Cinématique et dynamique des machines

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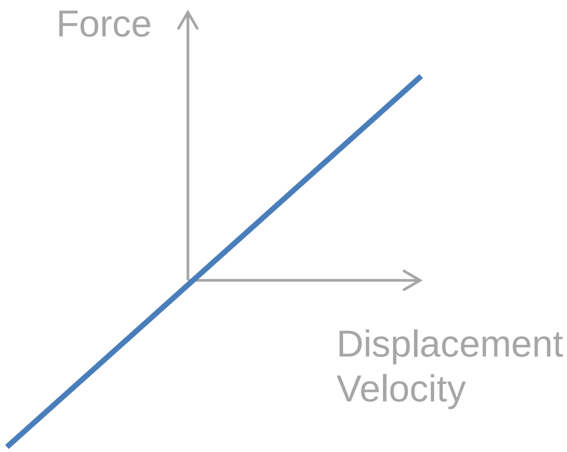
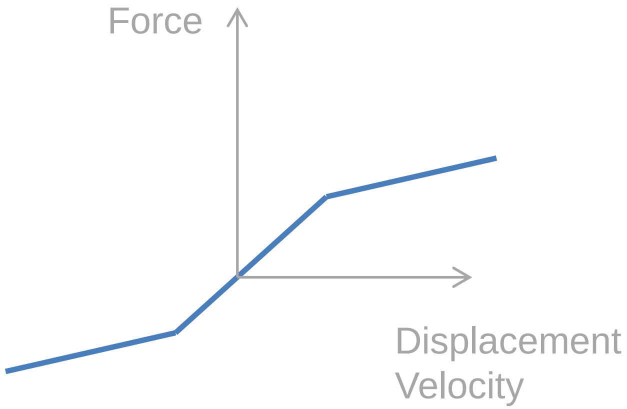
L06 NONLINEAR

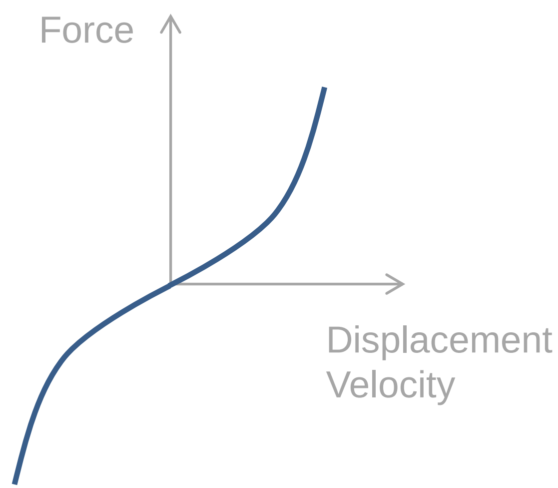
What is a nonlinearity ?

LINEAR



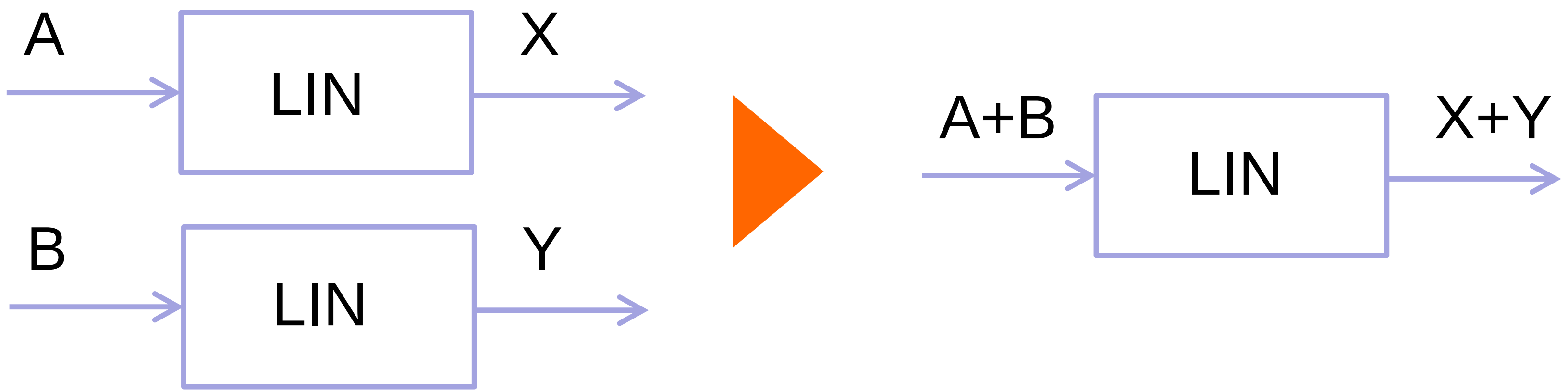
NONLINEARITY



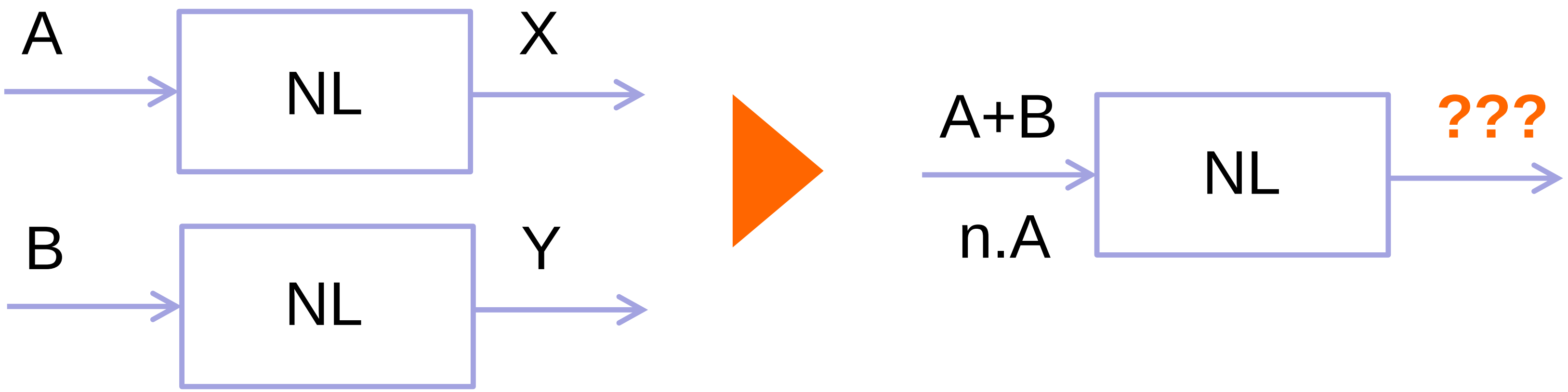


Superposition principle

Cornerstone of linear theory:



No superposition for a nonlinear system !



Outline

Introduction and motivation

Natural frequencies and frequency responses

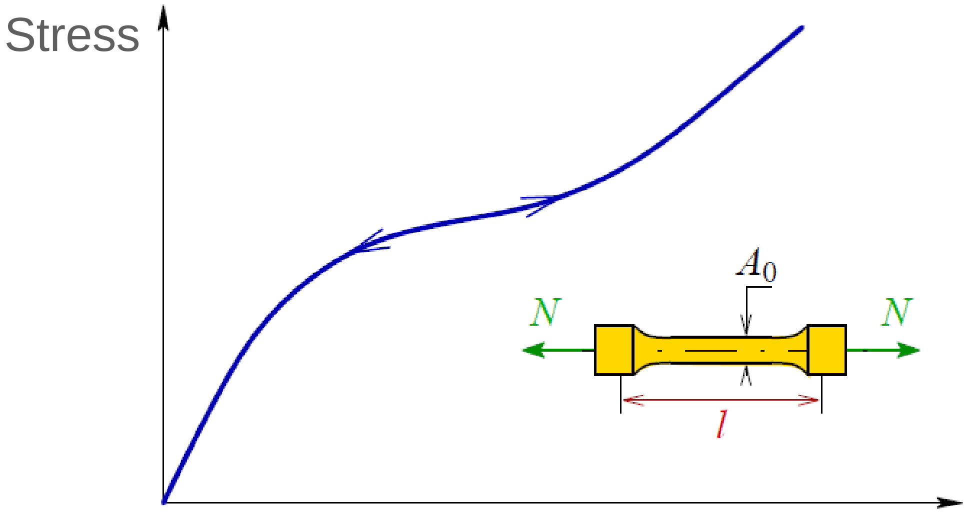
4 important properties illustrated on a nonlinear beam

3 main assumptions in linear structural dynamics

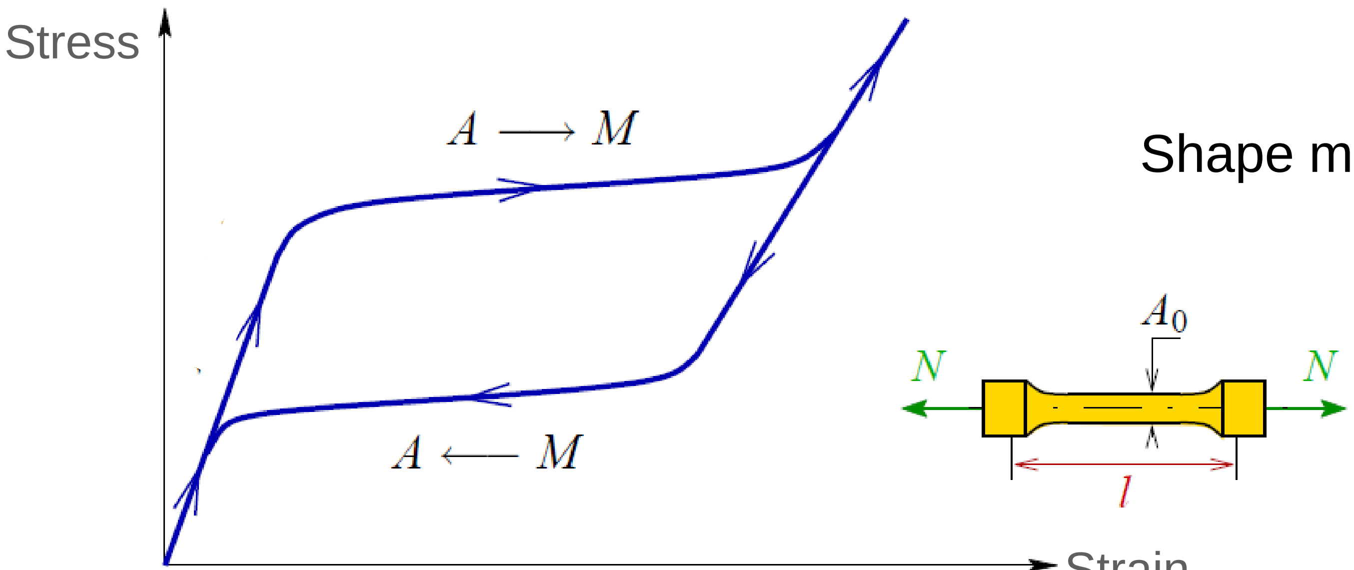
$$M\ddot{q} + C\dot{q} + Kq = 0$$

Linear elasticity	$\rightarrow$ nonlinear materials
Small displ. and rotations	$\rightarrow$ geometrical nonlinearity $\rightarrow$ nonlinear boundary conditions
Viscous damping	$\rightarrow$ nonlinear damping mechanisms

Assumption 1: nonlinear materials

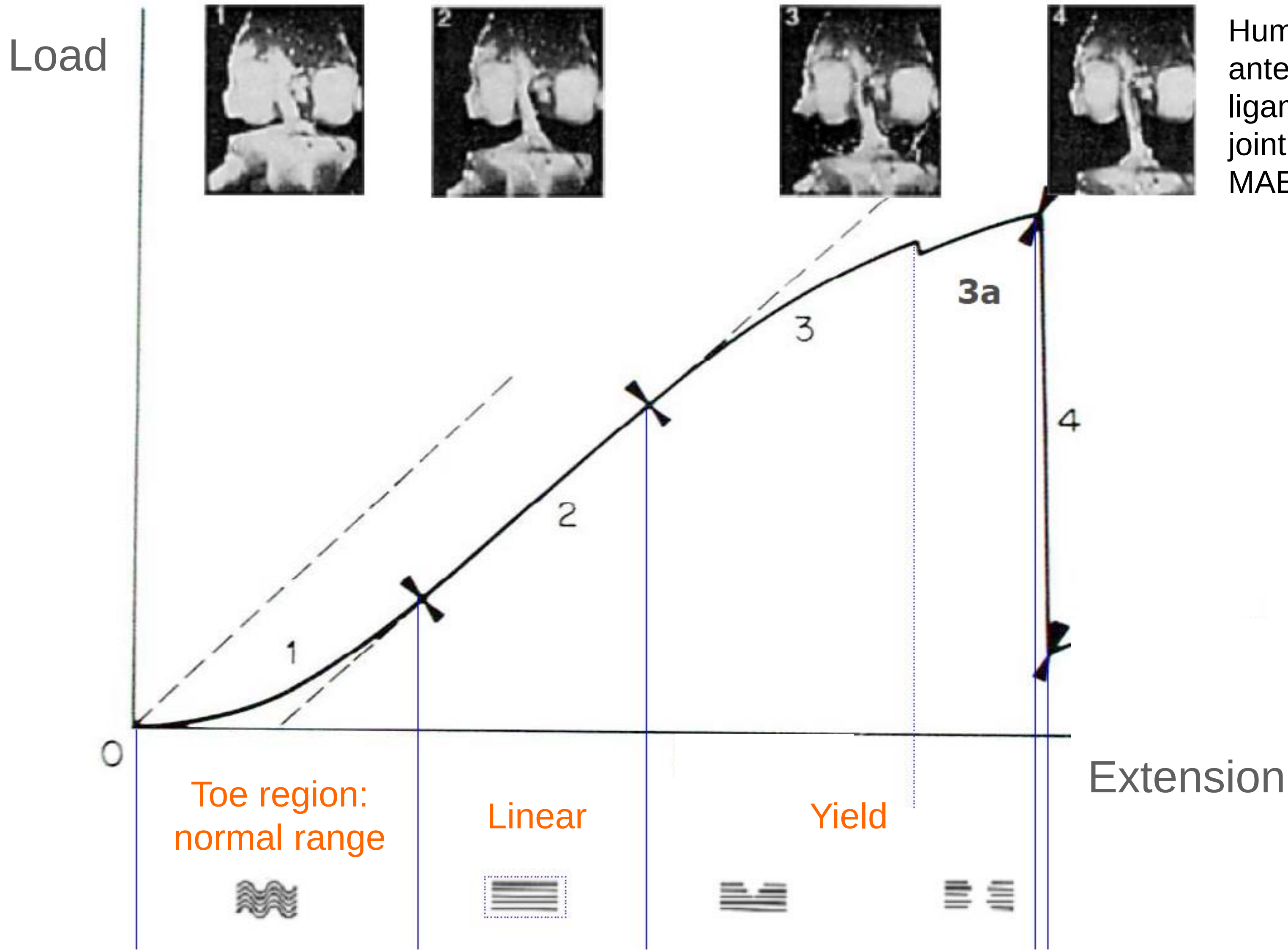


Hyperelastic material
(e.g., rubber)



