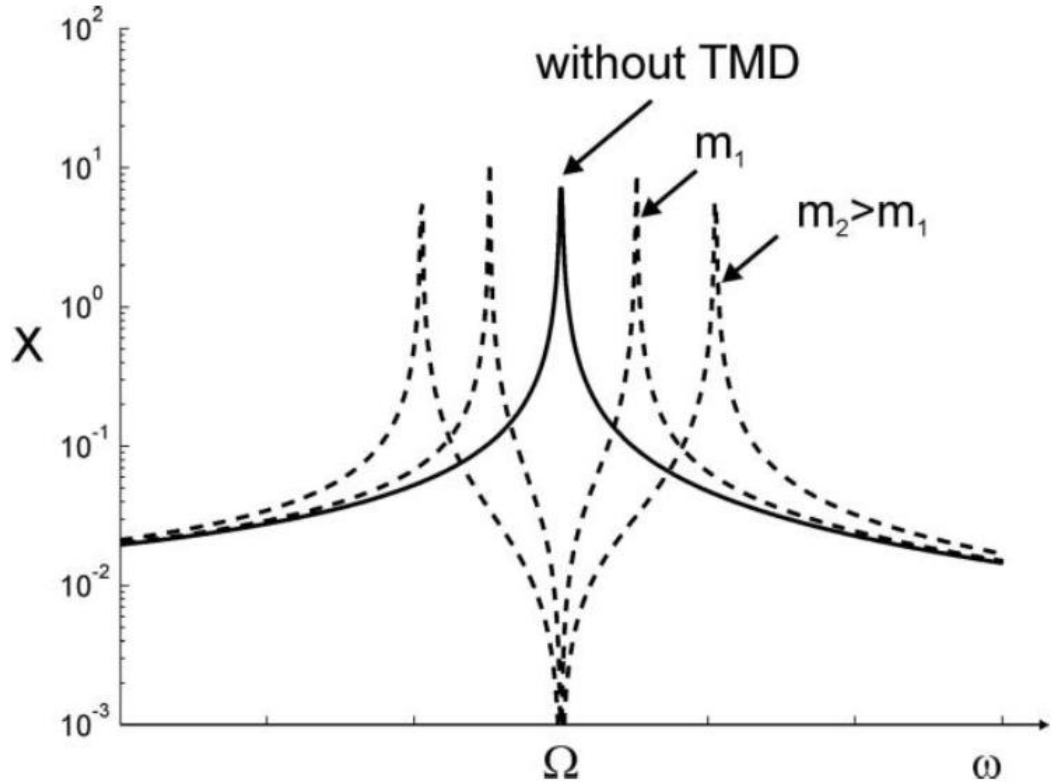
$$\begin{bmatrix} K + i\omega B - \omega^2(M+m) & -ml\omega^2 \\ -\omega^2 & g - \omega^2 l \end{bmatrix} \begin{Bmatrix} X \\ \Theta \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \end{Bmatrix}$$

$$\frac{X}{F} = \frac{g - \omega^2 l}{(K + i\omega B - \omega^2(M+m))(g - \omega^2 l) + \omega^4 ml}$$

$$X = 0 \quad \text{for} \quad \omega = \sqrt{\frac{g}{l}} = \omega_n$$

Undamped pendulum TMD

Tuning of the PTMD based on the length of the pendulum $\omega_n = \sqrt{\frac{g}{l}}$



