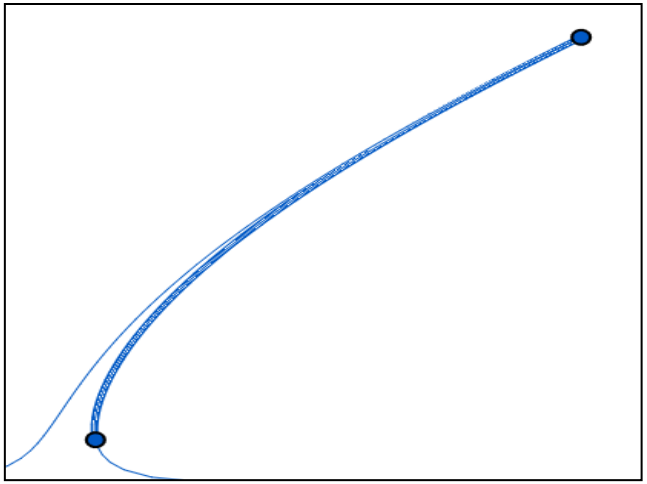
Nonlinear Vibrations of Aerospace Structures

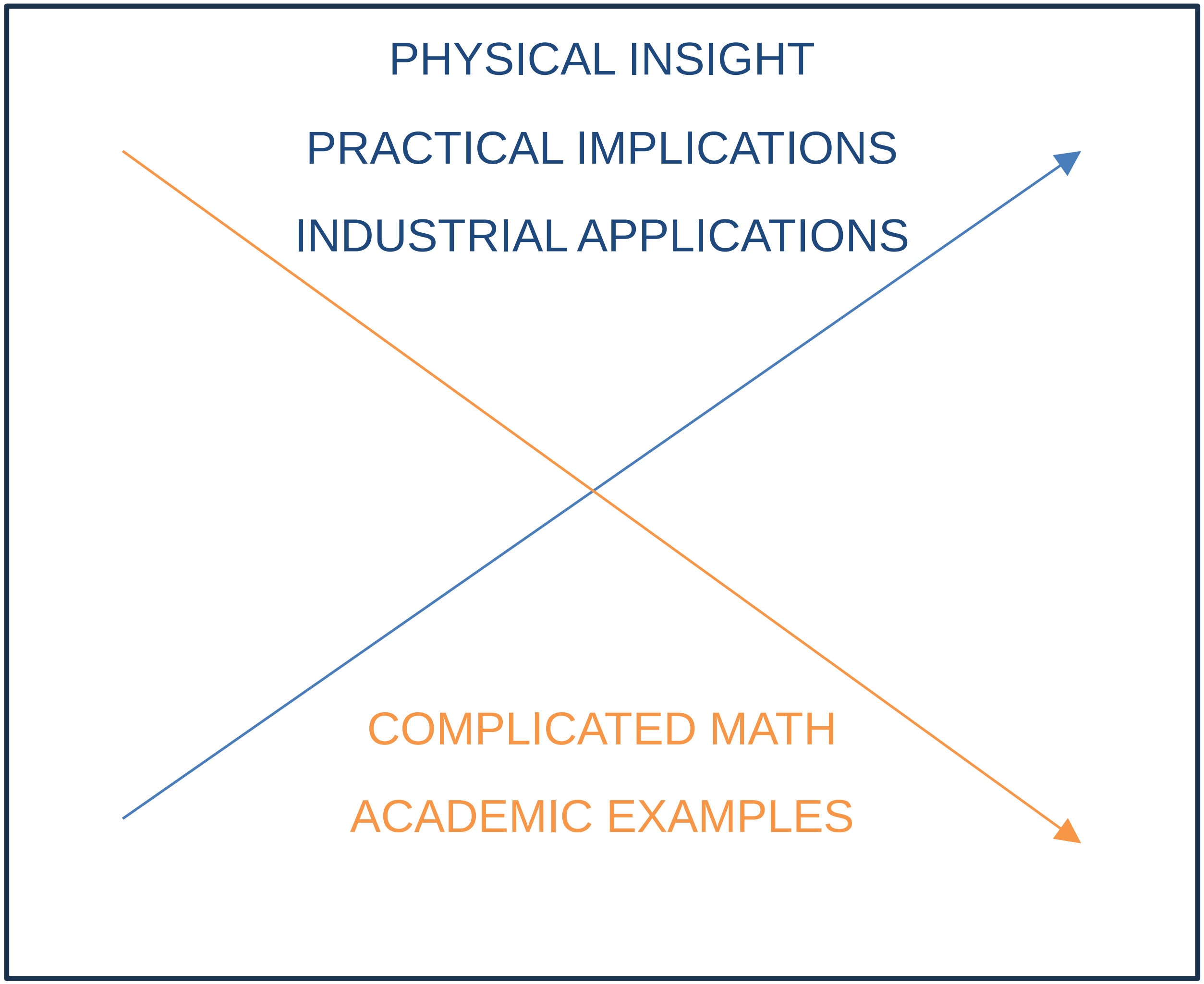
University of Liège, Belgium

L02 Impact of NL

Undamped, unforced dynamics



Today: maths but hopefully useful/meaningful maths



Outline ("nonlinear version of DSM course")

Focus on a 1DOF oscillator

Linear vs. nonlinear

Undamped, unforced dynamics

Damped, unforced dynamics

Undamped/damped, harmonic forcing

Going beyond…

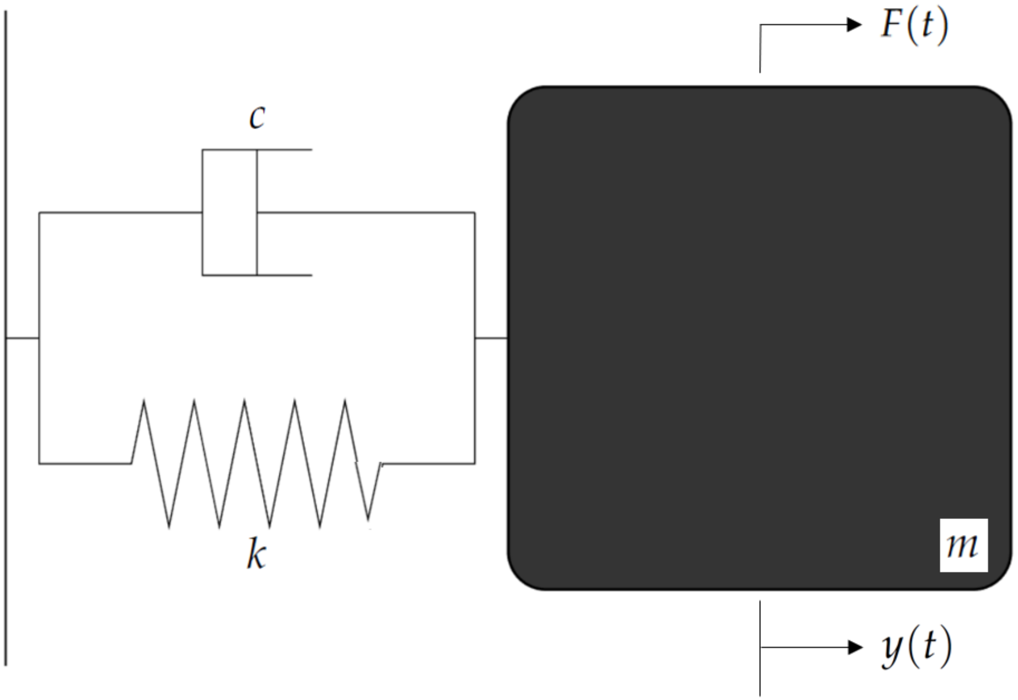
Colored boxes

Important take-away message

I need your opinion

FYI
(not on the critical path)

Back to basics: the spring-mass-damper-oscillator



$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = F(t), \quad \dot{y}(0) = \dot{y}_0, \; y(0) = y_0$$

Important dynamical quantities

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = F(t), \quad \dot{y}(0) = \dot{y}_0,\ y(0) = y_0$$

Divide by m

$$\ddot{y}(t) + 2\xi\omega_0\dot{y}(t) + \omega_0^2 y(t) = f(t), \quad \dot{y}(0) = \dot{y}_0,\ y(0) = y_0$$

$\omega_0 = \sqrt{k/m},$ *Natural frequency*	$\xi = \frac{c}{2\sqrt{km}},$ *Damping ratio*	$f(t) = \frac{F(t)}{m}$ *Mass-normalized forcing*

Linear system: undamped, unforced case

$$\ddot{y}(t) + \omega_0^2 y(t) = 0, \quad \dot{y}(0) = \dot{y}_0, \; y(0) = y_0$$

Multiply by velocity & integrate

$$\frac{\dot{y}^2(t)}{2} + \frac{\omega_0^2 y^2(t)}{2} = \frac{\dot{y}_0^2}{2} + \frac{\omega_0^2 y_0}{2}$$

Energy conservation

$$\omega_0 \, t = \pm \int_{y_0}^{y} \frac{dy}{\sqrt{\left(y_0^2 + \frac{\dot{y}_0^2}{\omega_0^2}\right) - y^2}}$$

$$\int \frac{dy}{\sqrt{a^2 - y^2}} = \sin^{-1}\left(\frac{y}{a}\right)$$

$$y(t) = \sqrt{y_0^2 + \frac{\dot{y}_0^2}{\omega_0^2}} \sin\left(\omega_0 t + \tan^{-1}\frac{\omega_0 y_0}{\dot{y}_0}\right)$$

The response of a linear oscillator takes the form of harmonic motion at the natural frequency $\omega_0 = \sqrt{k/m}$

The natural frequency depends only on k and m !