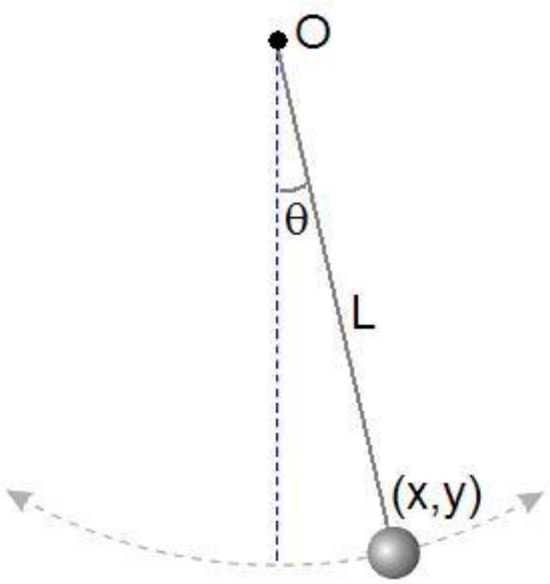
Shape memory alloy

Assumption 1: ligament in your knee joint



Human cadaveric anterior cruciate ligament in knee joint (Dr. Ziv, MAE, Buffalo)

Assumption 2: geometrical nonlinearities



$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0$$

$$\ddot{\theta} + \frac{g}{L}\left(\theta - \frac{\theta^3}{6} + \cdots\right) = 0$$

$\theta \ll 1$

$$\ddot{\theta} + \frac{g}{L}\theta = 0$$

$$Period = 2\pi\sqrt{\frac{L}{g}}$$

Assumption 3: nonlinear damping

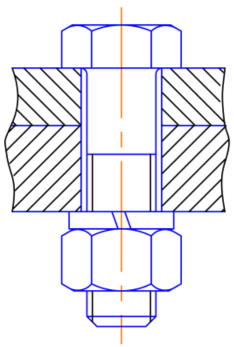
Viscous damping but also…

Coulomb friction

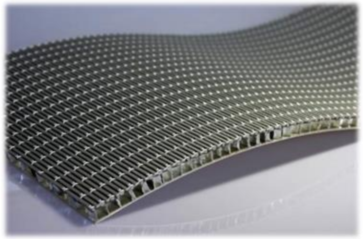
Aerodynamic damping

Common sources of nonlinearity

Bolts, joints and gaps



Elastomers and composites

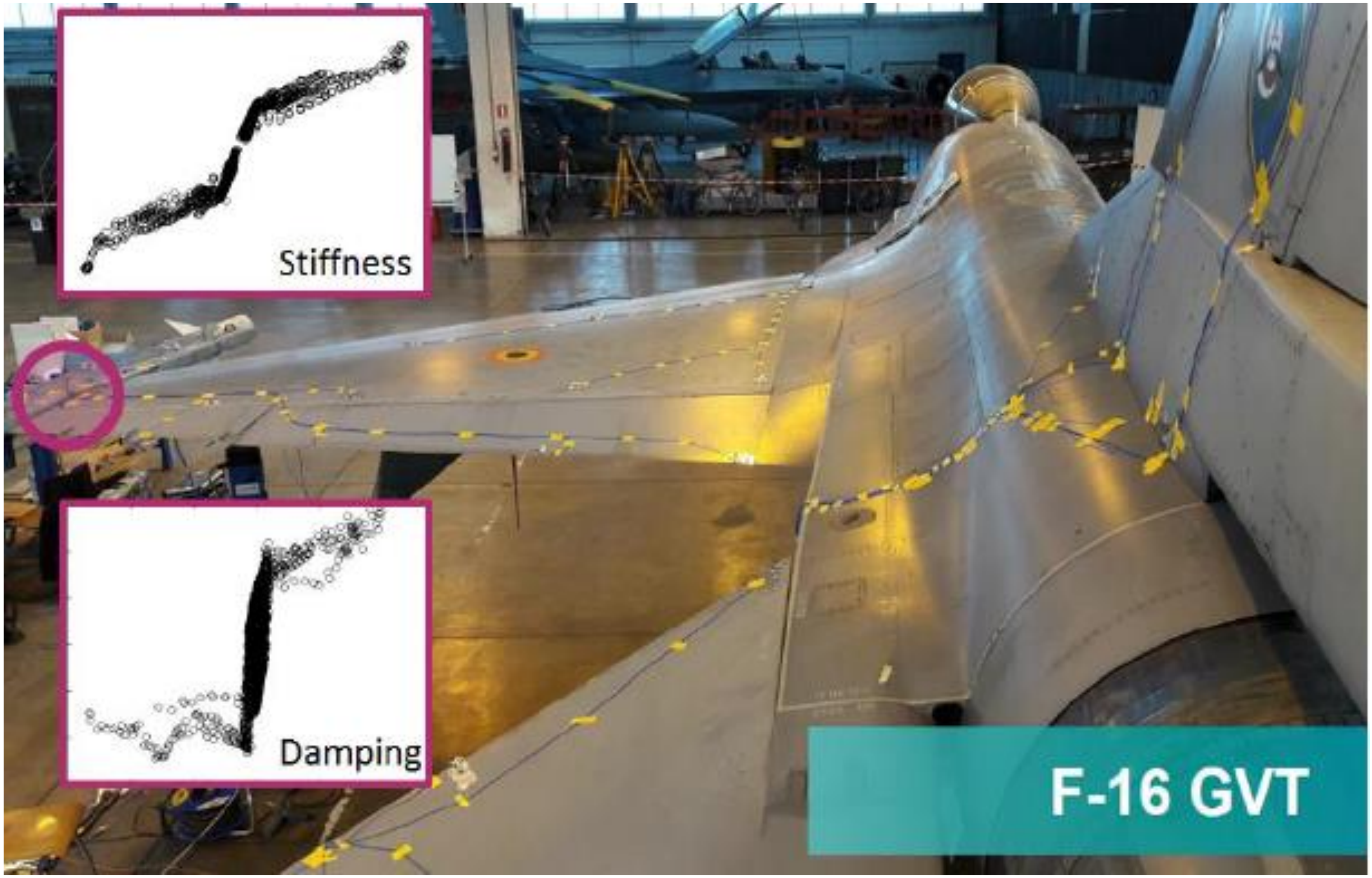


Friction and contact

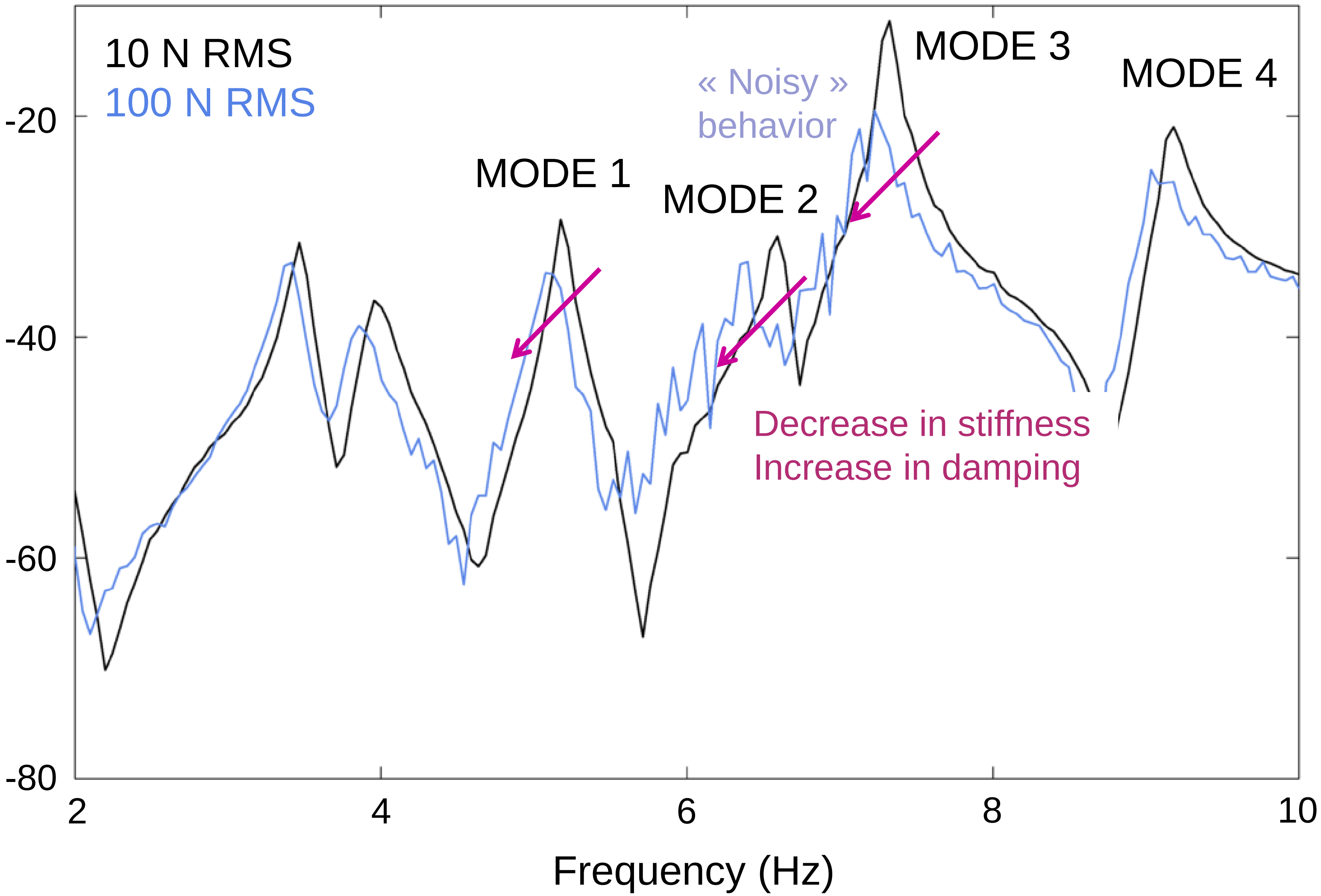
Large amplitudes

F-16 aircraft has a nonlinear sliding joint

Identified nonlinearities



It is nonlinear, so what ?



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Natural frequency of a linear oscillator

$$m\ddot{x} + kx = 0$$

$x(t) = Xcos\omega t$

$$-m\omega^2 Xcos\ \omega t + kXcos\ \omega t = 0$$

$$\omega = \sqrt{\frac{k}{m}}$$

Natural frequency of a Duffing oscillator

$$m\ddot{x} + kx + k_3 x^3 = 0$$

Natural frequency of a Duffing oscillator

$$m\ddot{x} + kx + k_3x^3 = 0$$

$x(t) = Xcos\ \omega t$

$$-m\omega^2 Xcos\ \omega t + kXcos\ \omega t + k_3X^3cos^3\omega t = 0$$

$cos^3\omega t = (3cos\ \omega t + cos\ 3\omega t)/4$

$$-m\omega^2 + k + \frac{3}{4}k_3X^2 = 0$$

1-term approximation

$$\omega = \sqrt{\frac{k + \frac{3}{4}k_3X^2}{m}}$$

Amplitude dependent !

Sign of k3 matters

How do we calculate the NL frequency response ?

$$m\ddot{x} + kx + k_3x^3 = Fsin\,\omega t$$

$$x = x(F, \omega, t)?$$

The harmonic balance method

$$m\ddot{x} + kx + k_3x^3 = F sin\ \omega t$$

$x(t) = X sin\ \omega t$

$$-m\omega^2 X sin\ \omega t + kX sin\ \omega t + k_3 X^3 sin^3 \omega t = F sin\ \omega t$$

$sin^3 \omega t = (3 sin\ \omega t - sin\ 3\omega t)/4$

Nonlinear relation between *X* and *F*

OPTION 1: EXACT

$$x(t) = X sin\ \omega t + X_3 sin\ 3\omega t$$

Solution: infinite series of harmonics

OPTION 2: APPROXIMATION

$$-m\omega^2 X + kX + \frac{3}{4} k_3 X^3 = F$$

Solve a 3rd order polynomial in *X*
(! BIFURCATIONS !)