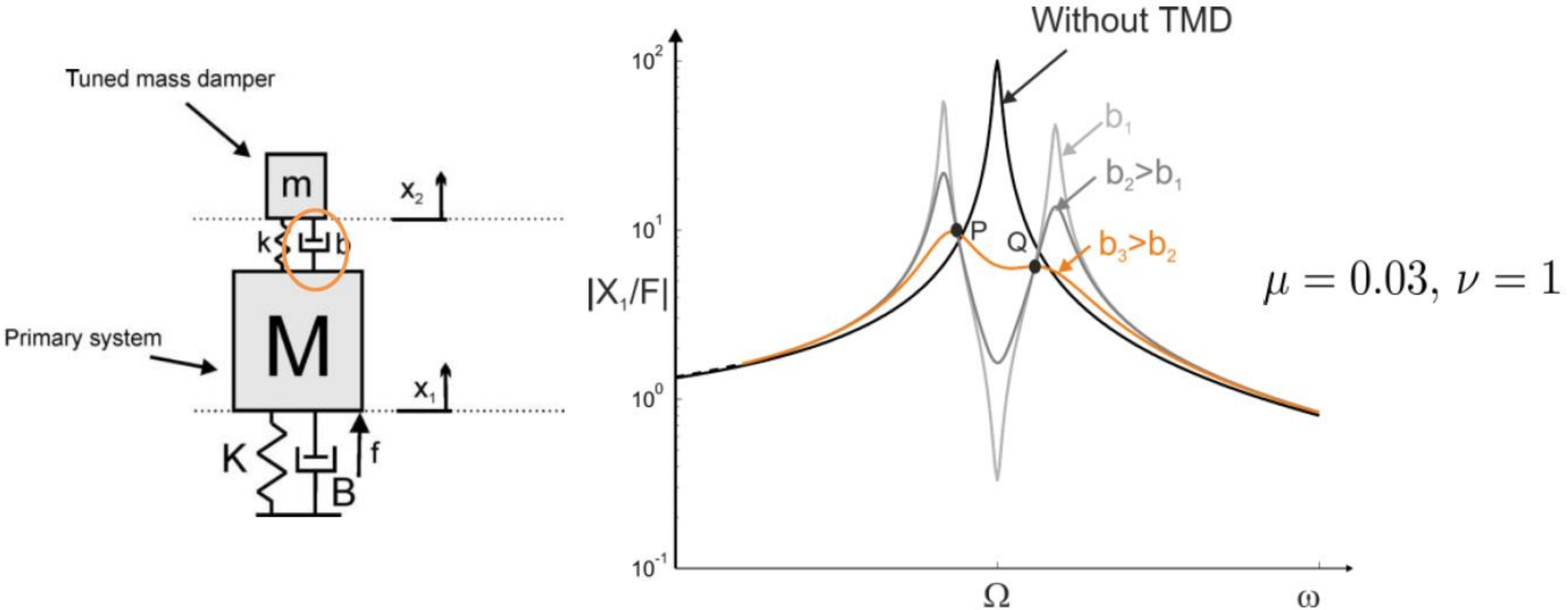
$\nu = 1$

Effect of the mass mainly on the spreading of the peaks

Pendulum TMD

https://www.youtube.com/watch?v=GzMuF-LMGaM

Damped TMD



-Reduction of vibration is lower around eigenfrequency with b increasing
-Reduces the amplification outside of the narrow frequency band
-Existence of P and Q : points where all curves cross

Optimal design of TMD

P and Q are at equal height for

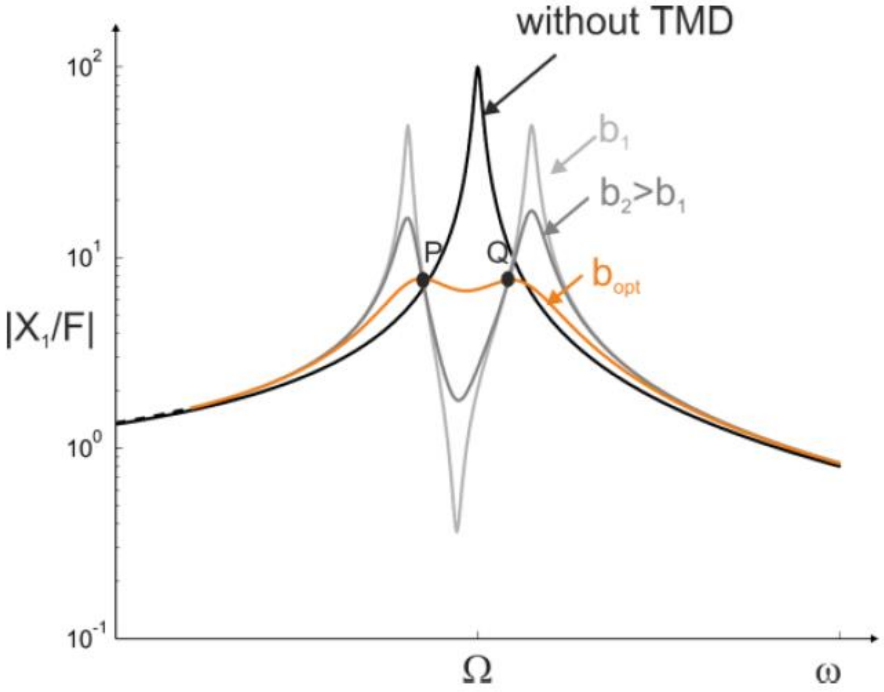
$$\nu = \frac{1}{1+\mu}$$

Optimum damping is given by

$$\xi = \sqrt{\frac{3\mu}{8(1+\mu)}}$$

$$\longrightarrow \quad b = 2\xi\sqrt{km}$$

[Den Hartog, 1954]