Comparison against direct time integration

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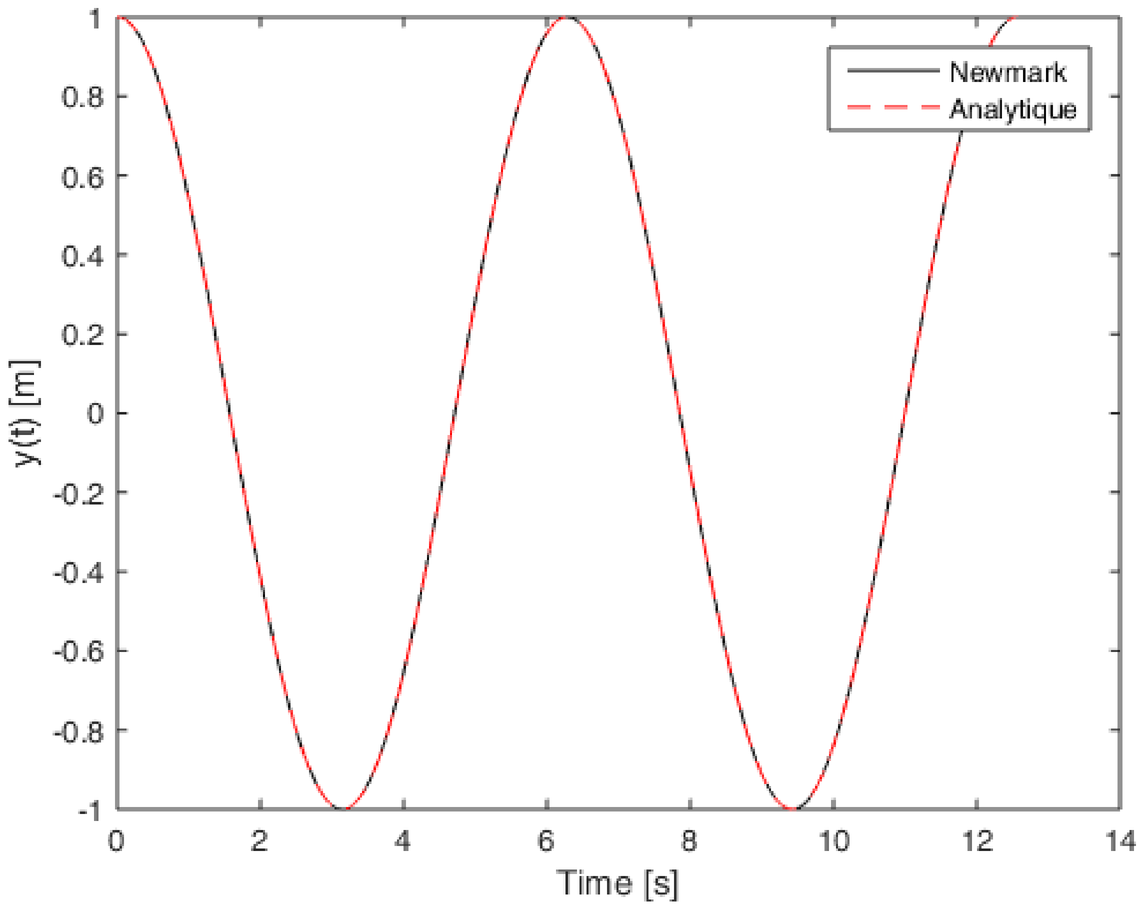
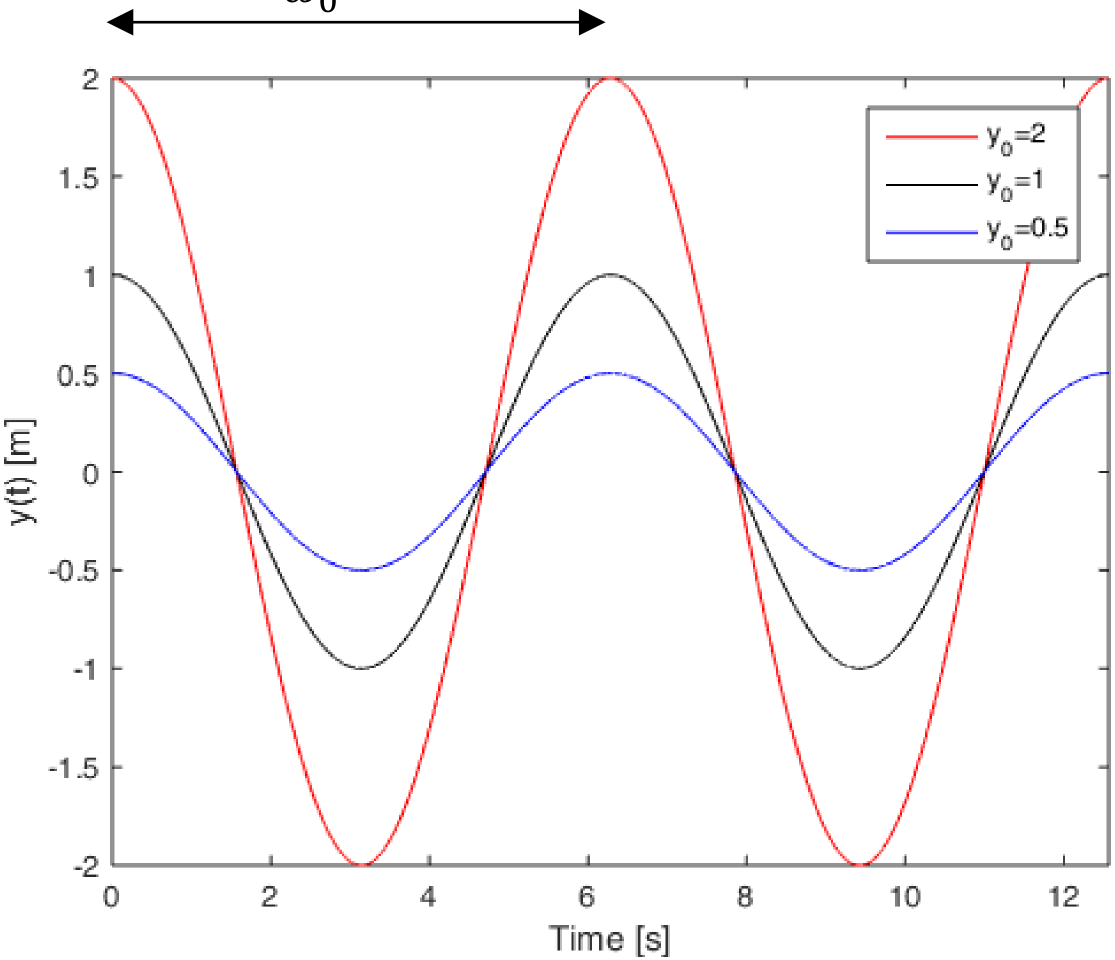


Response to different initial displacements ($\omega_0 = 1$)

$$T = \frac{2\pi}{\omega_0} = 6.28s$$

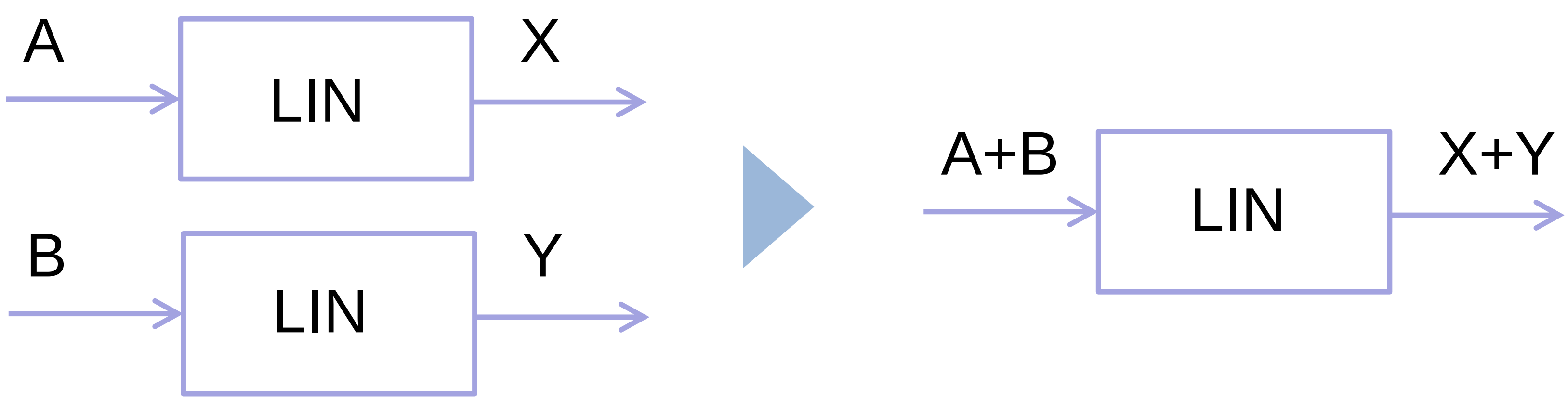


What do you observe ?

The principle of superposition

The principle of superposition is the cornerstone of linear theory:

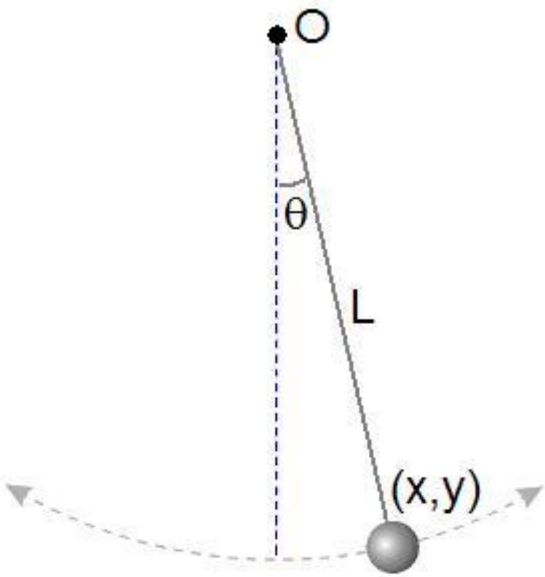
The response caused by two or more inputs is the sum of the responses that would have been caused by each input individually.