Can you draw $X=X(\omega)$?

The Harmonic Balance Method (with Damping)

$$m\ddot{x} + c\dot{x} + kx + k_3x^3 = F sin\ \omega t$$

$x(t) = X sin\ \omega t + X cos\ \omega t$

Same machinery

In summary

1. Frequency-amplitude dependence: $\omega = \sqrt{\frac{k + \frac{3}{4}k_3 X^2}{m}}$

2. Harmonics: $sin^3\omega t = (3sin\,\omega t - sin\,3\omega t)/4$

3. Bifurcations: $-m\omega^2 X + kX + \frac{3}{4}k_3 X^3 = F$

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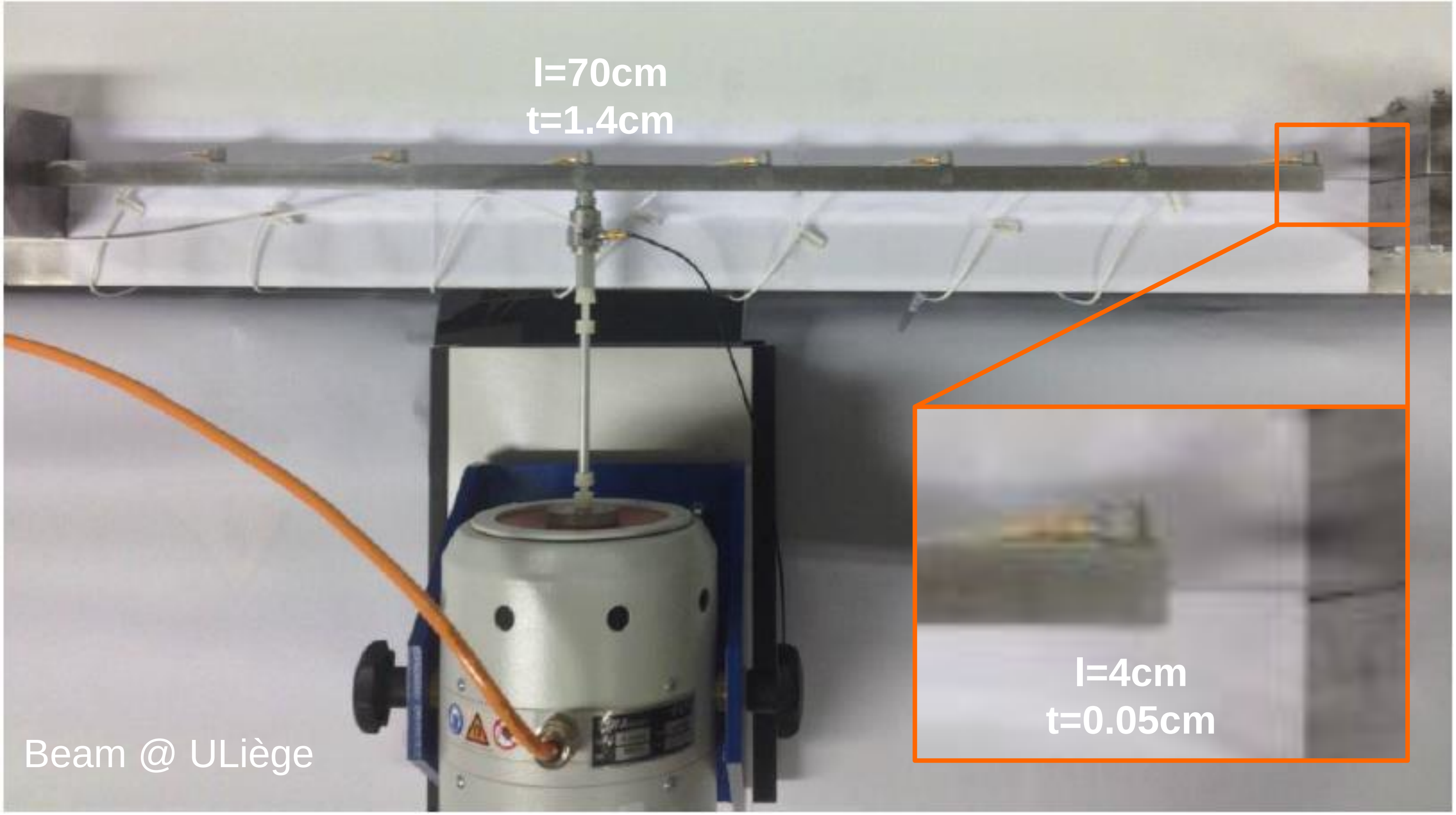
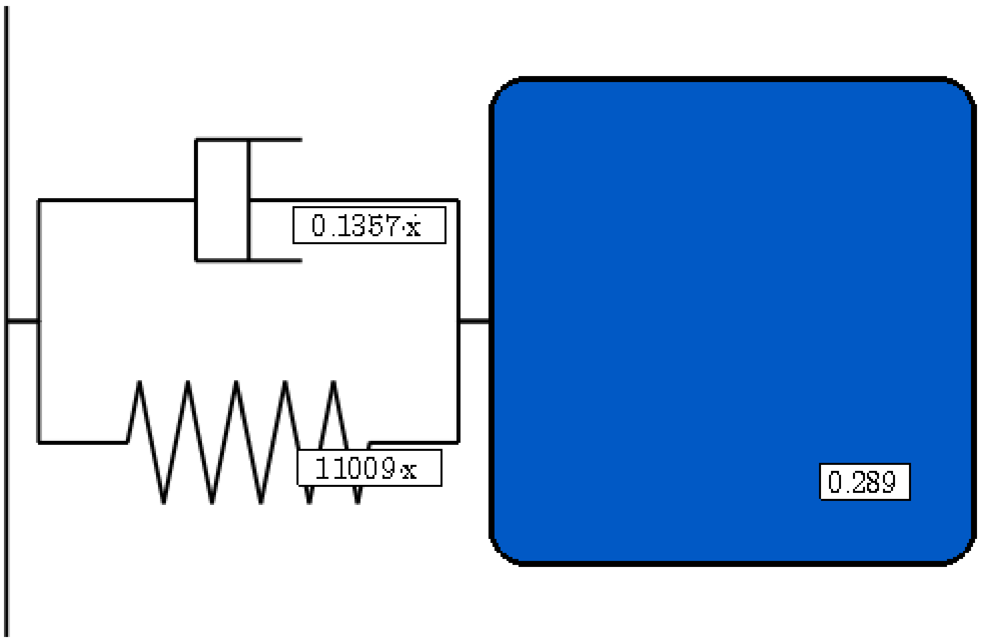
Cantilever beam example

With a very thin beam at its tip:



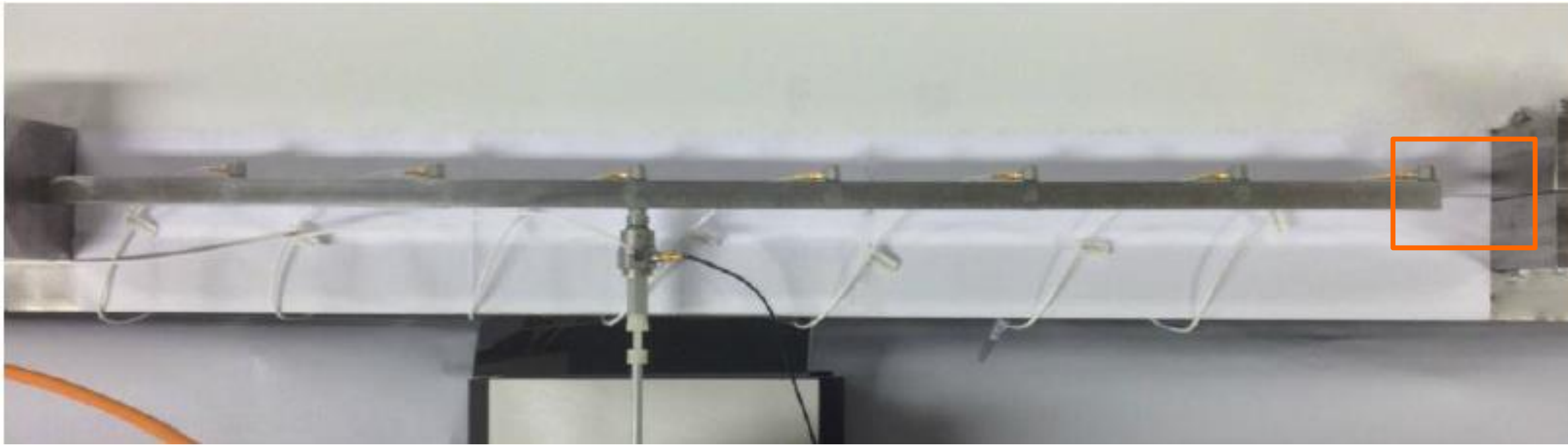
A 1DOF model of the first beam mode identified

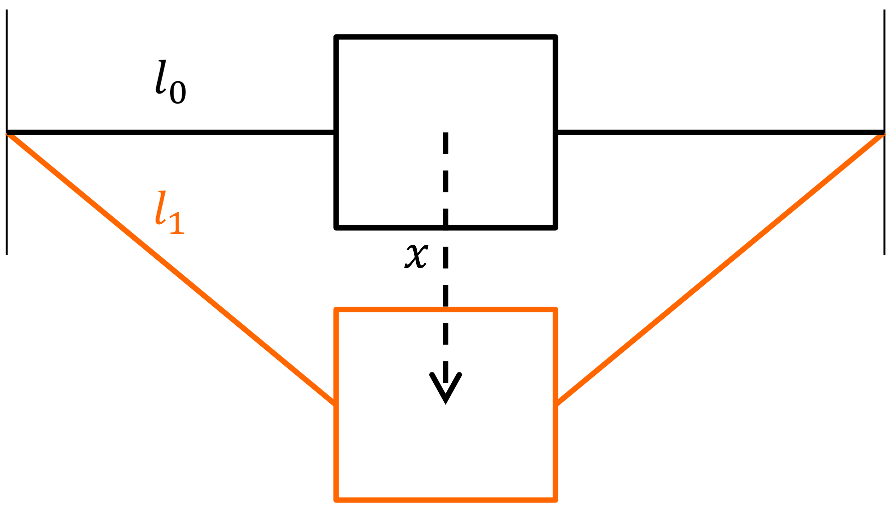
$$0.289\ddot{x} + 0.1357\dot{x} + 11009x = Fsin\omega t$$



mode 1: 31.0631 Hz / 0.12 %

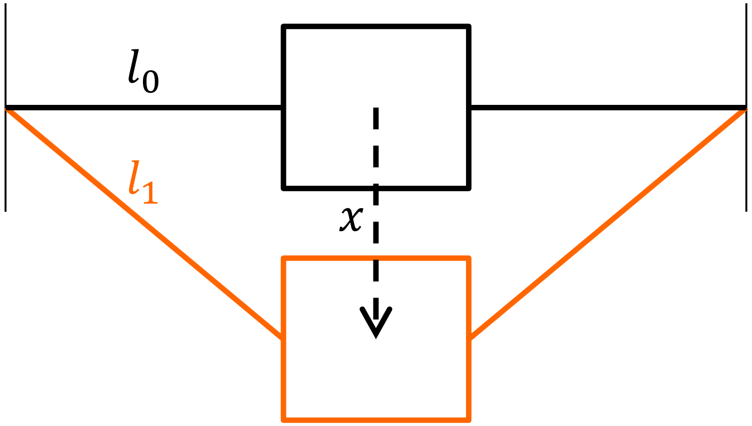
However, the thin beam is nonlinear





Behavior of a cubic spring

Evidence of a geometrical nonlinearity



elongation projection

$$F_{vert} = 2k(l_1 - l_0)\frac{x}{l_1} = 2kx\left(1 - \frac{l_0}{\sqrt{x^2 + {l_0}^2}}\right)$$

$$\frac{l_0}{\sqrt{x^2 + {l_0}^2}} = 1 - \frac{x^2}{2{l_0}^2} + \frac{3x^4}{8{l_0}^4} + O(x^6)$$