Optimal design of TMD

- The maximum mass of the device is decided fixing $\mu = \frac{m}{M}$

- Based on this value, the frequency of the TMD is tuned : $\nu = \frac{1}{1+\mu}$

- Which allows to compute the stiffnes of the TMD

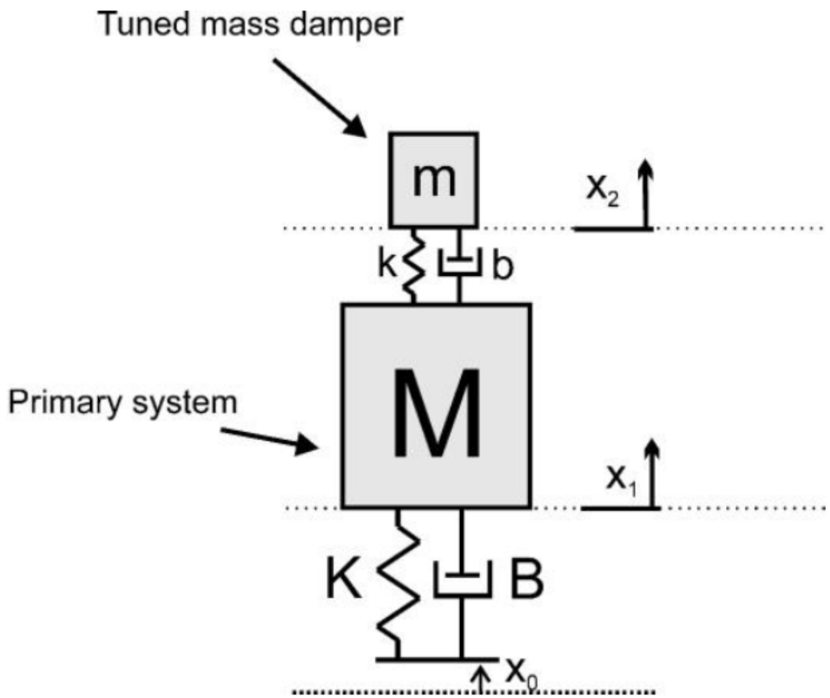
$$k = \nu^2 K \mu = K \frac{\mu}{(1+\mu)^2}$$