The undamped, unforced Duffing oscillator

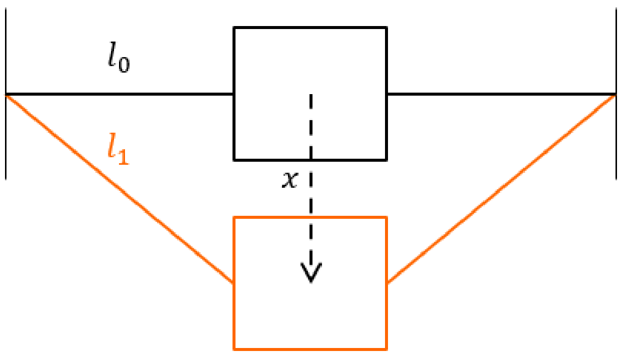
$$\ddot{y}(t) + \omega_0^2 y(t) + \alpha_3 y^3(t) = 0$$

WHY ?



$$\ddot{\theta} + {\omega_0}^2 \sin\theta = 0$$

$$\sin\theta = \theta - \frac{\theta^3}{6} + \dots$$



$$F = 2k(l_1 - l_0)\frac{x}{l_1} = 2kx\left(1 - \frac{l_0}{\sqrt{x^2 + {l_0}^2}}\right)$$

$$\frac{l_0}{\sqrt{x^2 + {l_0}^2}} = 1 - \frac{x^2}{2{l_0}^2} + \frac{3x^4}{8{l_0}^4} + O(x^6)$$

Response to an initial displacement

$$\ddot{y}(t) + \omega_0^2 y(t) + \alpha_3 y^3(t) = 0 \qquad \dot{y}(0) = 0 \; , \; y(0) = y_0$$

Multiply by the velocity and integrate

$$\frac{\dot{y}^2(t)}{2} + \frac{\omega_0^2 y^2(t)}{2} + \frac{\alpha_3 y^4(t)}{4} = \frac{\omega_0^2 y_0^2(t)}{2} + \frac{\alpha_3 y_0^4(t)}{4}$$

$$t = \int_{y_0}^{y} \frac{-dy}{\sqrt{\left(y_0^2 - y^2\right)\left(\omega_0^2 + \frac{\alpha_3}{2}\left(y_0^2 + y^2\right)\right)}}$$

$$y = y_0 \cos\phi$$

$$\Omega t = \int_0^{\phi} \frac{d\phi'}{\sqrt{1 - m\sin^2\phi'}}$$

$$\Omega = \sqrt{\omega_0^2 + \alpha_3 y_0^2}$$

$$m = \alpha_3 y_0^2 / 2\Omega^2$$

Response to an initial displacement

$$\left(\frac{dy}{dt}\right)^2 = \omega_0^2 y_0^2 - \omega_0^2 y^2 + \frac{\alpha_3}{2} y_0^4 - \frac{\alpha_3}{2} y^4 \rightarrow dt = \sqrt{\frac{dy^2}{\omega_0^2\left(y_0^2 - y^2\right) + \frac{\alpha_3}{2}\left(y_0^2 - y^2\right)\left(y_0^2 + y^2\right)}}$$

$$t = \int_{y_0}^{y} \frac{-dy}{\sqrt{\left(y_0^2 - y^2\right)\left(\omega_0^2 + \frac{\alpha_3}{2}\left(y_0^2 + y^2\right)\right)}}$$

$$y = y_0 \cos\phi$$

$$t = \int_0^{\phi} \frac{\cancel{y_0 \sin\phi}\, d\phi}{\sqrt{\cancel{y_0^2 \sin^2\phi}\left(\omega_0^2 + \frac{\alpha_3}{2}\left(y_0^2 + y_0^2 \cos^2\phi\right)\right)}} = \int_0^{\phi} \frac{d\phi}{\omega_0^2 + \frac{y_0^2}{2}\alpha_3 + \frac{y_0^2}{2}\alpha_3 \cos^2\phi}$$

Response to an initial displacement

$$t = \int_0^{\phi} \frac{\cancel{y_0 \sin\phi}\, d\phi}{\sqrt{\cancel{y_0^2 \sin^2\phi}\left(\omega_0^2 + \frac{\alpha_3}{2}\left(y_0^2 + y_0^2 \cos^2\phi\right)\right)}} = \int_0^{\phi} \frac{d\phi}{\omega_0^2 + \frac{y_0^2}{2}\alpha_3 + \frac{y_0^2}{2}\alpha_3 \cos^2\phi}$$

$$\rightarrow \omega_0^2 + \frac{y_0^2\alpha_3}{2} + \frac{y_0^2\alpha_3}{2} + \frac{y_0^2\alpha_3}{2}\left(\cos^2\phi - 1\right)$$

$$= \underbrace{\left(\omega_0^2 + \frac{y_0^2\alpha_3}{2}\right)}_{\Omega} + \frac{y_0^2\alpha_3}{2}\left(-\sin^2\phi\right) = \Omega^2 + \frac{y_0^2\alpha_3}{2}\left(-\sin^2\phi\right)$$

$$= \Omega^2\left(1 - \frac{y_0^2\alpha_3}{2\Omega^2}\sin^2\phi\right)$$

$$t = \int_0^{\phi} \frac{d\phi}{\Omega\sqrt{1 - \left(\frac{y_0^2\alpha_3}{2\Omega^2}\right)\sin^2\phi}} \rightarrow \Omega t = \int_0^{\phi} \frac{d\phi}{\sqrt{1 - m\sin^2\phi}}$$

Let's compare the linear and nonlinear cases

$$\ddot{y}(t) + \omega_0^2 y(t) + \alpha_3 y^3(t) = 0 \qquad \dot{y}(0) = 0\ ,\ y(0) = y_0$$

LINEAR

$$\omega_0\, t = \int_{y_0}^{y} \frac{dy}{\sqrt{\left(y_0^2 + \frac{\dot{y}_0^2}{\omega_0^2}\right) - y^2}}$$

$$\omega_0 = \sqrt{k/m}$$

NONLINEAR

$$\Omega t = \int_0^{\phi} \frac{d\phi'}{\sqrt{1 - m \sin^2 \phi'}}$$

$$\Omega = \sqrt{\omega_0^2 + \alpha_3 y_0^2}$$

What can you conclude ?

The solution is expressed as an elliptic cosine

$$u = F(\phi|m) = \int_0^{\phi} \frac{d\phi'}{\sqrt{1 - m\sin^2\phi'}}$$
$$= \Omega t$$

$$\mathrm{cn}(u|m) = \cos\phi$$

$$y = y_0 \cos\phi$$

$$y(t) = y_0\,\mathrm{cn}\,(\Omega t|m)$$

The response does not take the form of a harmonic function.

The frequency depends on k and m but also on the initial displacement and nonlinear coefficient

Handbook of Mathematical Functions

With

Formulas, Graphs, and Mathematical Tables

Edited by
Milton Abramowitz and Irene A. Stegun

The Jacobian elliptic functions can also be defined with respect to certain integrals. Thus if

16.1.3 $u=\int_0^{\varphi}\frac{d\theta}{(1-m\sin^2\theta)^{1/2}},$

the angle φ is called the *amplitude*

16.1.4 $\varphi=\mathrm{am}\ u$

and we define

16.1.5

$\mathrm{sn}\ u=\sin\varphi$, $\mathrm{cn}\ u=\cos\varphi$,

569

$\mathrm{dn}\ u=(1-m\sin^2\varphi)^{1/2}$

It is the exact solution: confirmation in Matlab

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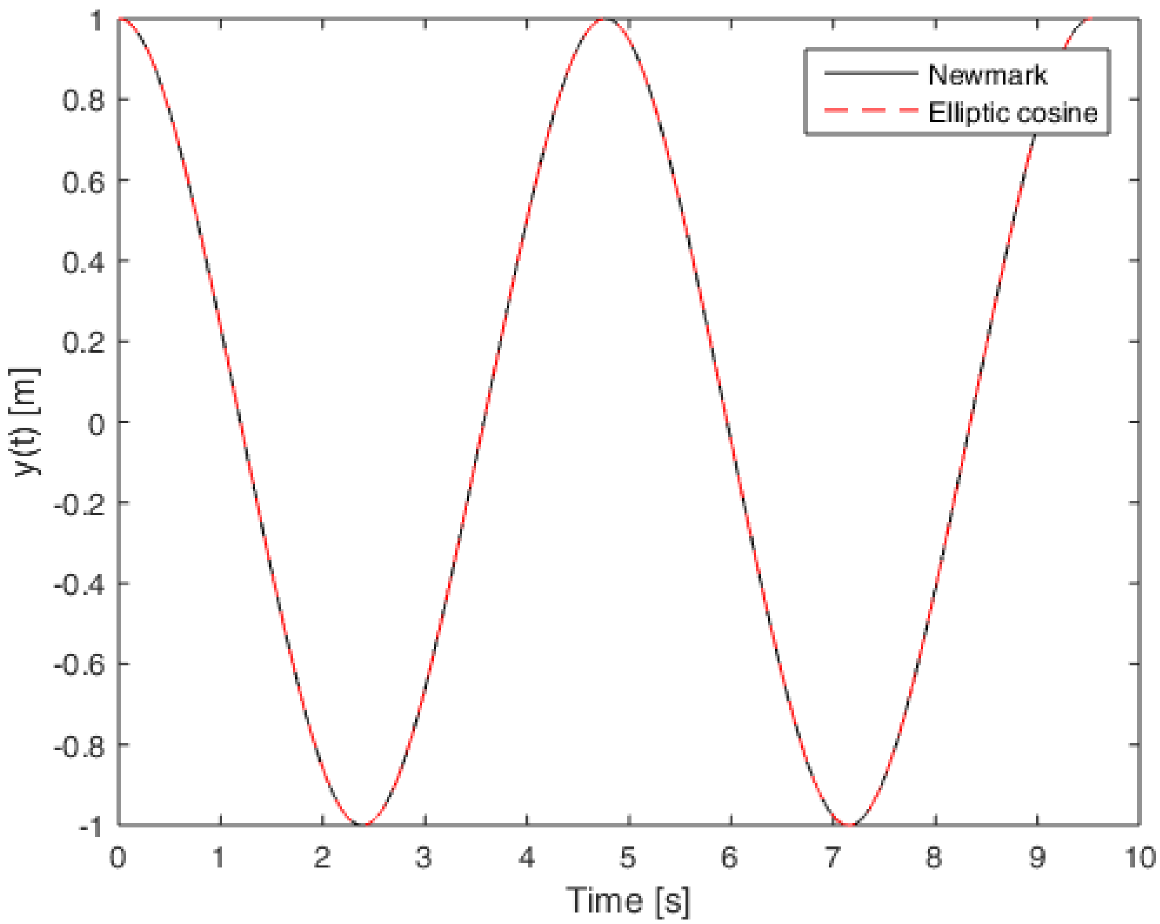


$$\ddot{y}(t) + y(t) + y^3(t) = 0$$

Function ellipj.m

More on elliptic functions

We recall that trigonometric functions can be defined in terms of the functional inverse of specific integrals. For example,

$$\theta = \int_0^y \frac{dy'}{\sqrt{1-y'^2}} = \arcsin y \;\; \rightarrow \sin\theta = y \qquad (3.63)$$

Similarly, Jacobi elliptic functions result from the inversion of the elliptic integral of the first kind. For instance,

$$u = \int_0^y \frac{dy'}{\sqrt{(1-y'^2)\,(1-k^2y'^2)}} \;\; \rightarrow \mathrm{sn}(u,k) = y \qquad (3.64)$$

or, if $y = \sin\phi$,

$$F(\phi,k) = \int_0^\phi \frac{d\phi'}{\sqrt{1-k^2\sin^2\phi'}} \;\; \rightarrow \mathrm{sn}(u,k) = \sin\phi \qquad (3.65)$$

While trigonometric functions are defined with reference to a circle, the previous section has shown that the Jacobi elliptic functions refer to the ellipse. But their geometrical interpretation is similar:

$$\cos\theta = \frac{x}{r}, \ \sin\theta = \frac{y}{r} \tag{3.66}$$

$$\mathrm{cn}(u,k) = \frac{x}{r}, \ \mathrm{sn}(u,k) = \frac{y}{r} \tag{3.67}$$

with $r = 1$ on the unit circle whereas r varies along the unit ellipse.