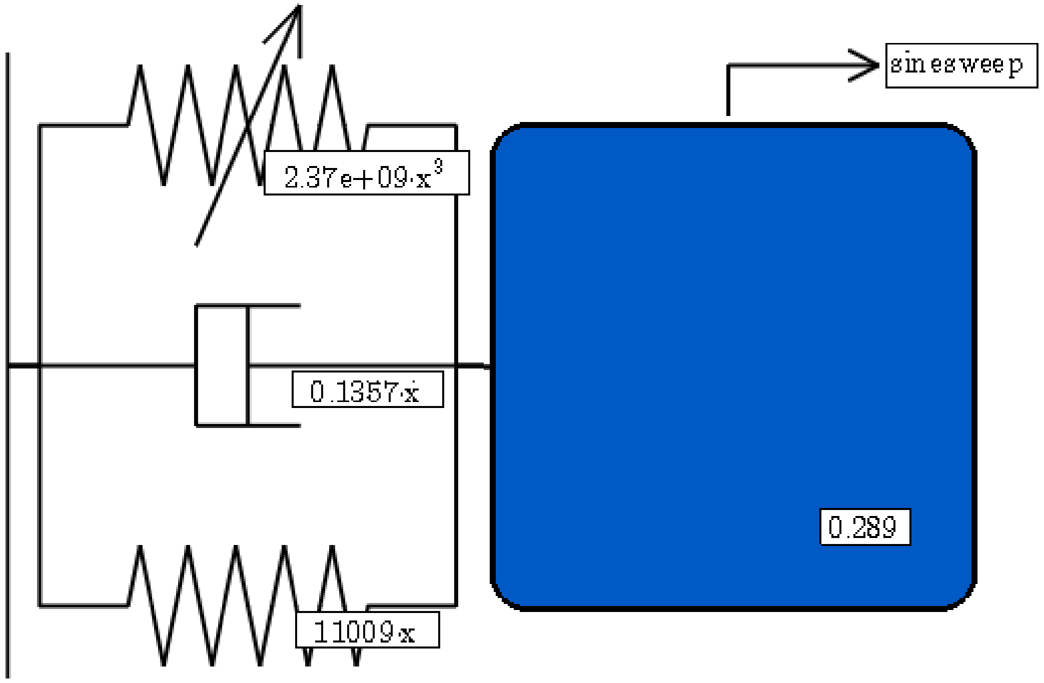
Horizontal: spring forces cancel out

Vertical: spring forces add

$$F = k\frac{x^3}{l_0^2} + O(x^5)$$

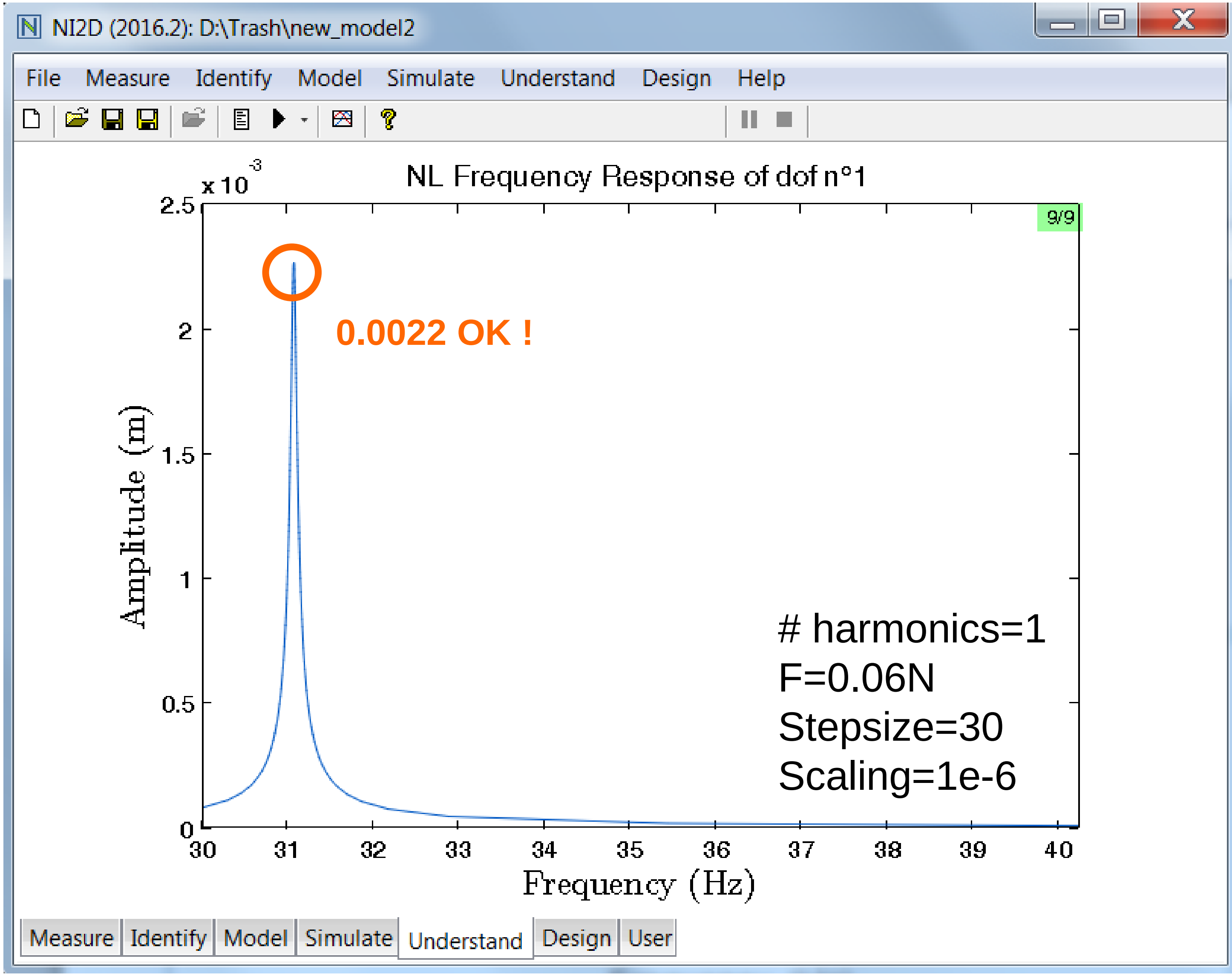
A 1DOF nonlinear model of the first beam mode