- And finally the optimal damping is computed

$$\xi = \sqrt{\frac{3\mu}{8(1+\mu)}} \longrightarrow b = 2\xi\sqrt{km}$$

Optimal design of TMD for a base excitation

Goal : minimize $\left|\frac{X_1 - X0}{-\omega^2 X_0}\right|$



$$\nu = \frac{\sqrt{1 - \mu/2}}{1 + \mu}$$

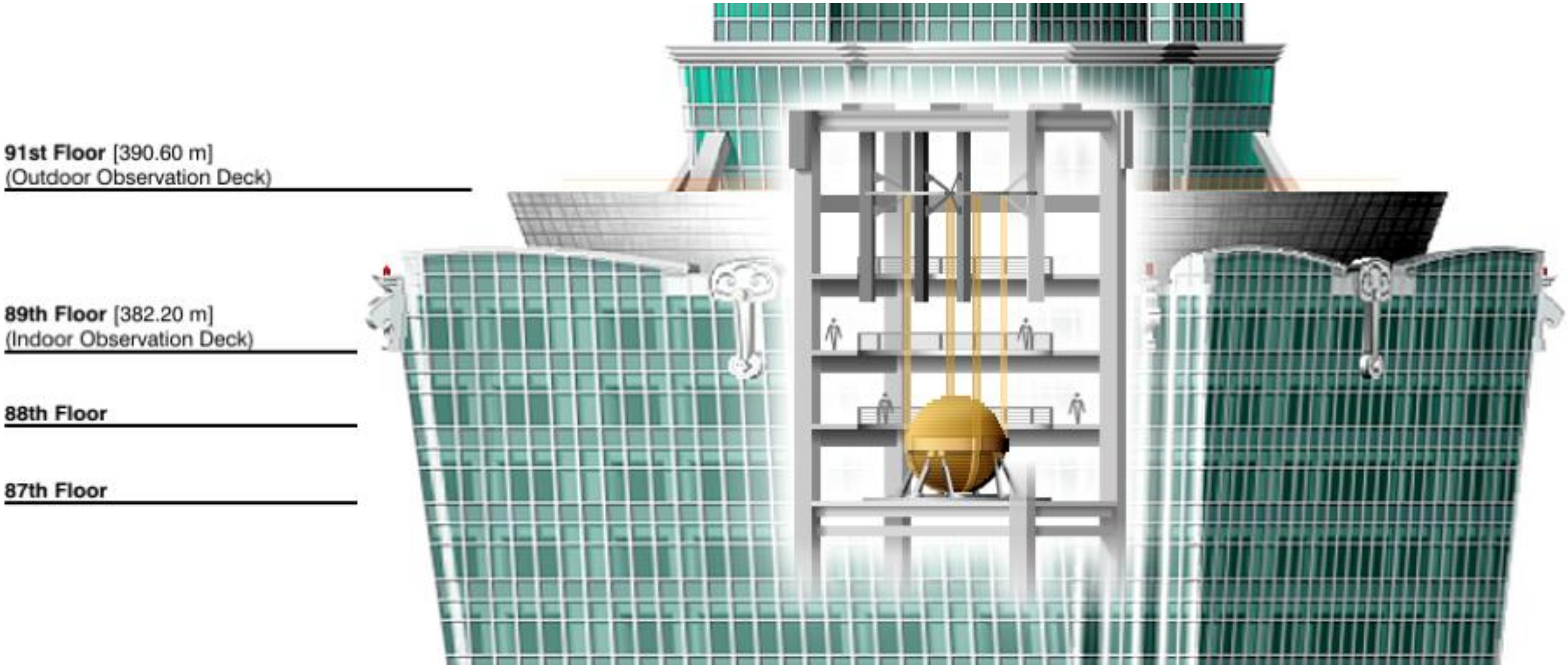
$$\xi = \sqrt{\frac{3\mu}{8(1 + \mu)(1 - \mu/2)}}$$

[Warburton 1982]