Finally, Jacobi elliptic functions include trigonometric and hyperbolic functions as special cases

$$k = 0 \quad : \quad \mathrm{sn}(u,0) = \sin(u), \ \mathrm{cn}(u,0) = \cos(u) \tag{3.68}$$

$$k = 1 \quad : \quad \mathrm{sn}(u,1) = \tanh(u), \ \mathrm{cn}(u,1) = \mathrm{sech}(u) \tag{3.69}$$

and

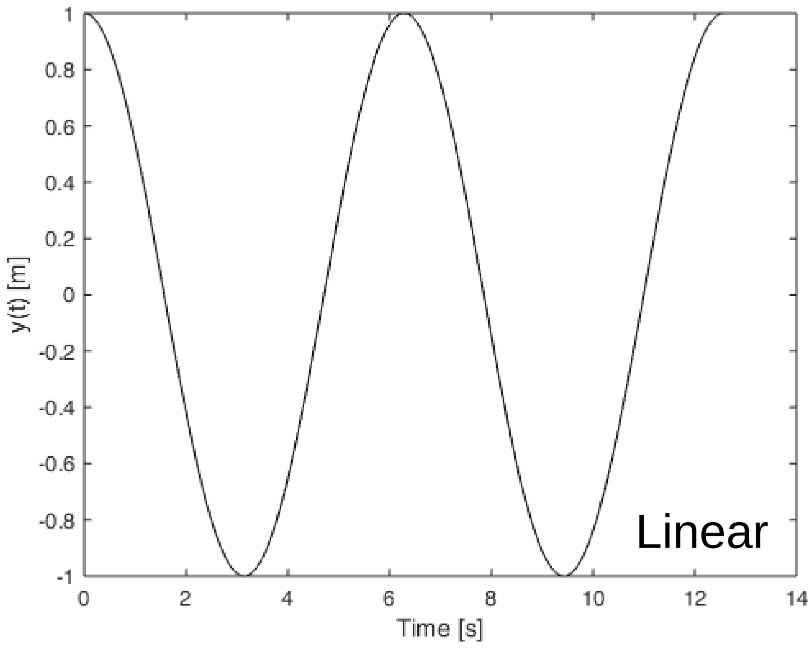
$$\mathrm{cn}(0,k) = 1, \quad \mathrm{sn}(0,k) = 0 \tag{3.70}$$

$$\mathrm{cn}^2(u,k) + \mathrm{sn}^2(u,k) = 1 \tag{3.71}$$

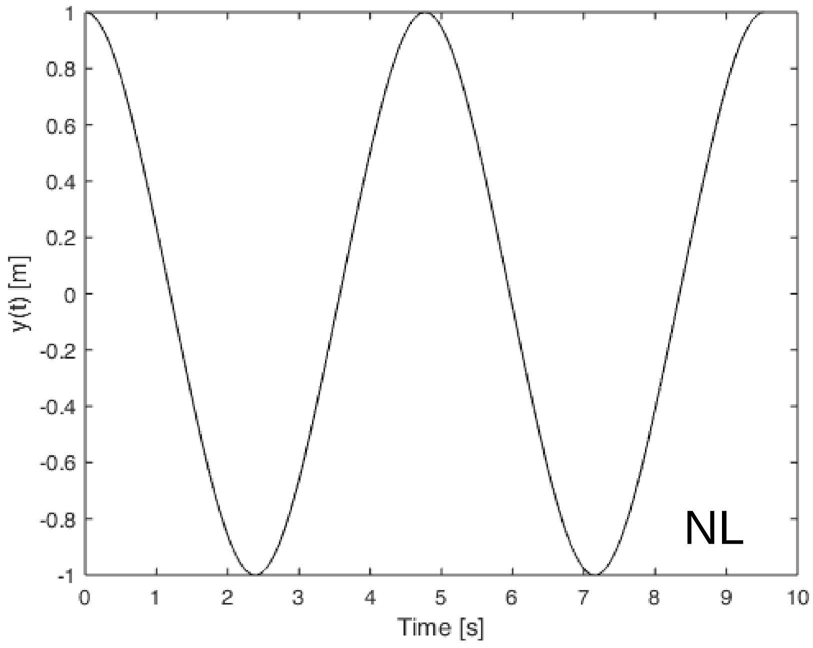
$$\frac{\mathrm{d}}{\mathrm{du}}\mathrm{cn} = -\mathrm{sn}\,\mathrm{dn}, \ \ \frac{\mathrm{d}}{\mathrm{du}}\mathrm{sn} = \mathrm{cn}\,\mathrm{dn}, \ \ \frac{\mathrm{d}}{\mathrm{du}}\mathrm{dn} = -k\,\mathrm{cn}\,\mathrm{sn} \tag{3.72}$$

Let’s compare the linear and nonlinear cases

$$\ddot{y}(t) + y(t) = 0$$



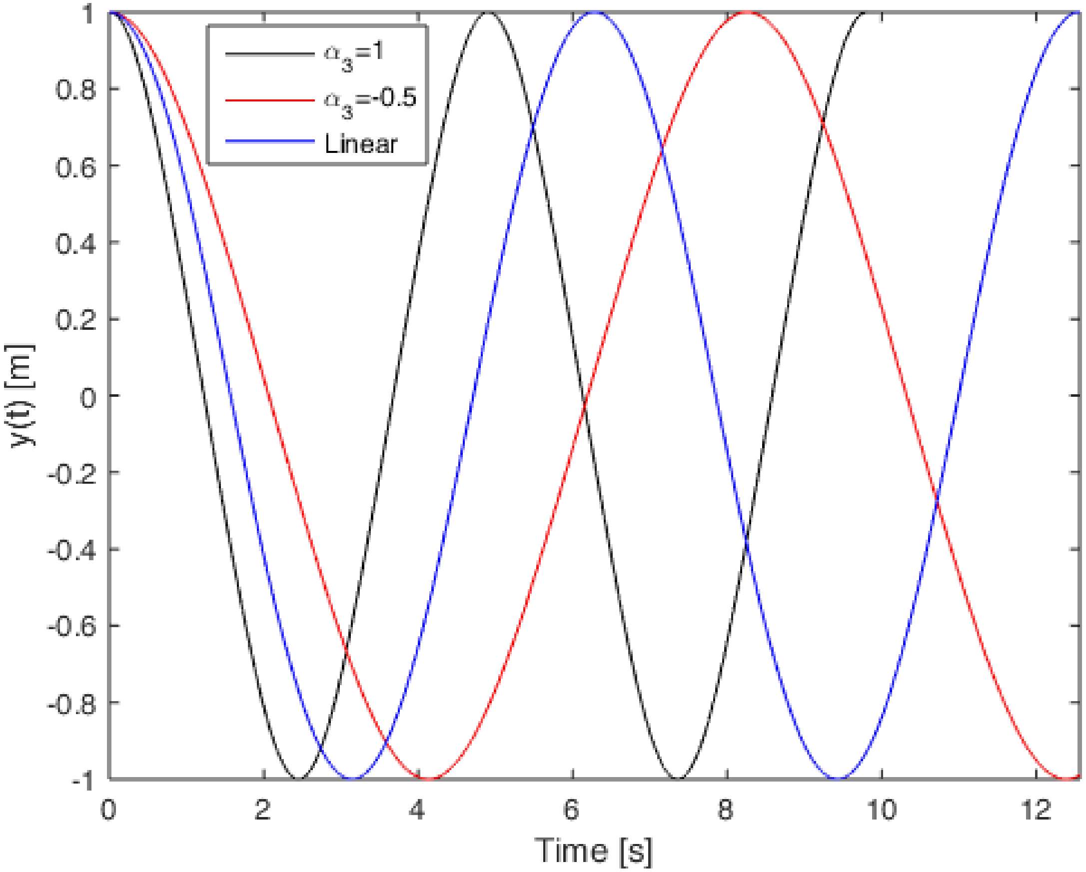
$$\ddot{y}(t) + y(t) + y^3(t) = 0$$



What do you observe ? (2)

The period depends on the nonlinear coefficient

$$\ddot{y}(t) + y(t) + \alpha_3 y^3(t) = 0$$



Hardening case,

« More stiffness »