$$0.289\ddot{x} + 0.1357\dot{x} + 11009x + 2.37.10^9\,x^3 = Fsin\omega t$$

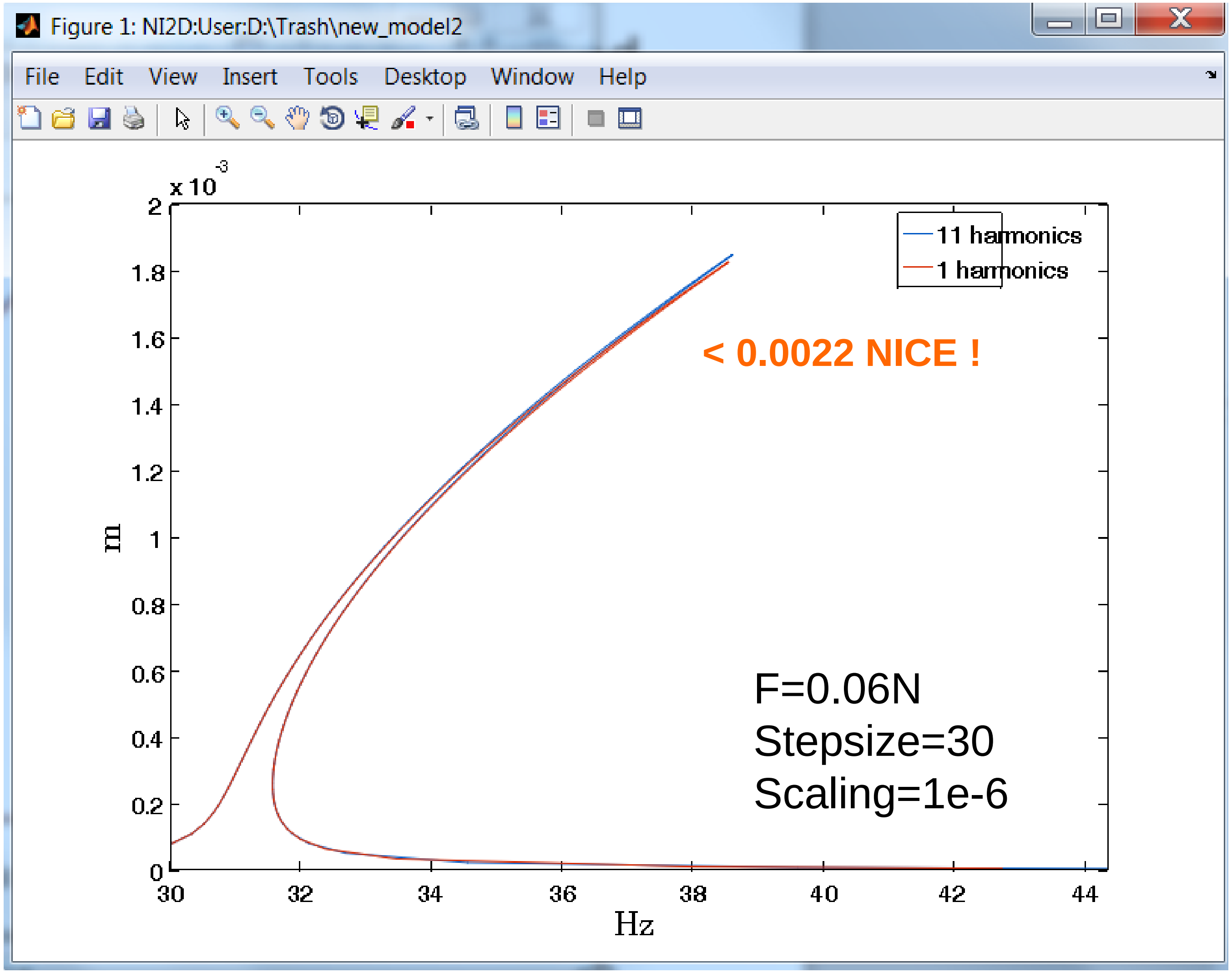


1. Harmonics

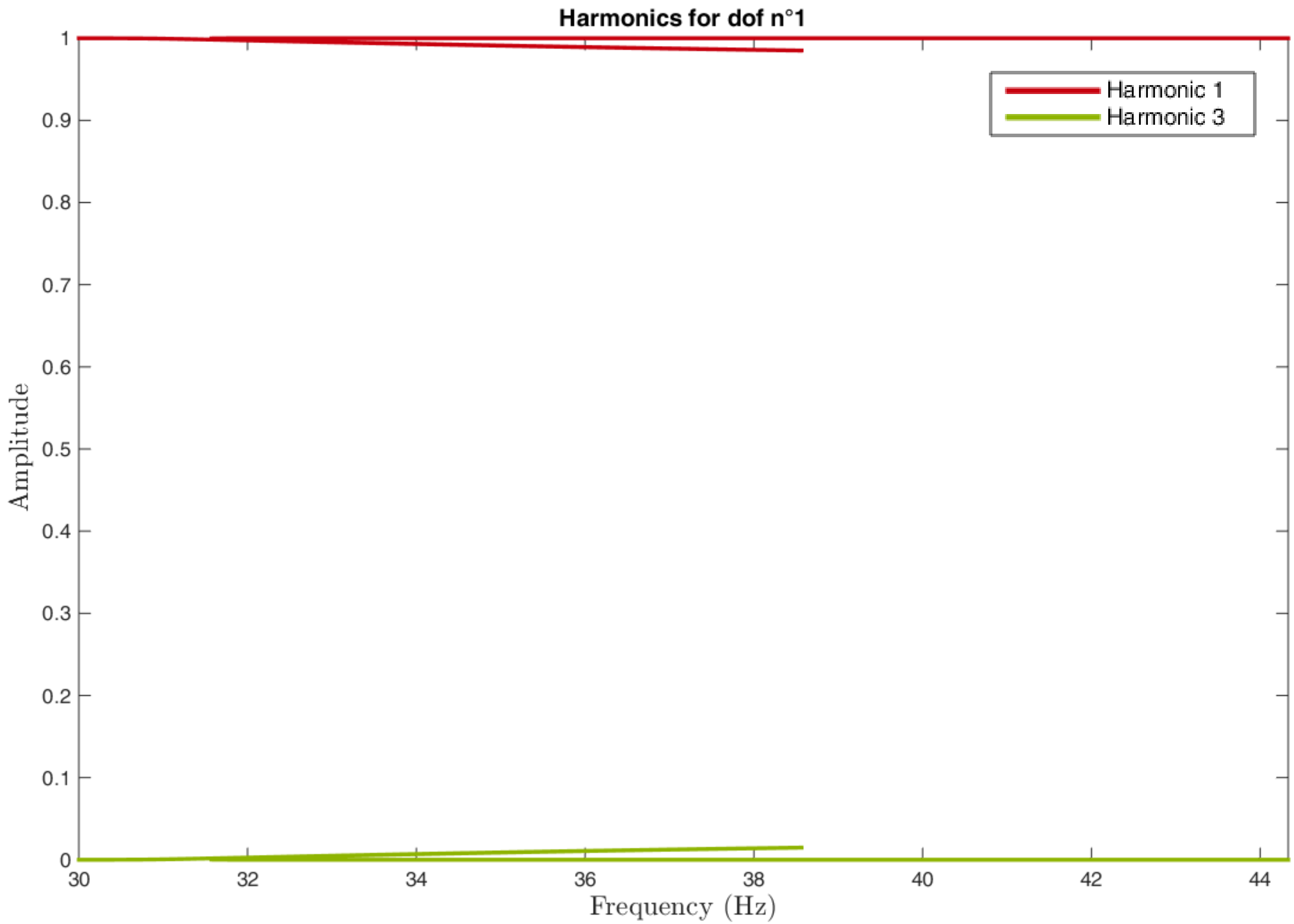
Harmonic balance for the linear system



Harmonic balance for the nonlinear system

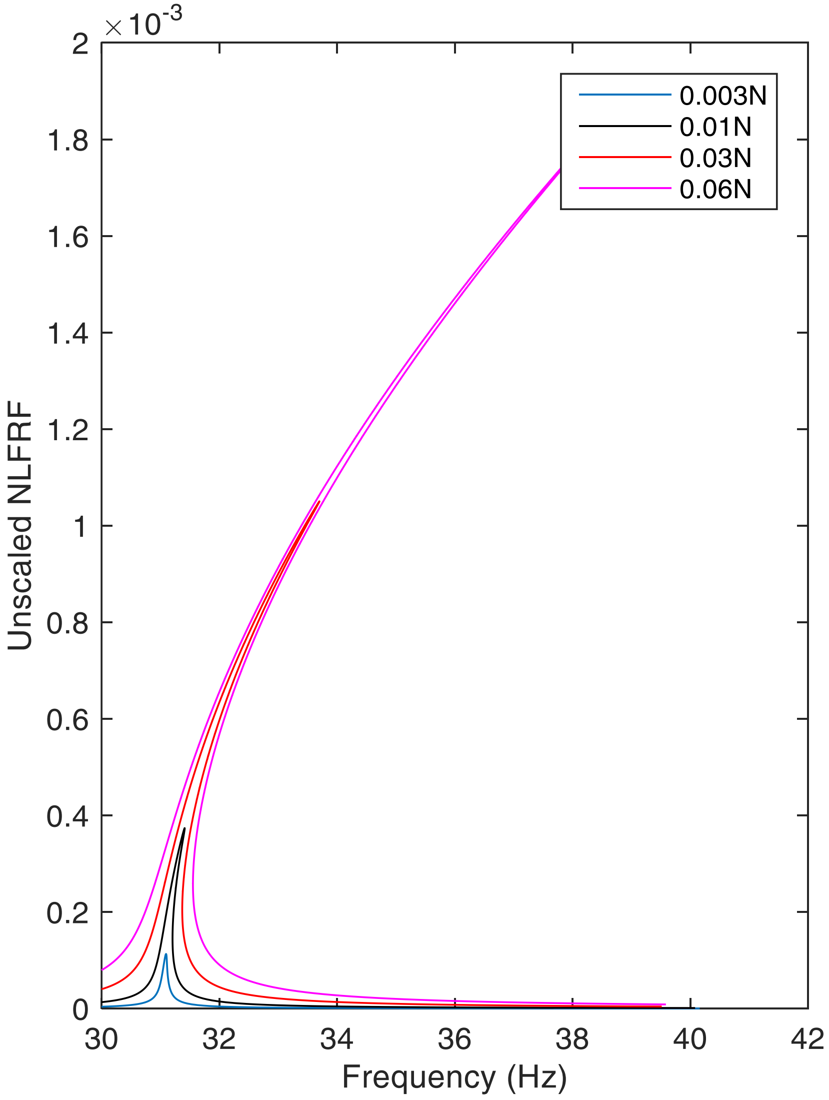


Importance of harmonics

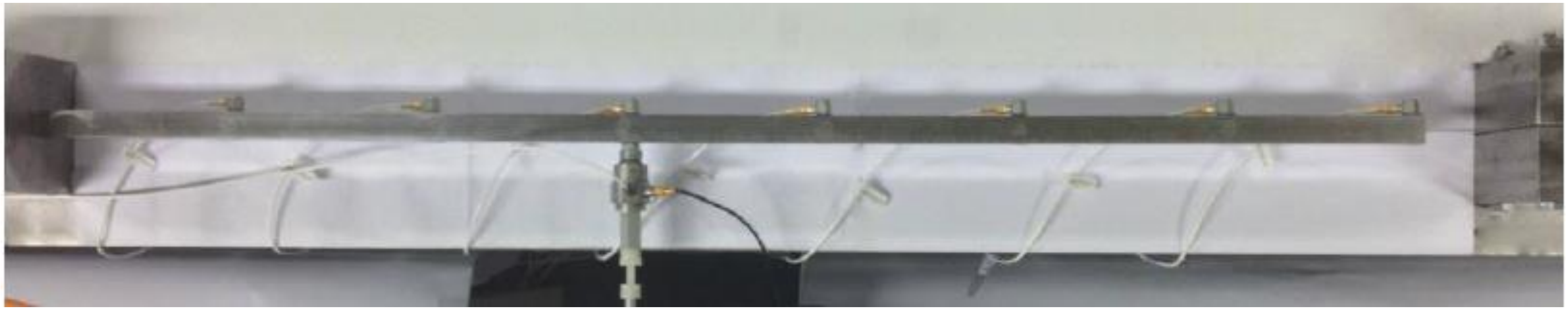


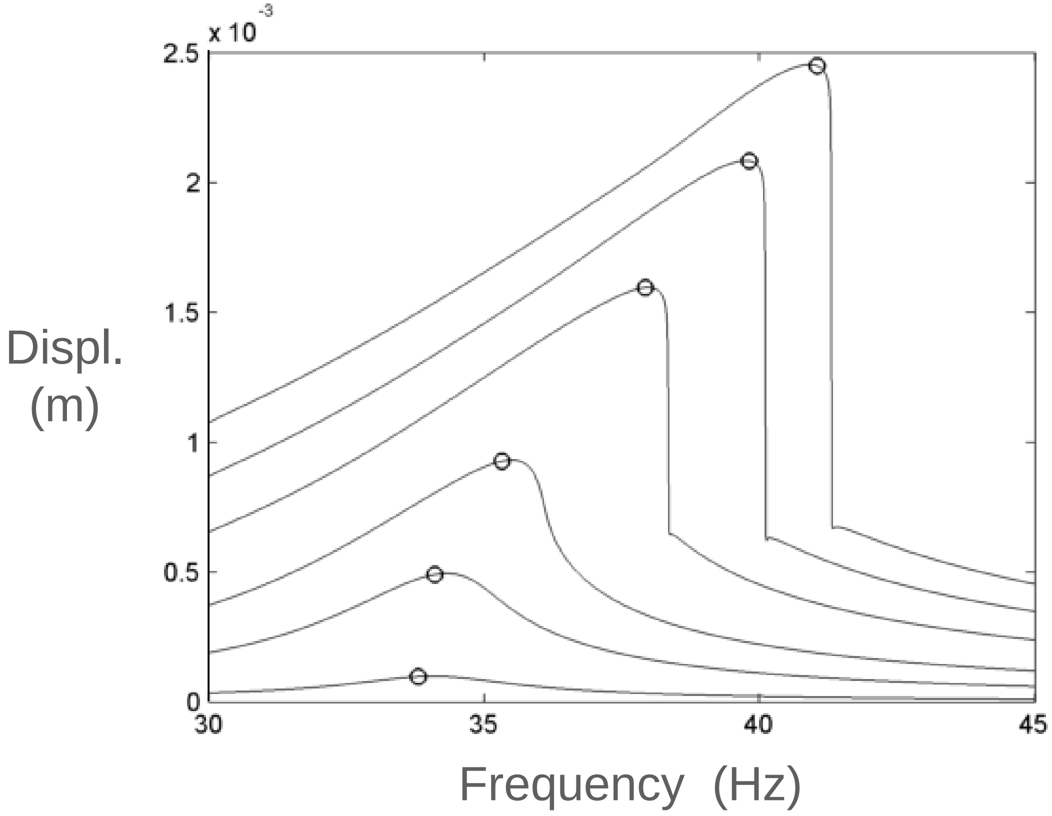
2. Frequency-amplitude dependence

Variation of nonlinear frequency responses



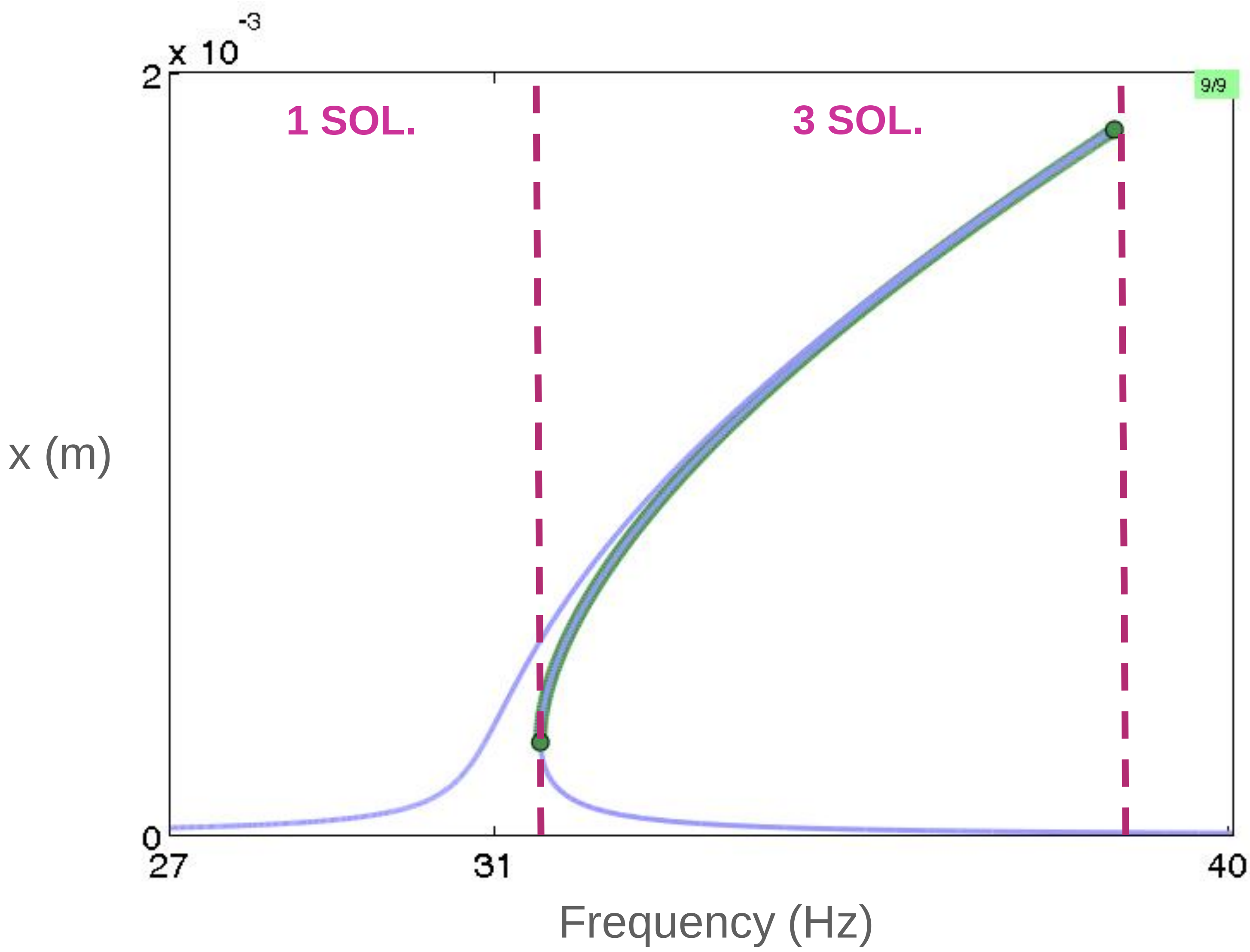
Measured nonlinear frequency responses



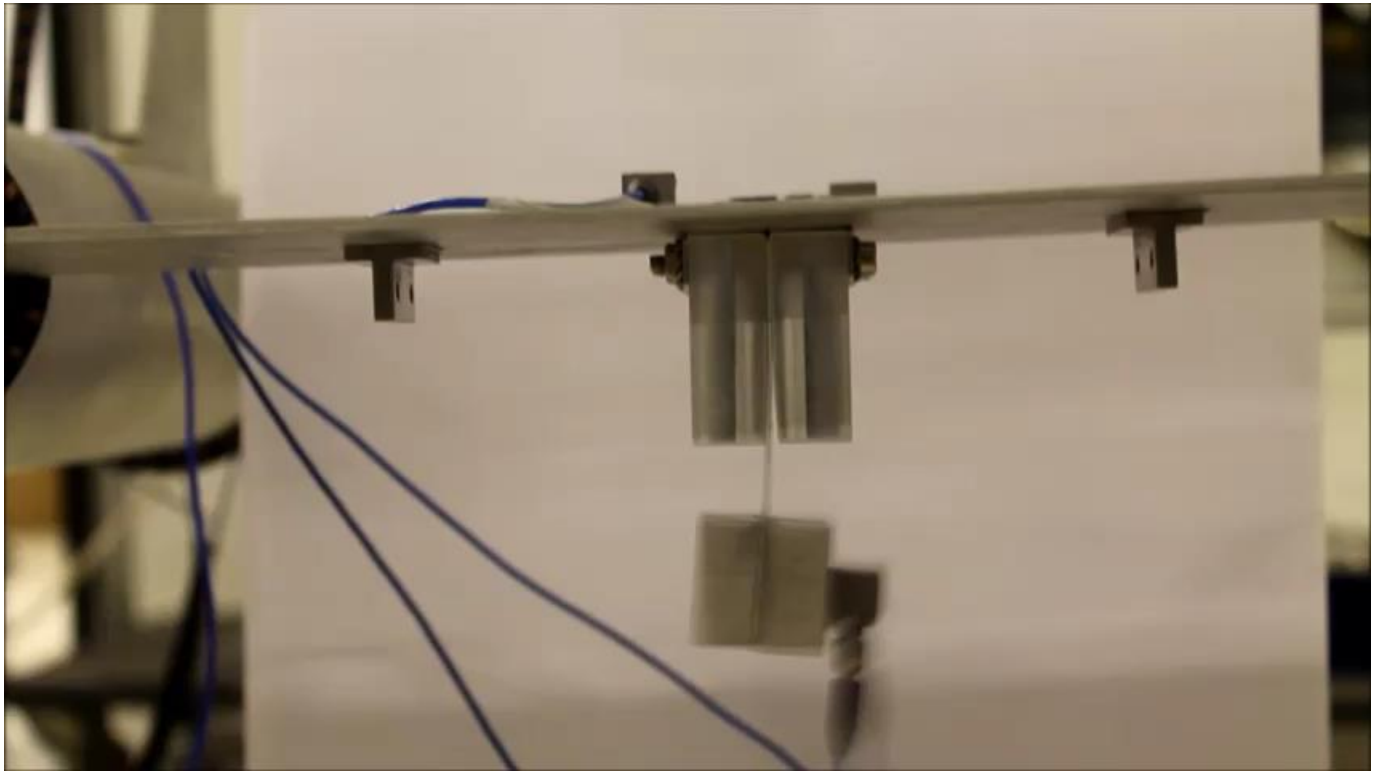


3. Bifurcations

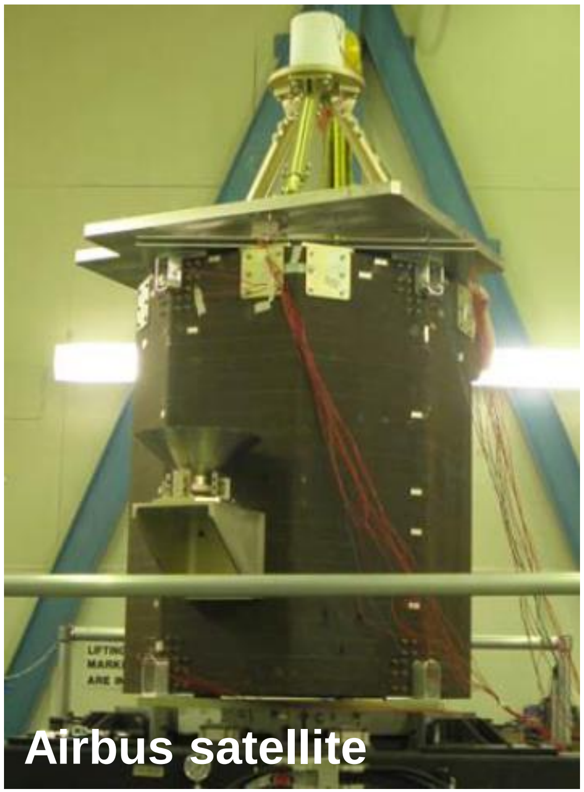
Bifurcations generate multi-valued response