TMD for pipe vibrations

https://www.youtube.com/watch?v=X25DJ1_po8s

Pendulum TMD



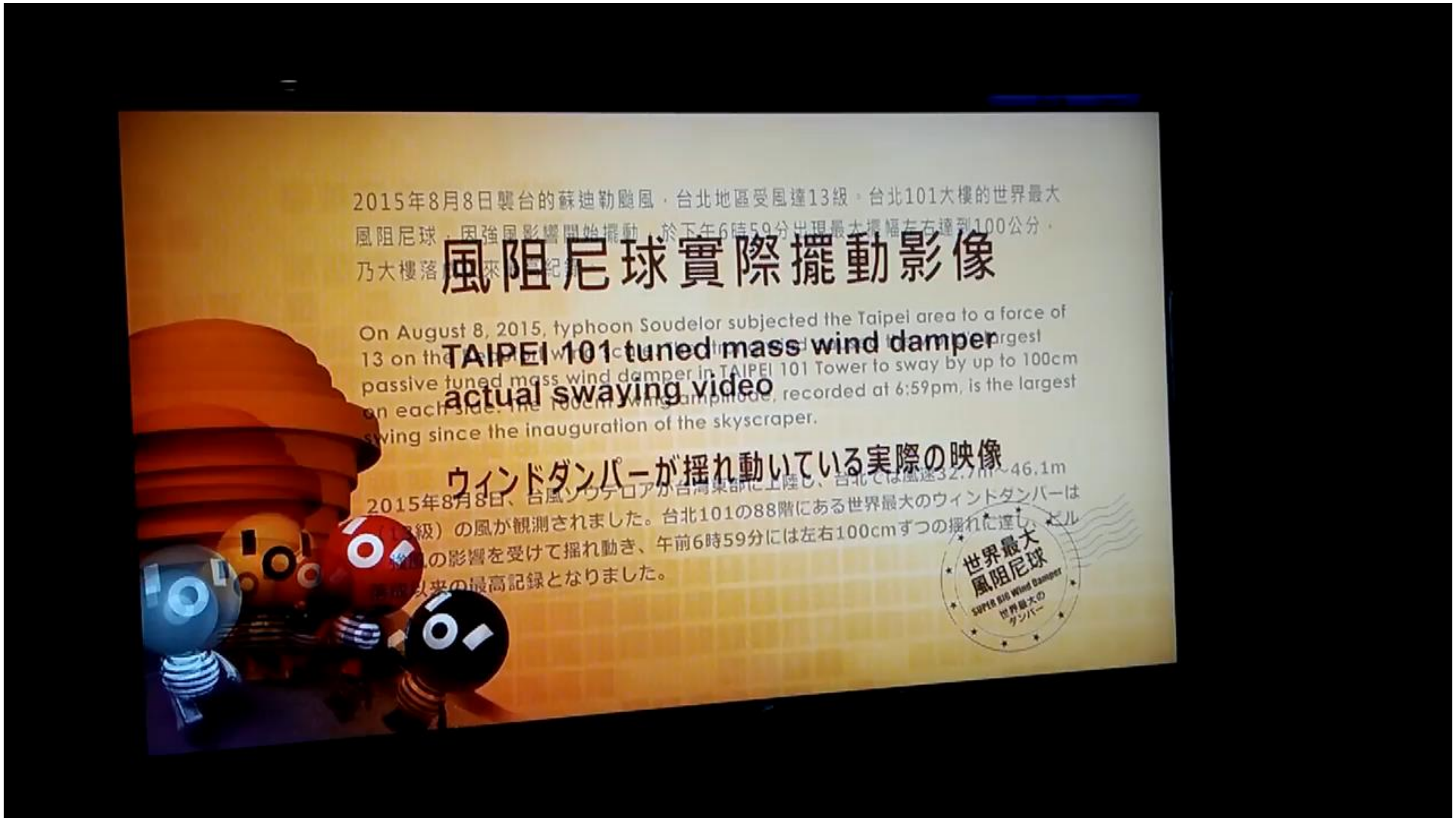
$$\omega = \sqrt{\frac{g}{l}} \approx 0.15Hz \approx 1rad/s$$