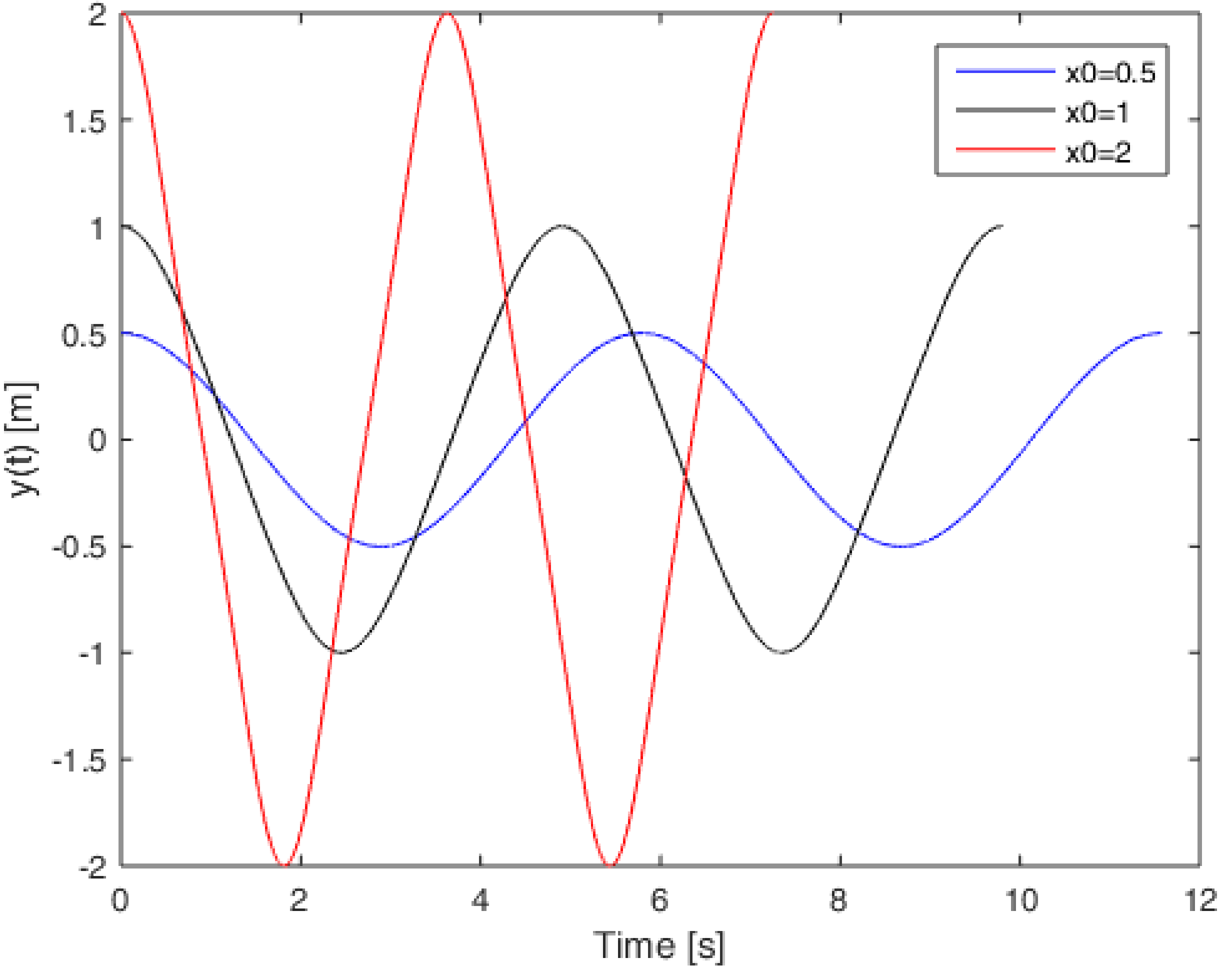
Softening case

« Less stiffness »

Linear case

The period depends on the initial displacement

$$\ddot{y}(t) + y(t) + y^3(t) = 0$$



Failure of superposition principle

The period can be calculated explicitly

The period is equal to 4 times the time to move from the initial position to the equilibrium position. The corresponding variation of ϕ is between 0 and $\pi/2$.

$$u = F(\phi|m) = \int_0^{\phi} \frac{d\phi'}{\sqrt{1 - m\sin^2\phi'}}$$

$$= \Omega t$$

$$T = \frac{4F\left(\frac{\pi}{2}|m\right)}{\Omega}$$

The period can be calculated explicitly

$$T = \frac{4F\left(\frac{\pi}{2}|m\right)}{\Omega} = \frac{4K\left(m\right)}{\Omega}$$

Function ellipke.m

17.3. Complete Elliptic Integrals of the First and Second Kinds

Referred to the canonical forms of **17.2**, the elliptic integrals are said to be *complete* when the amplitude is $\frac{1}{2}\pi$ and so $x=1$. These complete integrals are designated as follows

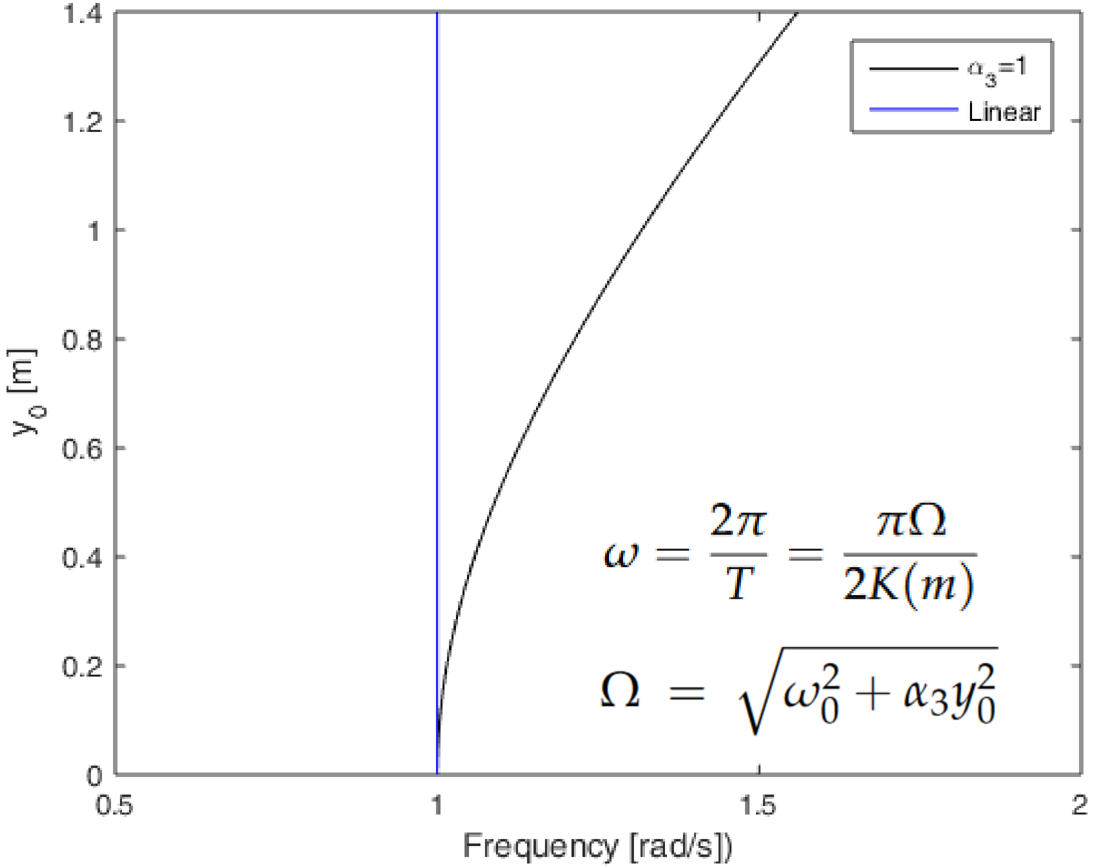
17.3.1

$$[K(m)]=K=\int_0^1[(1-t^2)(1-mt^2)]^{-1/2}dt$$

$$=\int_0^{\pi/2}(1-m\sin^2\theta)^{-1/2}d\theta$$

17.3.2 $K=F(\frac{1}{2}\pi|m)=F(\frac{1}{2}\pi\backslash\alpha)$

The natural frequency of the Duffing oscillator

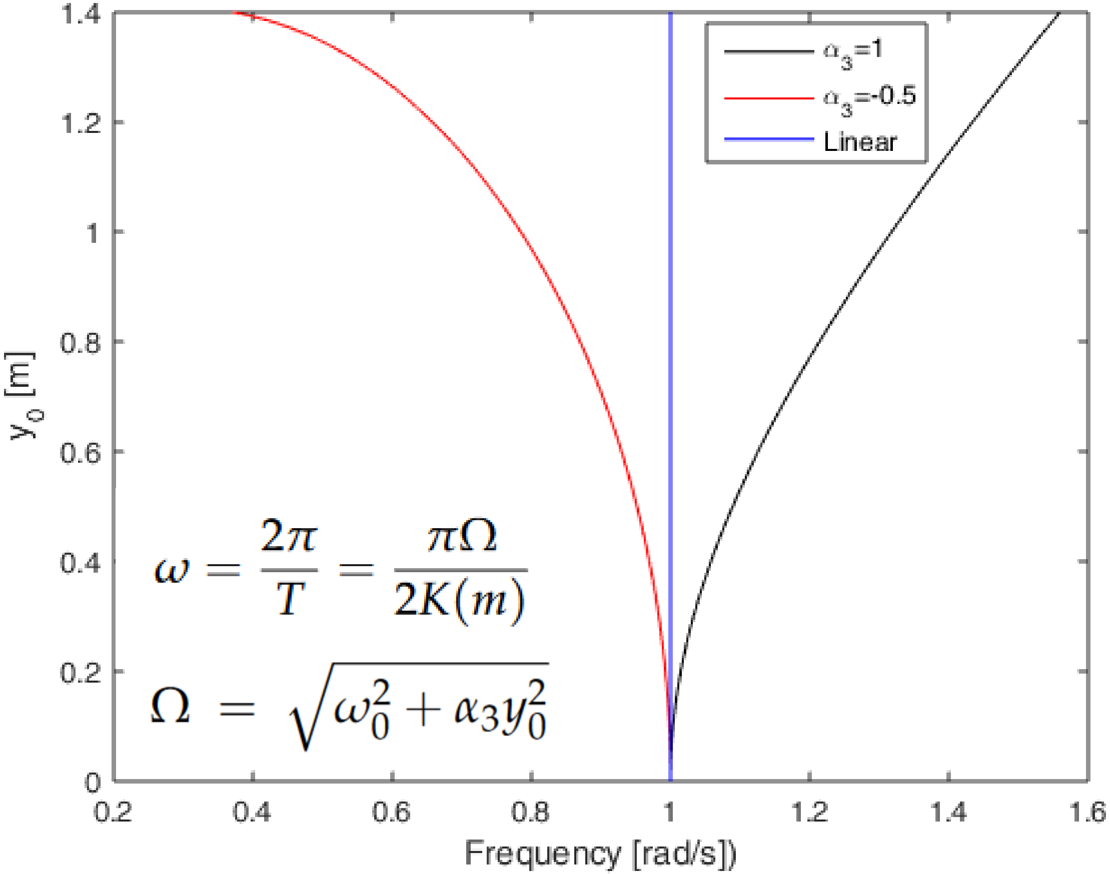


This highlights the frequency-amplitude dependence of nonlinear ocillations

Very important concept:
the backbone curve

What if the nonlinear coefficient is negative ?