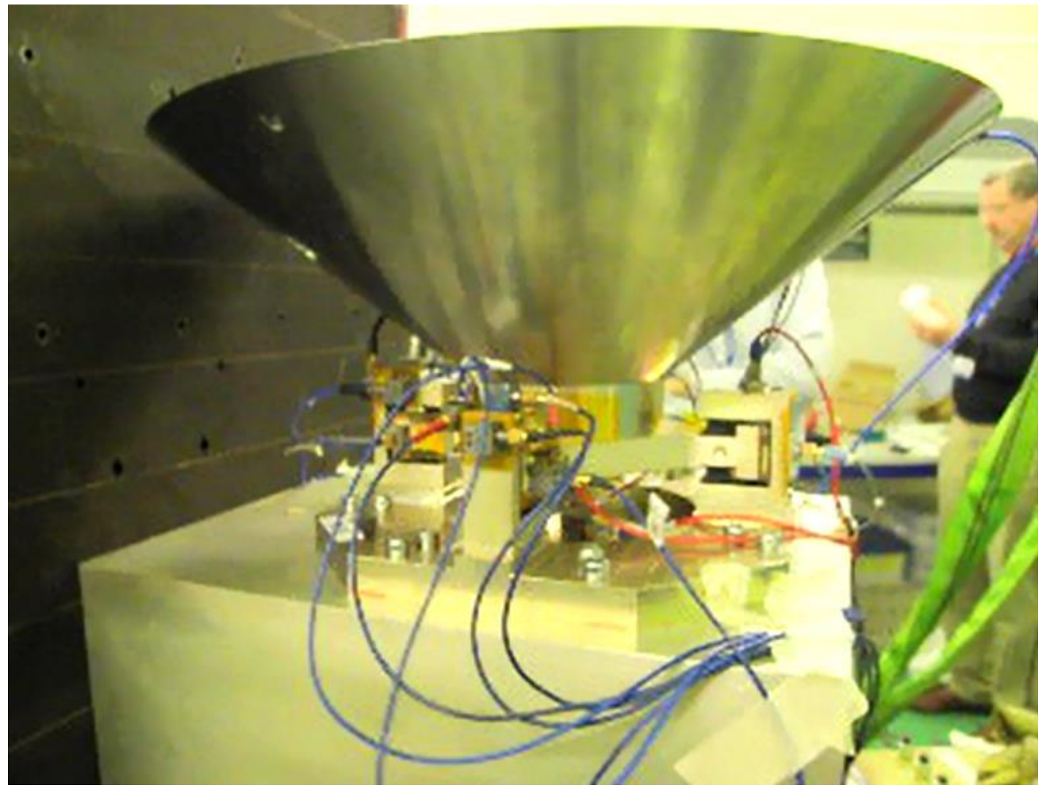


The jump phenomenon



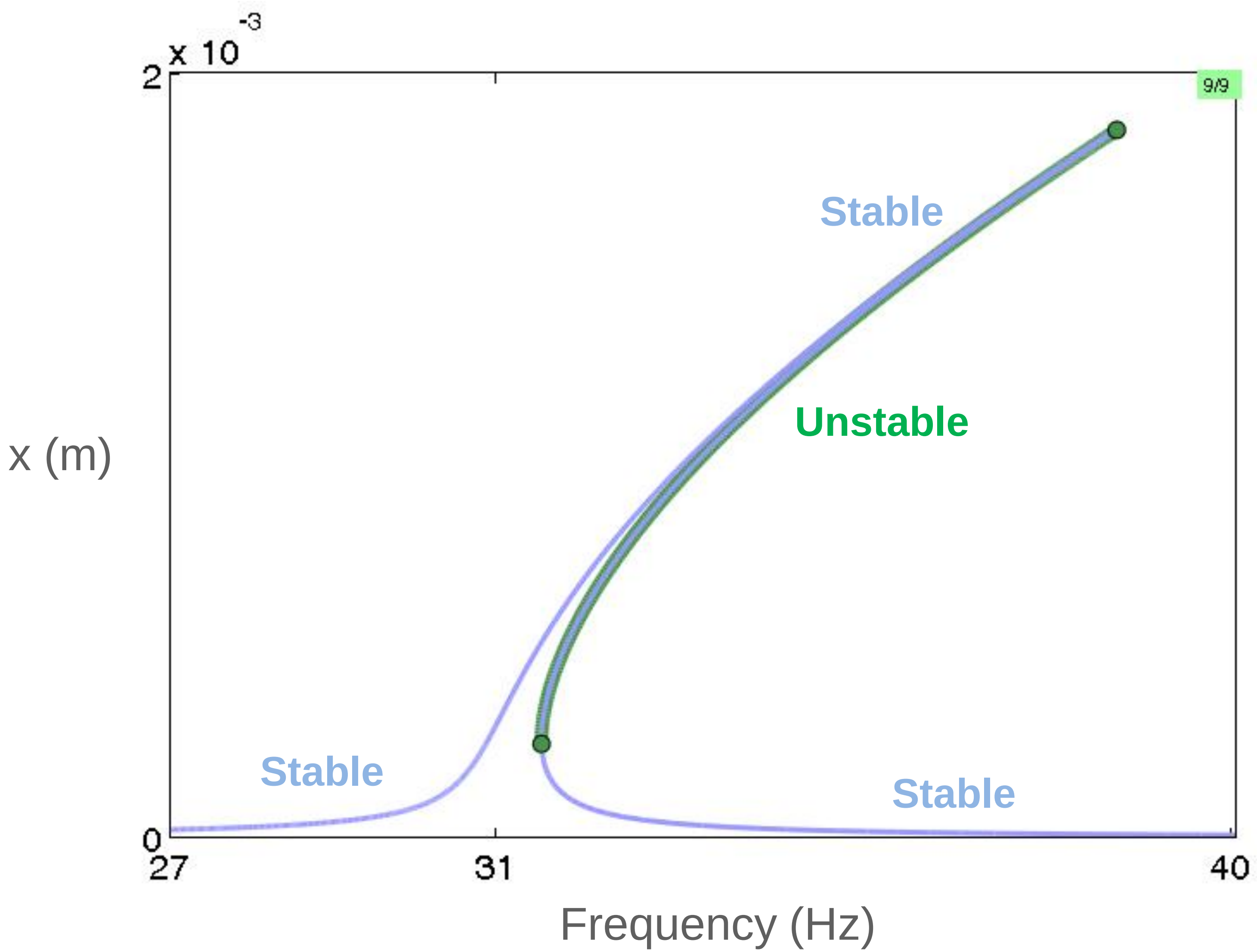
The jump phenomenon





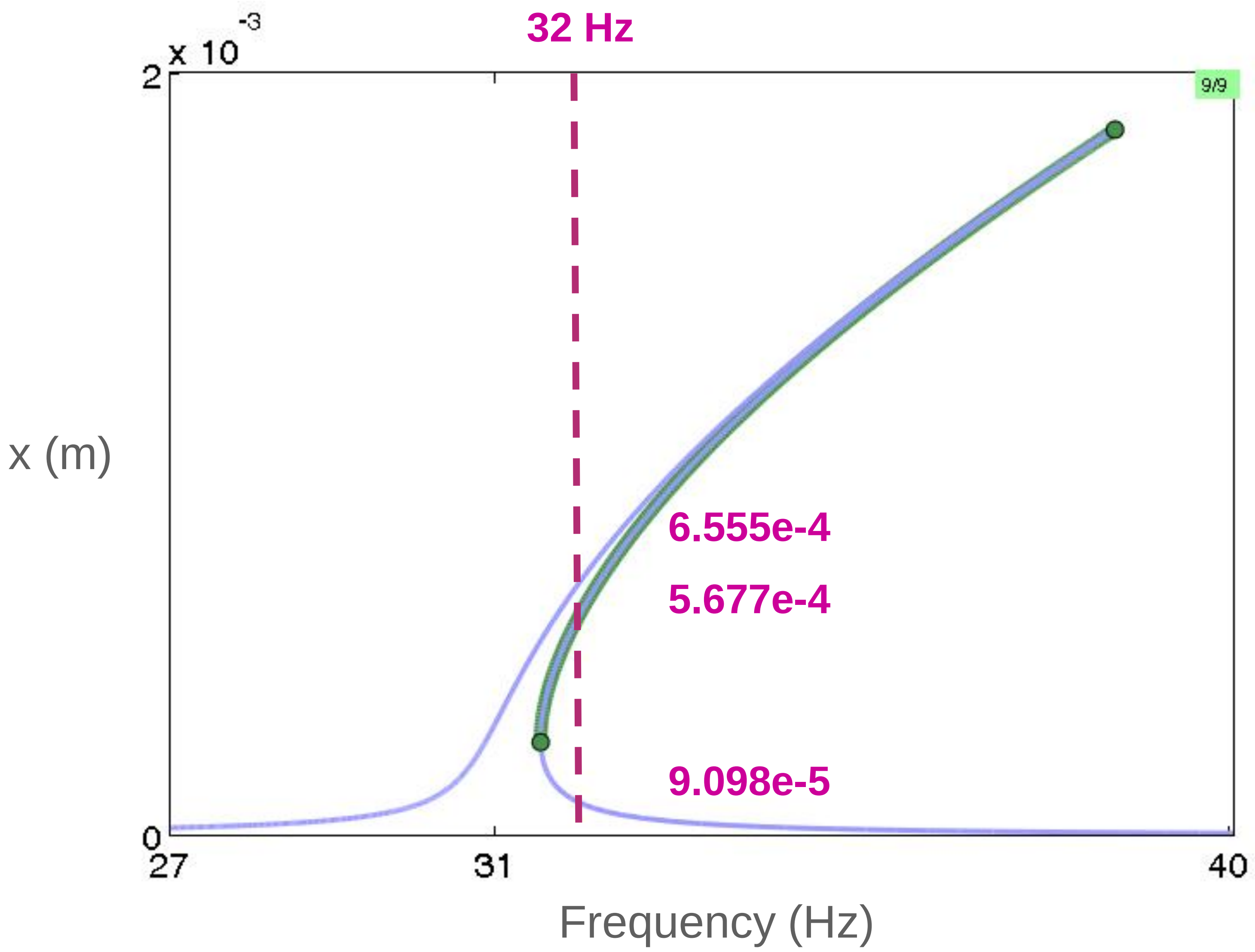
4. Stability

Bifurcations Change Stability

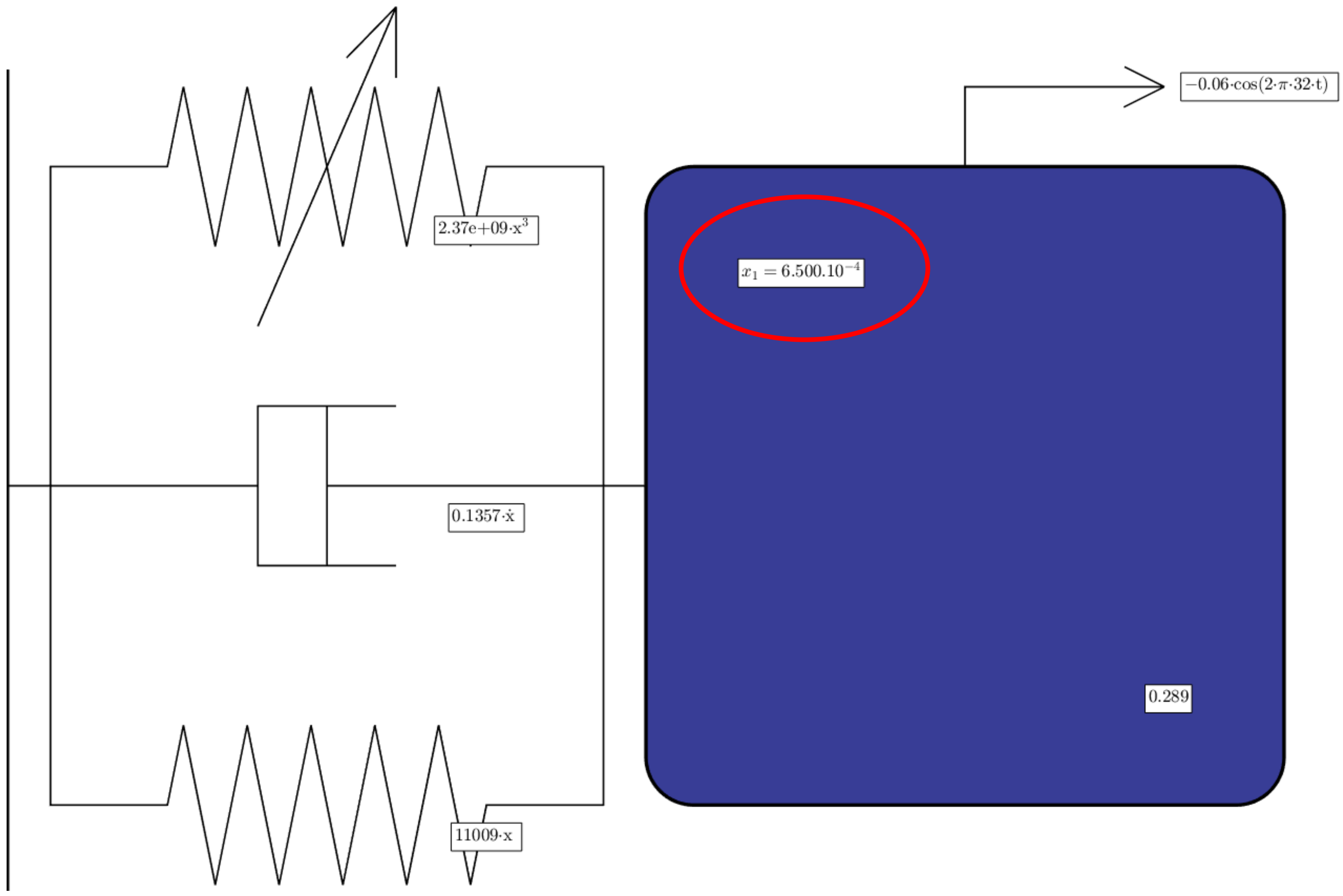


What Is Stability/Instability ?