$$l \approx 10m$$

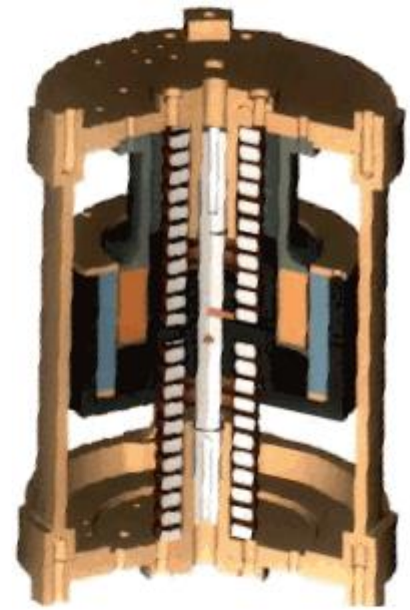
Damper

730 tons - $4 Million

Pendulum TMD



TVA in Formula 1: A Benefit of 0.25s a Lap



Sealed cylinder with a disc sandwiched between 2 coil springs

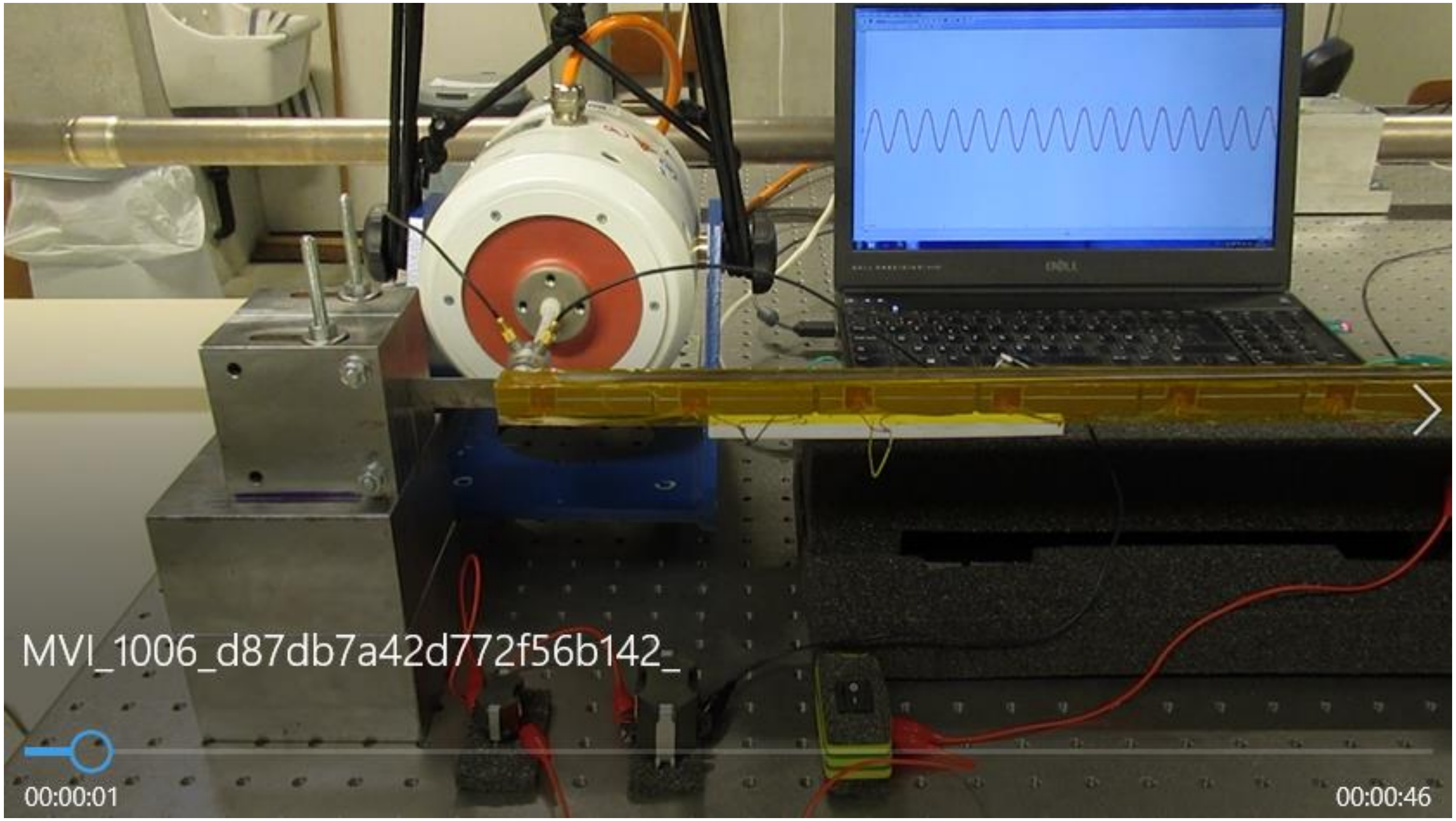
Stabilization of the front end, especially over a kerb (increased grip). One team even investigated a 30kg TMD.

Banned by FIA after the *mass damper affair* in 2006.