What if the nonlinear coefficient is negative ?



$$\omega = \frac{2\pi}{T} = \frac{\pi\Omega}{2K(m)}$$

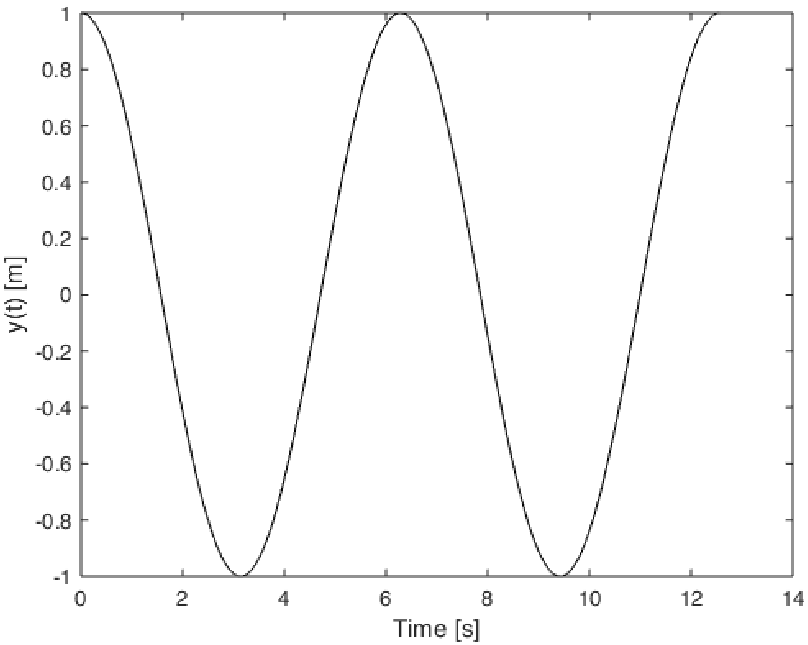
$$\Omega = \sqrt{\omega_0^2 + \alpha_3 y_0^2}$$

This highlights the frequency-amplitude dependence of nonlinear ocillations

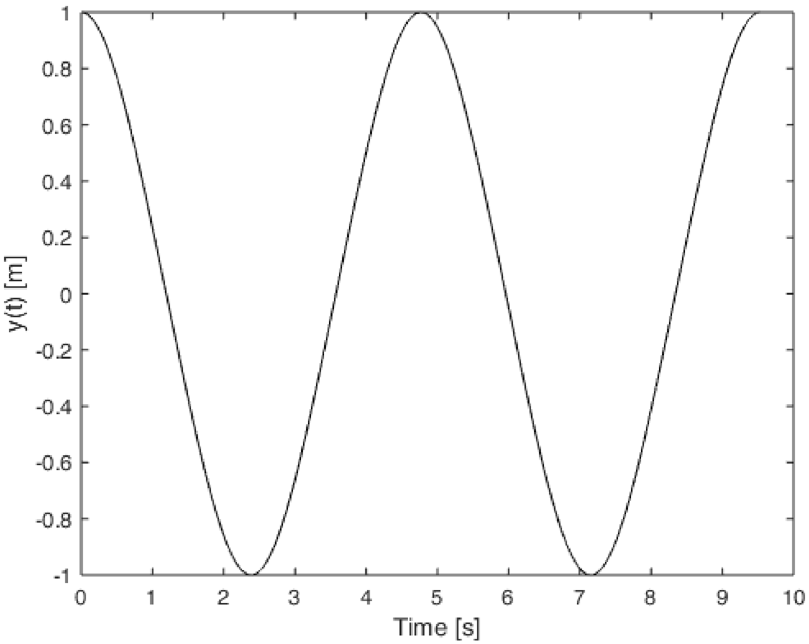
Very important concept:
the backbone curve

The elliptic cosine looks like a pure cosine

$$\ddot{y}(t) + y(t) = 0$$



$$\ddot{y}(t) + y(t) + y^3(t) = 0$$



Really ?

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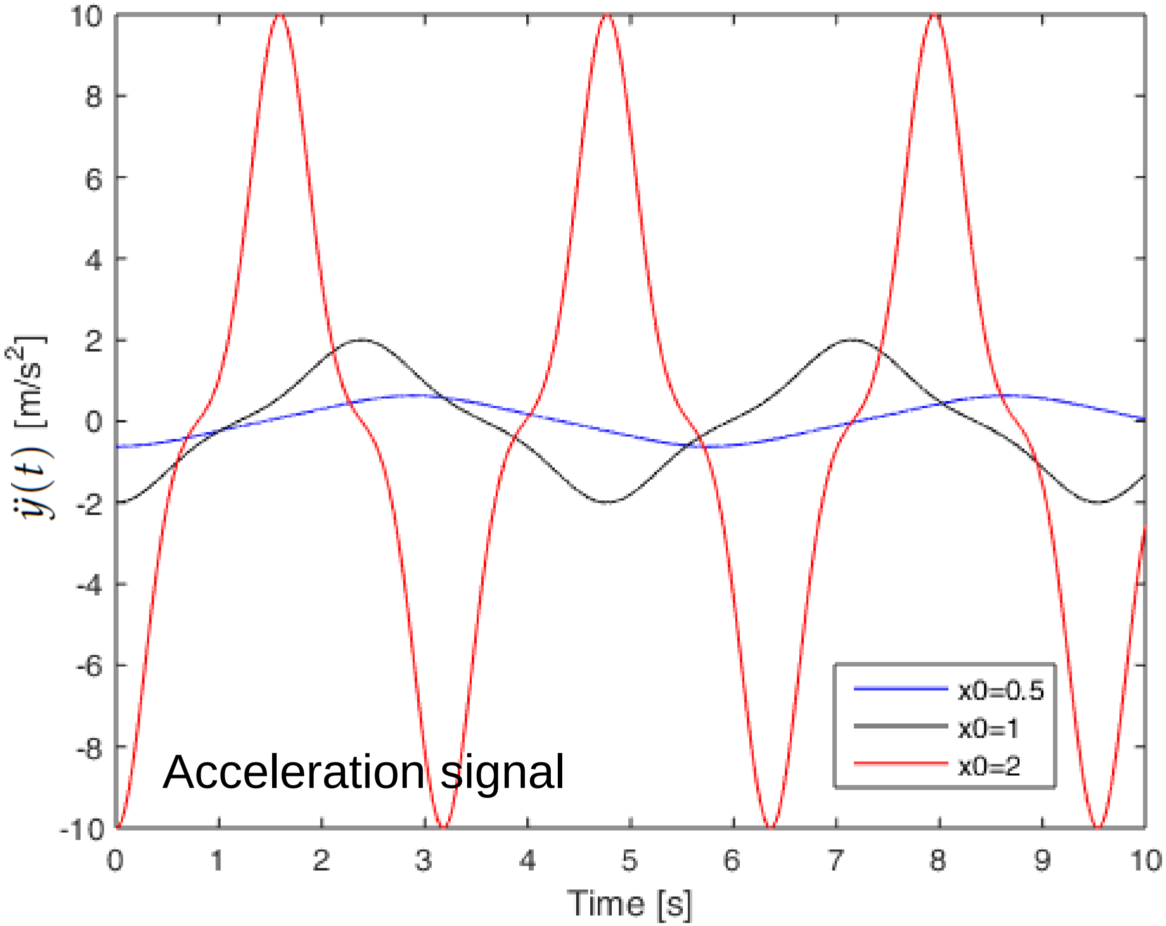

What do you observe ?

$$\ddot{y}(t) + y(t) + y^3(t) = 0$$

Analytical expression of the acceleration signal

$$\frac{\mathrm{d}}{\mathrm{d}u}\mathrm{cn} = -\mathrm{sn}\,\mathrm{dn}, \quad \frac{\mathrm{d}}{\mathrm{d}u}\mathrm{sn} = \mathrm{cn}\,\mathrm{dn}, \quad \frac{\mathrm{d}}{\mathrm{d}u}\mathrm{dn} = -k\,\mathrm{cn}\,\mathrm{sn}$$

$$y(t) = y_0\,\mathrm{cn}\left(\Omega t | m\right)$$

$$\ddot{y}(t) = y_0 \Omega^2 \mathrm{cn}\left(\Omega t | m\right)\left[m\,\mathrm{sn}^2\left(\Omega t | m\right) - \mathrm{dn}^2\left(\Omega t | m\right)\right]$$