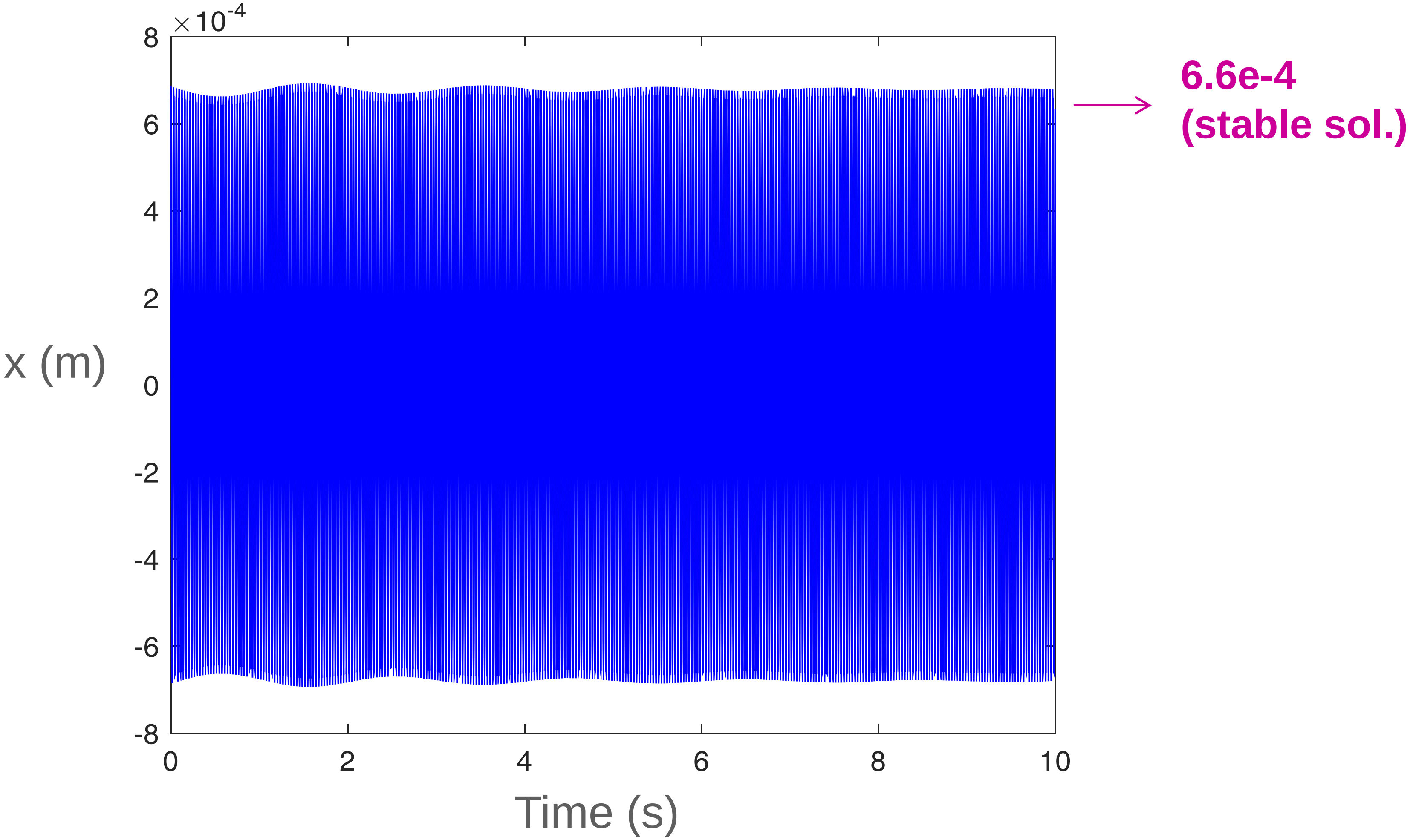
Starting from a Stable/Unstable Solutions



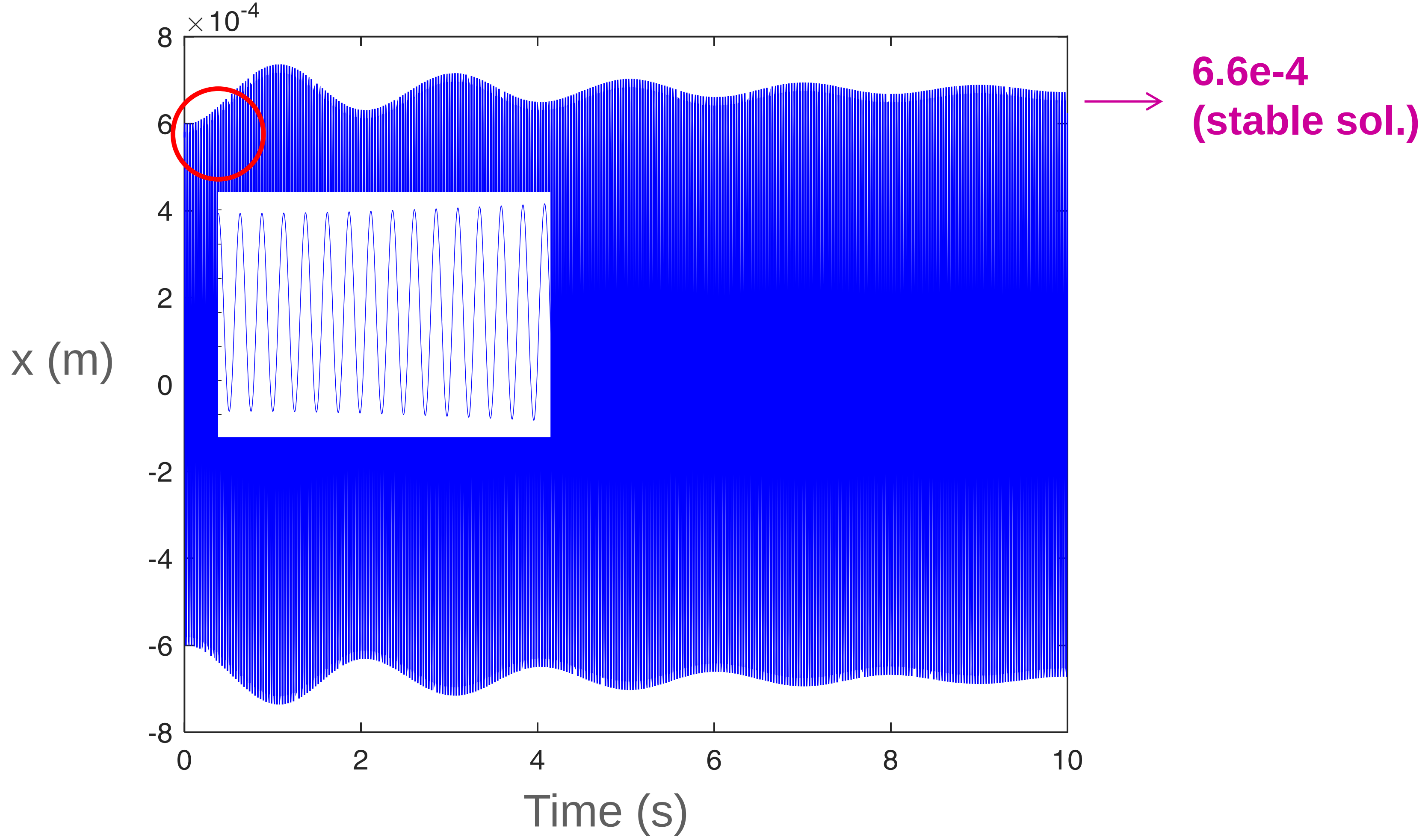
Starting from a Stable/Unstable Solutions



Starting from a Perturbed Stable Solution

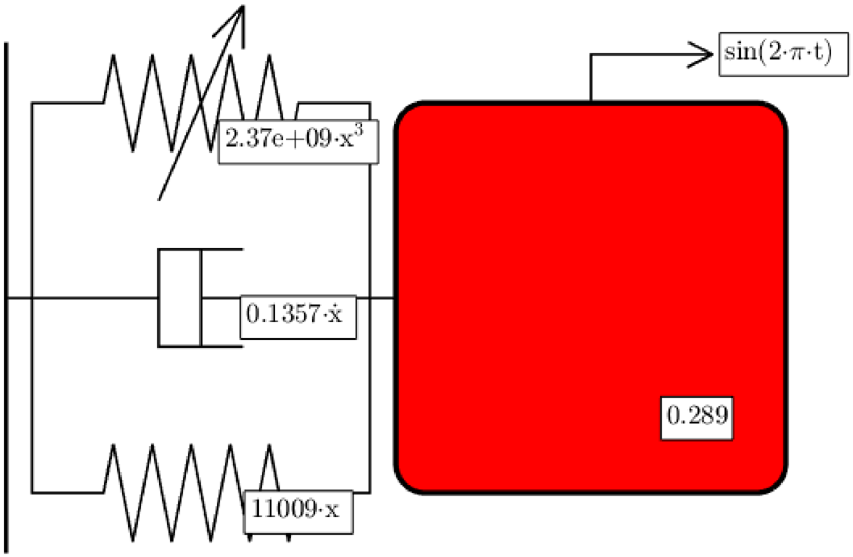


Starting from a Perturbed Unstable Solution

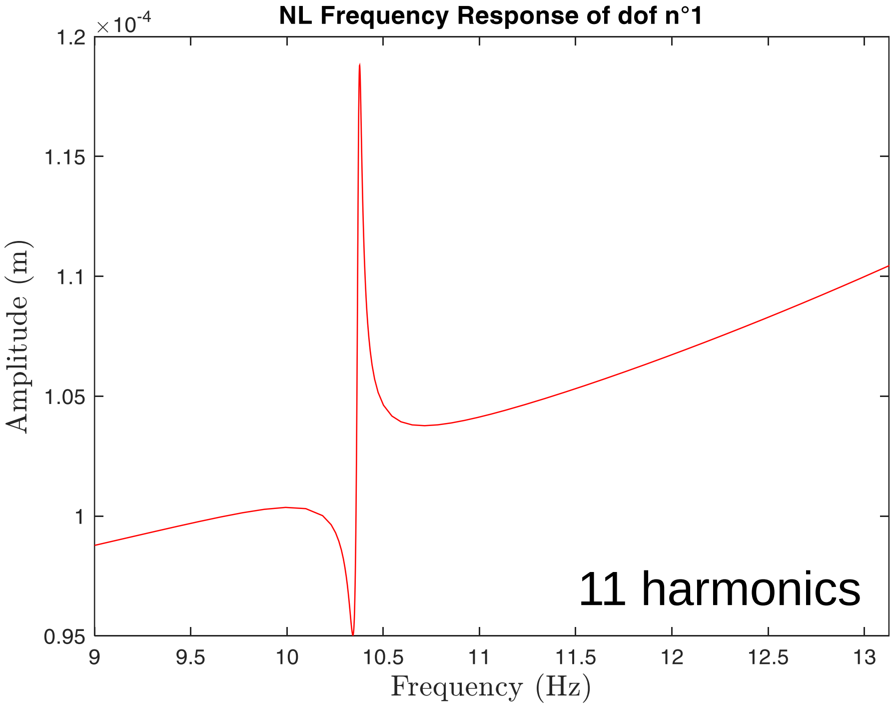


Is that all ?