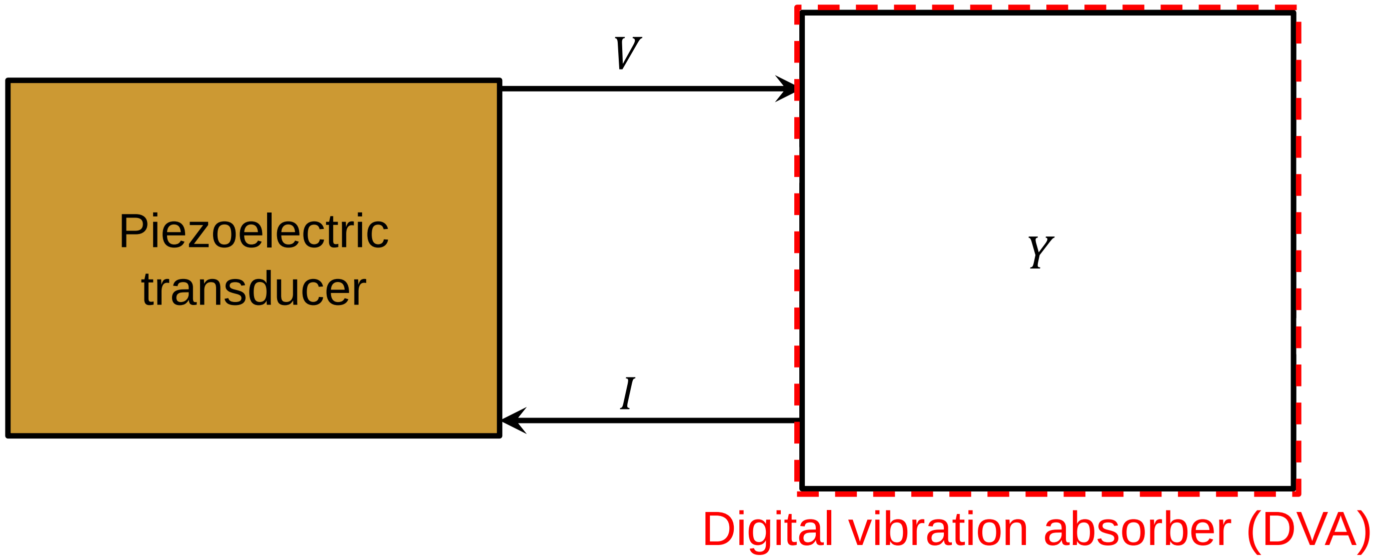
Electrical dampers @ ULiège

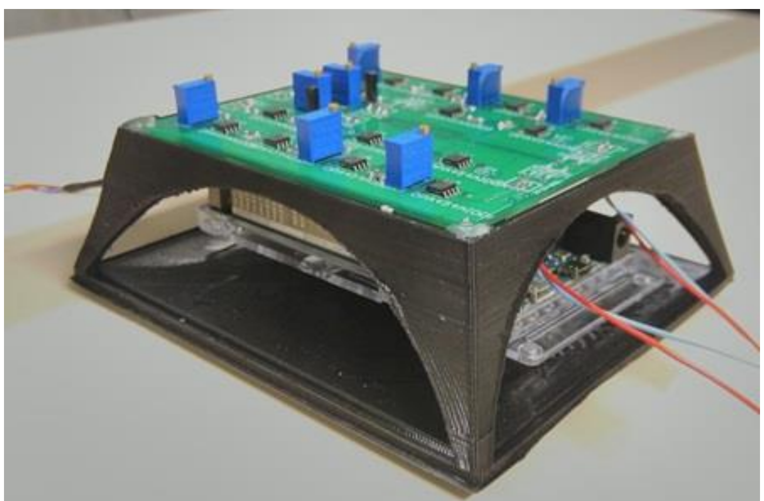
Electromechanical analogy

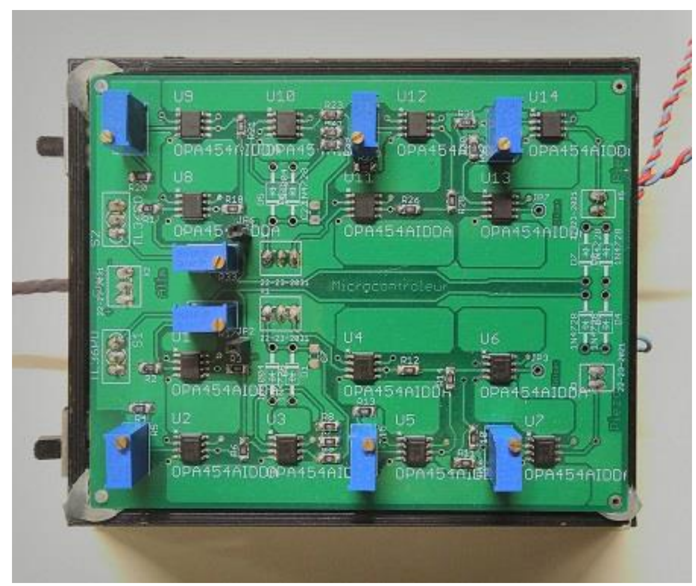
Mass	$m \Leftrightarrow L$	Inductor
Damping	$c \Leftrightarrow R$	Resistor
Stiffness	$k \Leftrightarrow \frac{1}{C}$	Inverse Capacitor
Displacement	$x \Leftrightarrow q$	Charge
Force	$F \Leftrightarrow V$	Voltage