Frequency analysis through FFT

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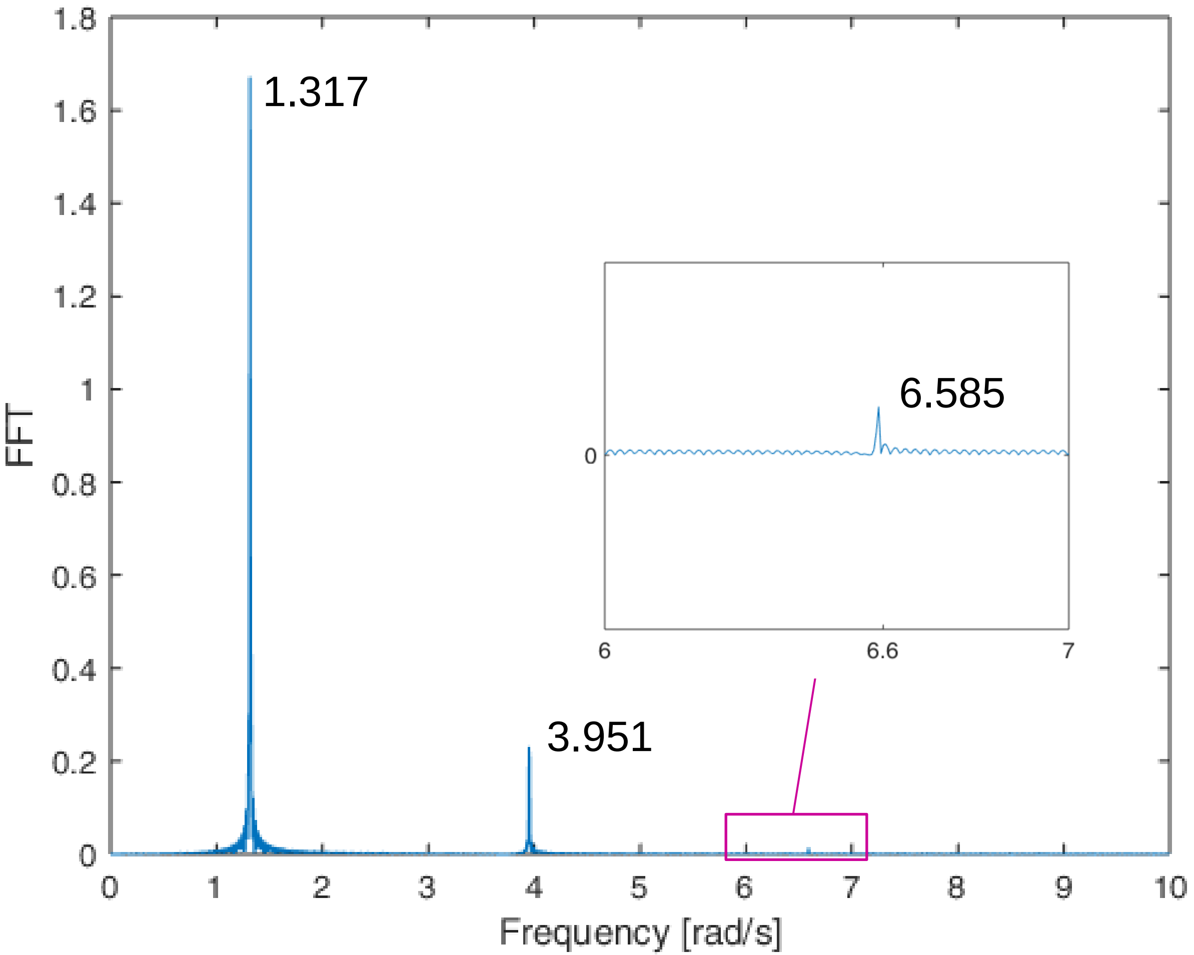
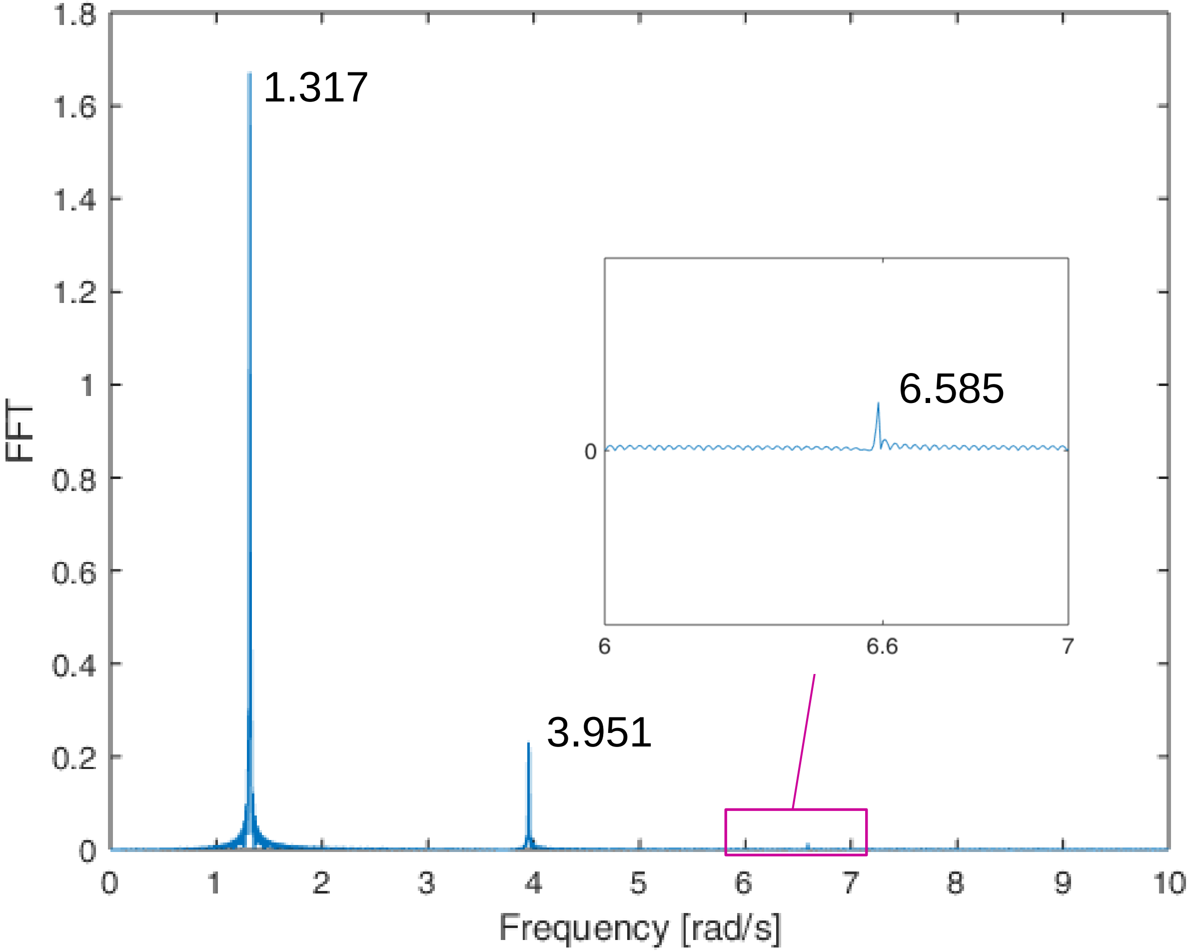


Relation between the numbers ?

Frequency analysis through FFT



Nonlinear system generates harmonics

An analysis based on the first harmonic only underestimates the actual amplitude

Response to an initial velocity

$$\ddot{y}(t) + \omega_0^2 y(t) + \alpha_3 y^3(t) = 0 \qquad \dot{y}(0) = \dot{y}_0,\ y(0) = 0$$

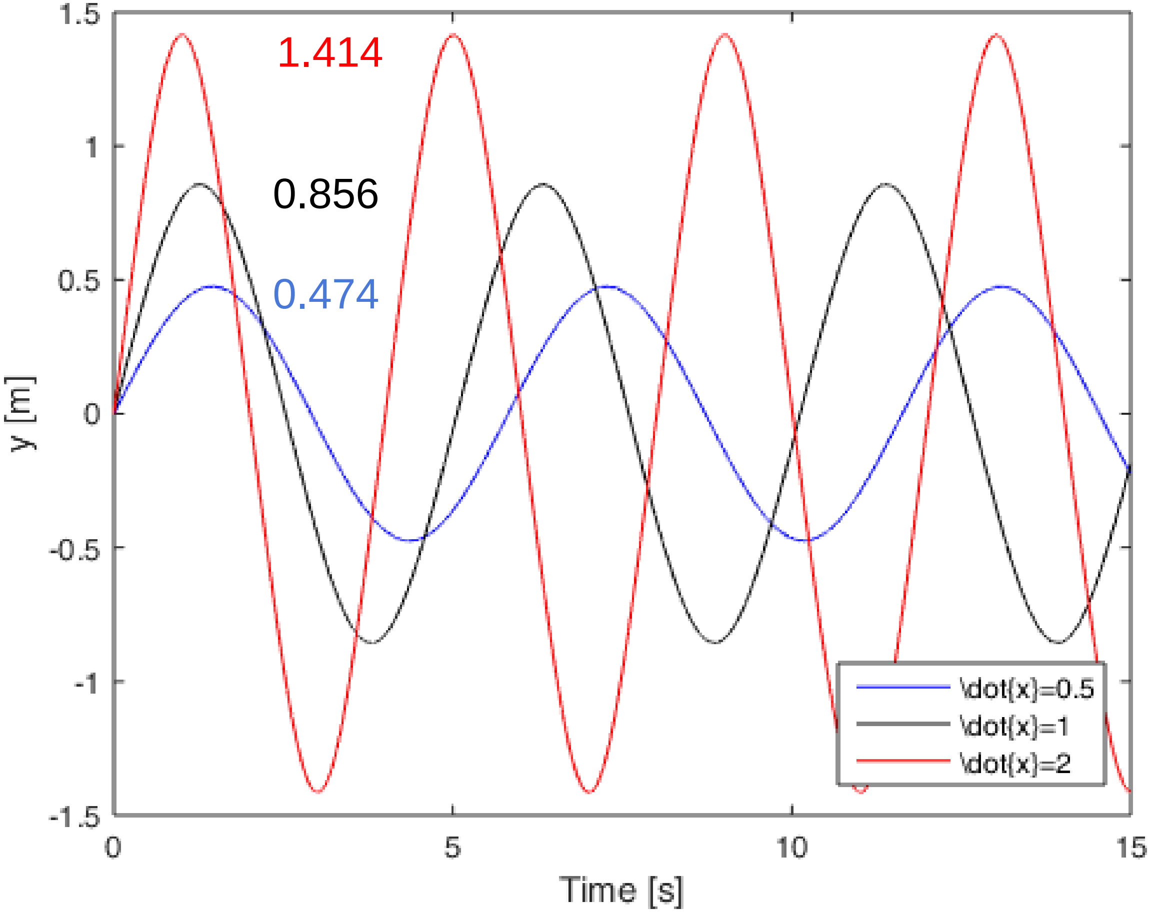
$$y(t) = \frac{\dot{y}_0}{\Omega} \mathrm{sn}\left(\Omega t | m\right)$$

$$\Omega = \sqrt{\frac{\omega_0^2 + \sqrt{\omega_0^4 + 2\alpha_3 \dot{y}_0^2}}{2}}$$

$$m = \frac{-\alpha_3 \dot{y}_0^2}{\omega_0^4 + \alpha_3 \dot{y}_0^2 + \omega_0^2 \sqrt{\omega_0^4 + 2\alpha_3 \dot{y}_0^2}}$$