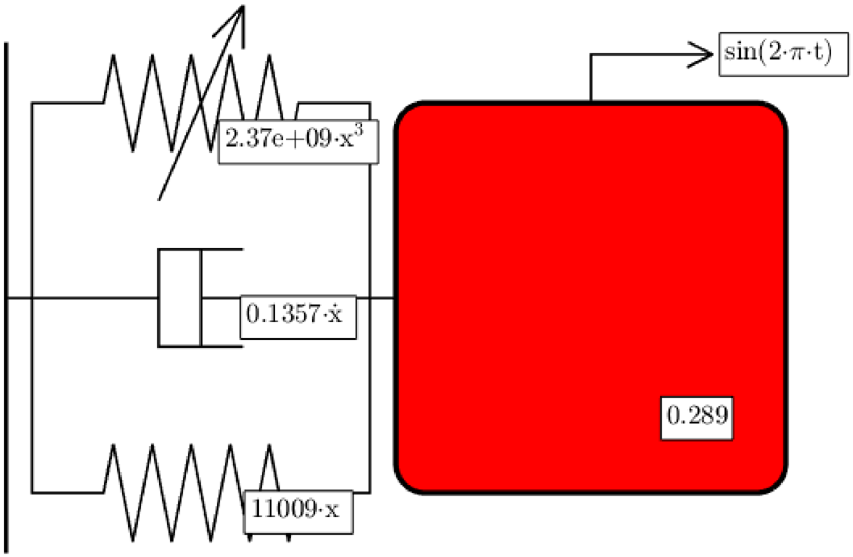
What's Going On @ 1N (Far From Resonance) ?



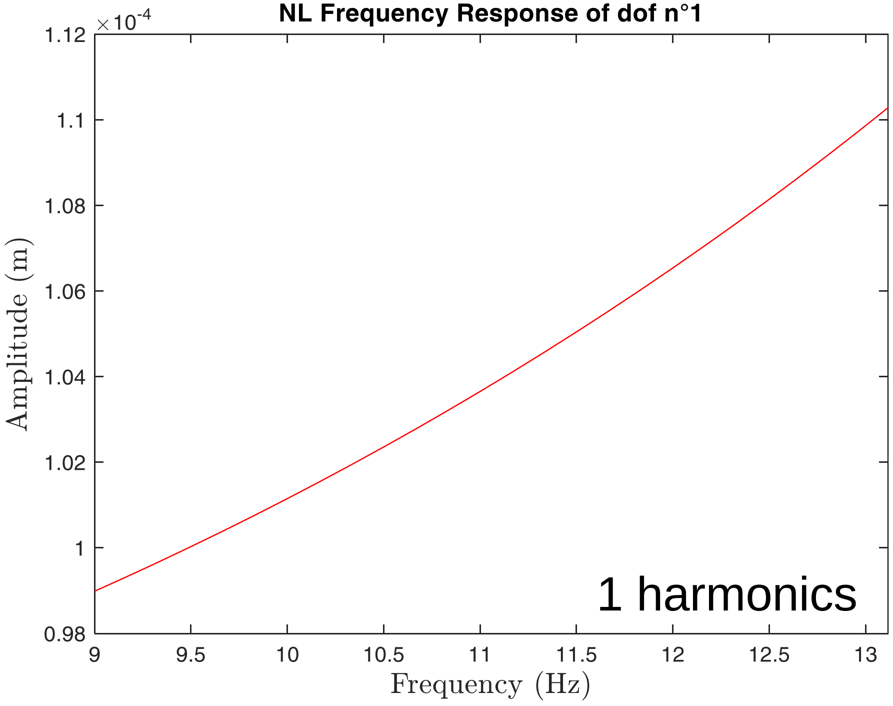
Adaptative stepsize



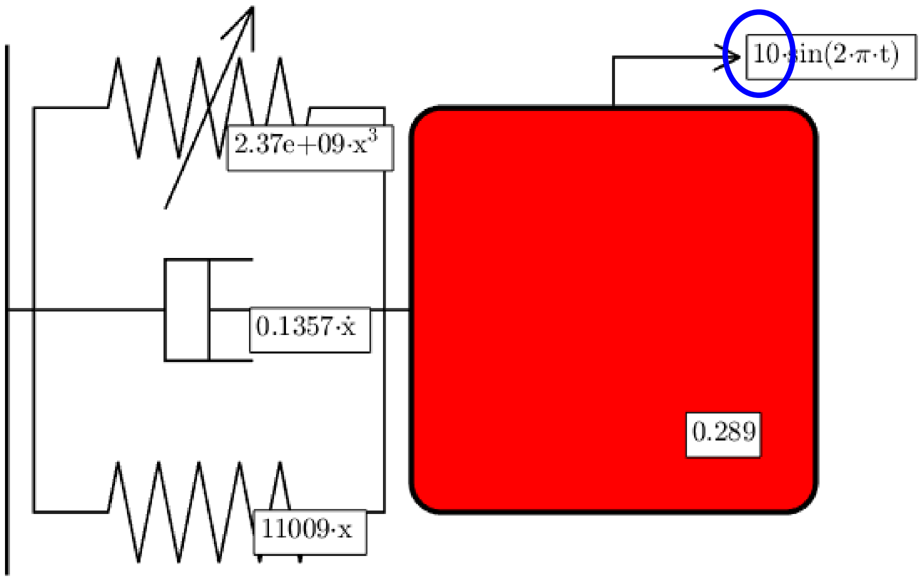
The New Resonance Has Disappeared !



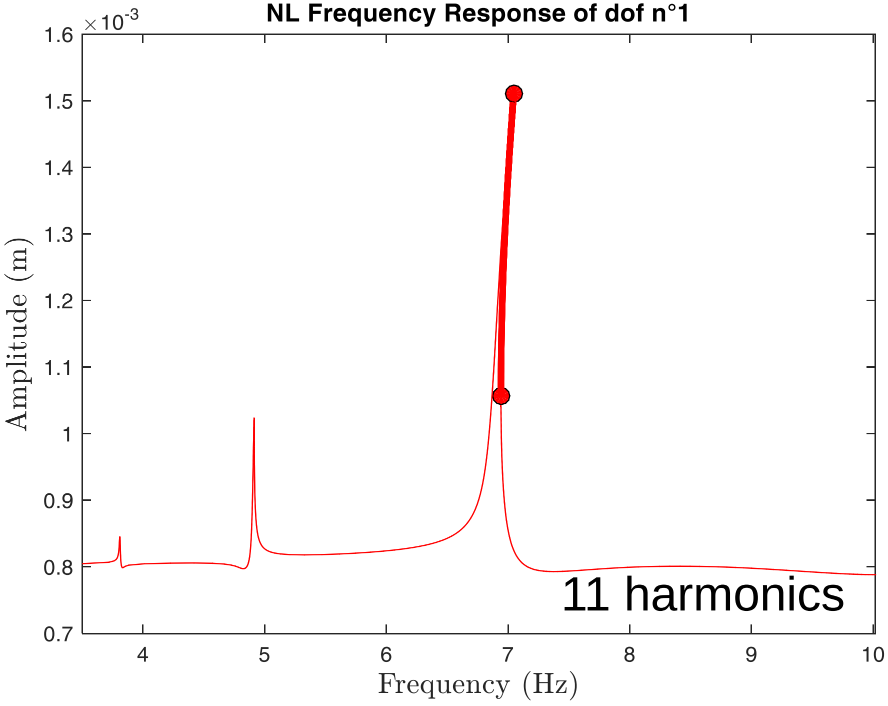
Adaptative stepsize



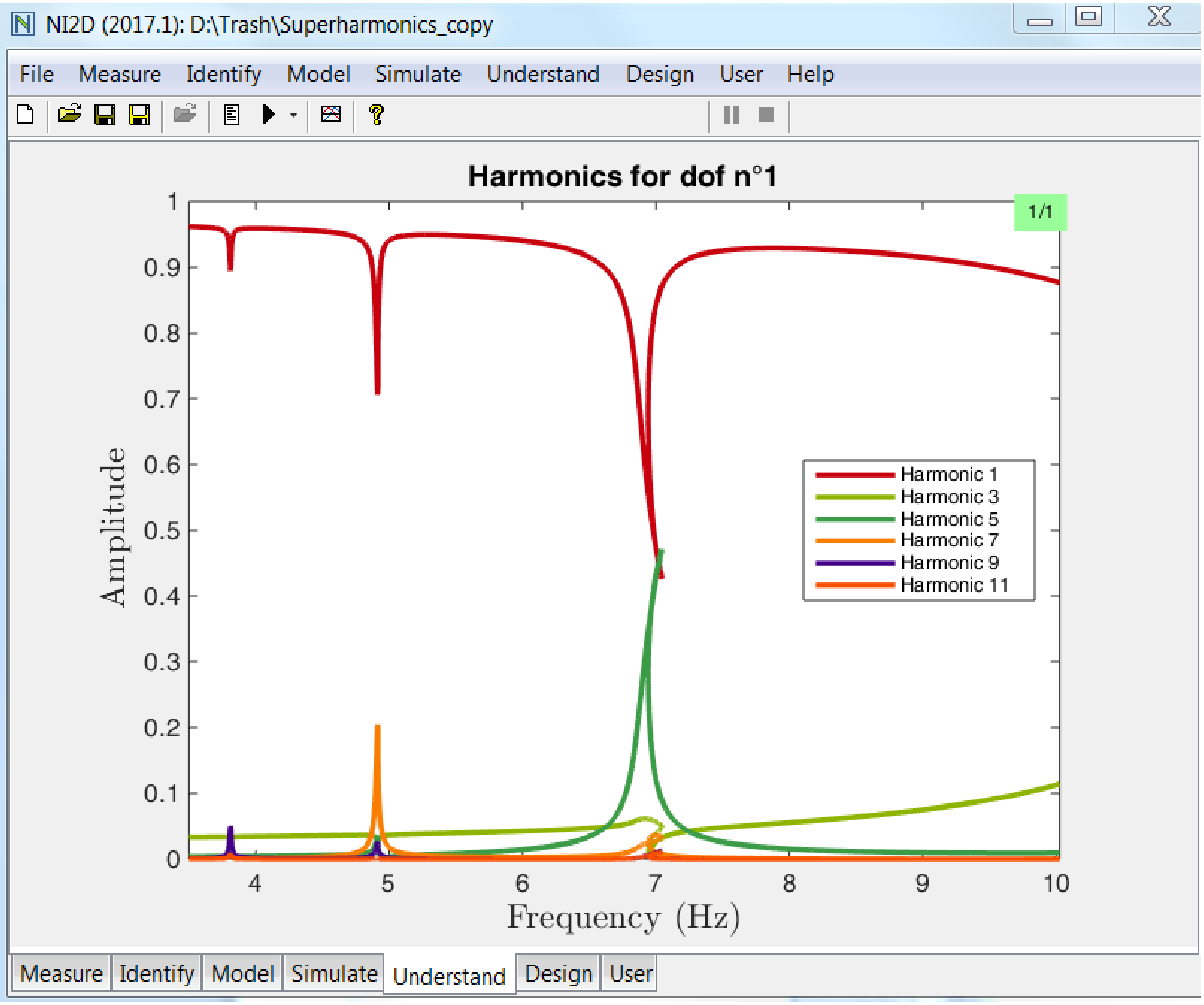
More Resonances @ 10N

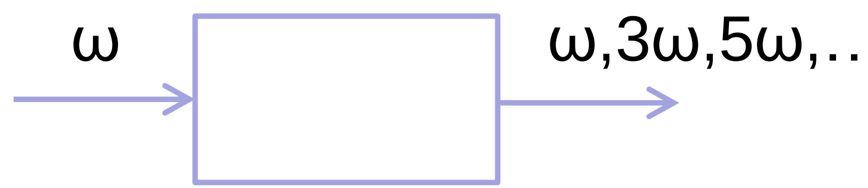


Stepsize=2



Superharmonic Resonances



Linear	Nonlinear
Superposition principle 	**NO !** By definition …
Simple frequency content 	**NO !** Harmonics 
Uniqueness of the solutions	**NO !** Bifurcations
Invariance of FRFs	**NO !** Freq.-energy dependence

4 New Key Concepts

- Harmonics
  - Rich frequency content
  - New resonances
- Frequency-amplitude dependence
  - Don’t extrapolate !
- Bifurcations
  - Nonuniqueness.
  - New resonances
- Stability
  - Stable/unstable solutions

Questions for the exam (L5)

Isolation des vibrations

17. Expliquer le problème de l'isolation directe des vibrations. A l'aide d'un modèle simplifié à 1ddl, écrire les équations du mouvement et établir l'expression du facteur d'isolation FT/F en fonction du pourcentage d'amortissement et de la fréquence propre. Tracer le facteur d'isolation en fonction de la fréquence de l'excitation et identifier le domaine d'isolation. Expliquer l'effet de l'amortissement.
18. Expliquer le problème de l'isolation inverse des vibrations. A l'aide d'un modèle simplifié à 1ddl, écrire les équations du mouvement et établir l'expression de la transmissibilité X/Y en fonction du pourcentage d'amortissement et de la fréquence propre. Tracer la transmissibilité en fonction de la fréquence de l'excitation et identifier le domaine d'isolation. Expliquer l'effet de l'amortissement.

Dynamique des rotors

19. Etablir la réponse libre et la réponse forcée d'un arbre en rotation décrit par le modèle de Jeffcott sans amortissement en utilisant les coordonnées complexes. Définir la notion de vitesse critique.

Questions for the exam (L6)

Systèmes non-linéaires

20. Citer 4 propriétés importantes des systèmes non-linéaires (absentes dans le cas linéaire). Illustrer ces propriétés en considérant la réponse fréquentielle d'un oscillateur cubique à un degré de liberté.
21. Calculer la fréquence de résonance d'un oscillateur cubique (Duffing) non amorti. Quelle conclusion tirez-vous de cette expression ? Montrer que la force élastique d'un système avec une masse entre deux câbles peut s'écrire sous la forme d'un ressort cubique.