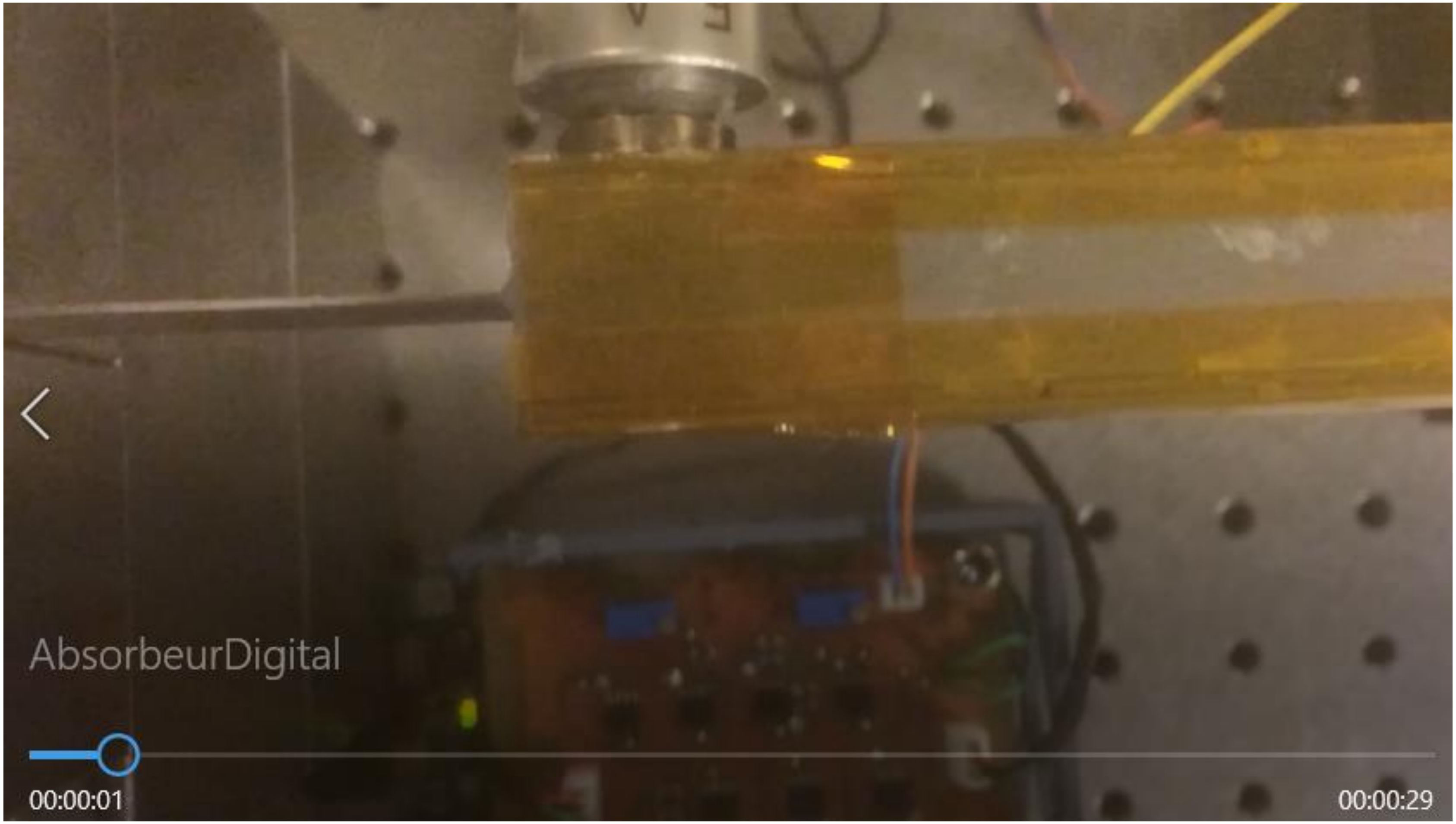
Digital dampers @ ULiège







Digital dampers @ ULiège



Industrial application: Safran Aero Boosters

In summary

Thorough analysis of a 2DOF oscillator with/without damping and with/without forcing.

Important concepts of mode shapes and FRFs.

Vibration absorption.