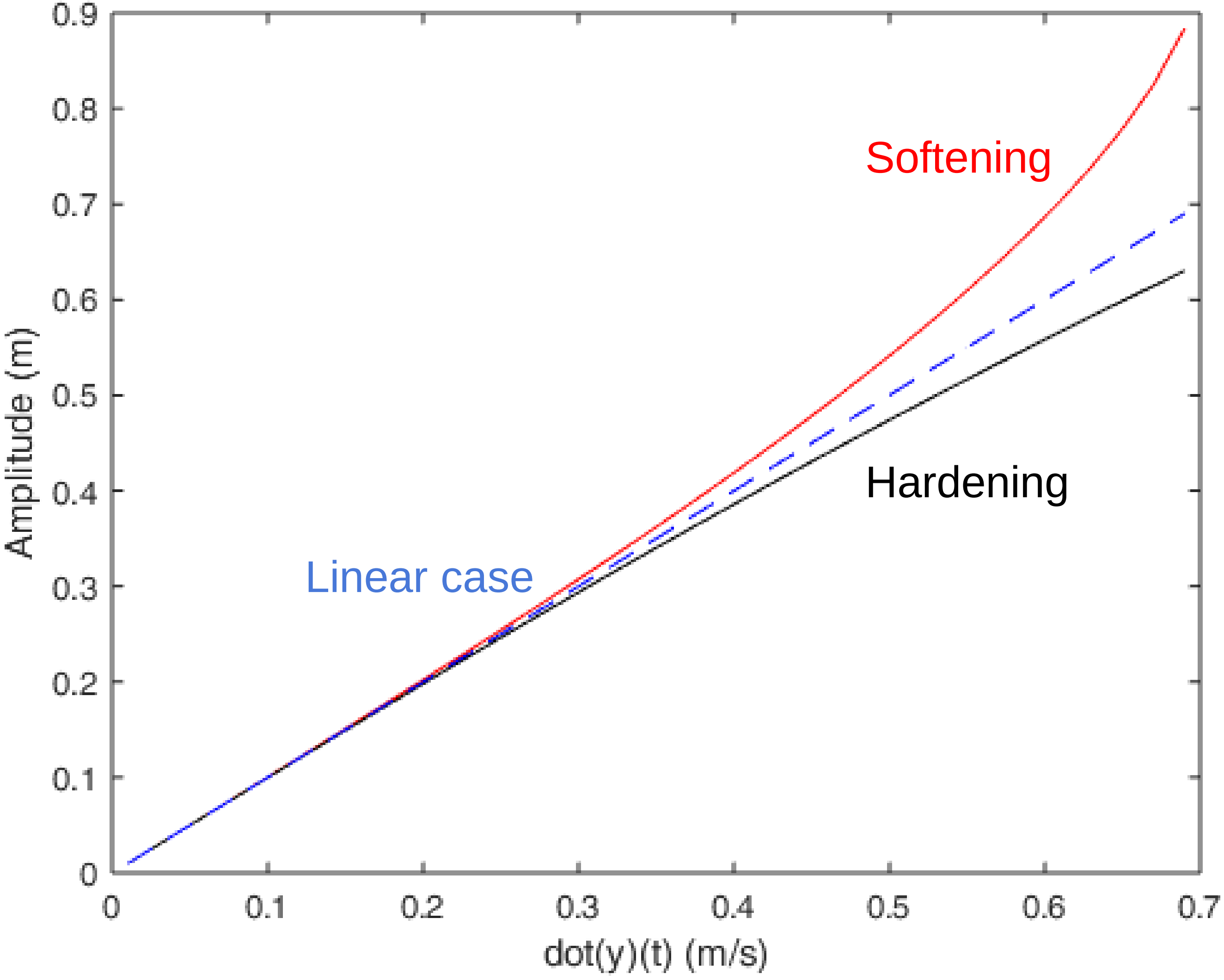
Response to an initial velocity

$$\ddot{y}(t) + y(t) + y^3(t) = 0$$



Failure of superposition principle

Beneficial or detrimental effect of nonlinearity



Free response: quite a number of lessons learned !

	LINEAR	NONLINEAR
$y(0) = y_0$	$y(t) = y_0 \cos(\omega_0 t)$ $\omega_0 = \sqrt{k/m}$	$y(t) = y_0 \,\text{cn}\,(\Omega t \| m)$ [1] [2] $\ddot{y}(t)$ [4] $\Omega = \sqrt{\omega_0^2 + \alpha_3 y_0^2}$ [3]
$\dot{y}(0) = \dot{y}_0$	$y(t) = \dfrac{\dot{y}_0}{\omega_0} \sin(\omega_0 t)$ $\omega_0 = \sqrt{k/m}$	$y(t) = \dfrac{\dot{y}_0}{\Omega} \text{sn}\,(\Omega t \| m)$ [2] [1] $\Omega = \sqrt{\dfrac{\omega_0^2 + \sqrt{\omega_0^4 + 2\alpha_3 \dot{y}_0^2}}{2}}$ [3]

1. The response is no longer purely harmonic
2. No superposition principle
3. Frequency-amplitude dependence: concept of backbone curve
4. Nonlinear systems generate harmonics

Jacobi elliptic functions: the way to go ?

Very useful for gaining insight into the dynamics of the undamped, unforced Duffing oscillator.

However, they have very limited applicability:

$$\ddot{y} + 2\xi\omega_0\dot{y}(t) + \omega_0^2 y(t) + \alpha_3 y^3(t) = 0$$

$$\ddot{y} + 2\xi\omega_0\dot{y}(t) + \omega_0^2 y(t) + \alpha_3 y^3(t) = f \sin \omega t$$

Elliptic functions do no longer represent the exact analytical solution (which does not exist anyway…).

Even if it will necessarily be an approximation, can you think of a simpler and more versatile mathematical function ?

Use a combination of sinusoids !

Idea at the root of the harmonic balance method (HBM).

Rationale: one can get a reasonably accurate approximation by keeping the first few terms (*truncated Fourier series*):

$$y(t) = c_0 + \sum_{k=1}^{N_H} (s_k \sin k\omega t + c_k \cos k\omega t) = \sum_{k=0}^{N_H} A_k \sin (k\omega t - \phi_k)$$

$$A_k = \sqrt{s_k^2 + c_k^2} \qquad \phi_k = \text{atan2}(-c_k, s_k)$$

Methodology:

1. *Substitute the approximation in the equations of motion;*
2. *Equate the coefficients associated with a specific harmonic;*
3. *Compute the unknowns by solving a nonlinear algebraic system with $2N_H$+1 equations.*

Harmonic balance as a Galerkin method

Weighted residual method

weight function

$$\int_0^T \boldsymbol{r}\left[\boldsymbol{q}_\mathrm{h}(t), \dot{\boldsymbol{q}}_\mathrm{h}(t)\right] \psi_k^*(t)\, \mathrm{d}t = \boldsymbol{0} \quad k = 0, 1, \ldots$$

Galerkin method: take ansatz functions as weight functions

In our case, the ansatz functions are $\psi_k^* = \mathrm{e}^{-\mathrm{i}k\Omega t}$. We thus obtain:

$$\int_0^T \boldsymbol{r}\left[\boldsymbol{q}_\mathrm{h}(t), \dot{\boldsymbol{q}}_\mathrm{h}(t)\right] \mathrm{e}^{-\mathrm{i}k\Omega t}\, \mathrm{d}t = \boldsymbol{0}$$

$$\int_0^T \Re\{\sum_{\ell=0}^{\infty} \boldsymbol{R}_\ell \mathrm{e}^{\mathrm{i}\ell\Omega t}\} \mathrm{e}^{-\mathrm{i}k\Omega t}\, \mathrm{d}t = \boldsymbol{0}$$

$$\int_0^{2\pi} \Re\{\sum_{\ell=0}^{\infty} \boldsymbol{R}_\ell \mathrm{e}^{\mathrm{i}\ell\tau}\} \mathrm{e}^{-\mathrm{i}k\tau}\, \mathrm{d}\tau = \boldsymbol{0}$$

with $\tau = \Omega t$

$$\int_0^{2\pi} \sum_{\ell=0}^{\infty} \left(\boldsymbol{R}_\ell \frac{\mathrm{e}^{\mathrm{i}\ell\tau}}{2} + \boldsymbol{R}_\ell^* \frac{\mathrm{e}^{-\mathrm{i}\ell\tau}}{2}\right) \mathrm{e}^{-\mathrm{i}k\tau}\, \mathrm{d}\tau = \boldsymbol{0}$$

$$\boldsymbol{R}_k = \boldsymbol{0}$$

since

$$\int_0^{2\pi} \mathrm{e}^{\mathrm{i}m\tau}\, \mathrm{d}\tau = \begin{cases} 2\pi & m = 0 \\ 0 & m \neq 0 \end{cases} \quad m \in \mathbb{Z}$$

So let's try…

$$\ddot{y}(t) + \omega_0^2 y(t) + \alpha_3 y^3(t) = 0$$

&

$$y(t) = A_1 \sin(\omega t - \phi_1)$$

$$\sin(3\theta) = -\sin^3\theta + 3\cos^2\theta\sin\theta$$
$$= -4\sin^3\theta + 3\sin\theta$$

$$-\omega^2 \sin(\omega t - \phi_1) + \omega_0^2 \sin(\omega t - \phi_1) + \frac{\alpha_3 A_1^2}{4}\left(3\sin(\omega t - \phi_1) - \sin(3\omega t - 3\phi_1)\right) = 0$$

Nonlinear systems generate harmonics

What are our
2 options at this stage ?

Option 1: we look for an approximation

$$-\omega^2 \sin(\omega t - \phi_1) + \omega_0^2 \sin(\omega t - \phi_1) + \frac{\alpha_3 A_1^2}{4}\left(3\sin(\omega t - \phi_1) - \cancel{\sin(3\omega t - 3\phi_1)}\right) = 0$$

$$\omega = \sqrt{\omega_0^2 + \frac{3\alpha_3}{4} A_1^2}$$

$$y(0) = -A_1 \sin\phi_1 = y_0$$

$$\dot{y}(0) = A_1 \omega \cos\phi_1 = 0$$

Initial displacement

$$y(t) = y_0 \cos\left(\sqrt{\omega_0^2 + \frac{3\alpha_3 y_0^2}{4}}\, t\right)$$

A 1-term HB approximation

Option 2: we enrich our assumption

Initial displacement

$$y(t) = A_1 \cos \omega t + A_3 \cos 3\omega t$$

$$y(0) = A_1 + A_3 = y_0$$

$$-\omega^2 A_1 \cos \omega t - 9\omega^2 A_3 \cos 3\omega t + \omega_0^2 A_1 \cos \omega t + \omega_0^2 A_3 \cos 3\omega t + ...$$
$$\alpha_3 \left(A_1^3 \cos^3 \omega t + 3A_1^2 A_3 \cos^2 \omega t \cos 3\omega t + 3A_1 A_3^2 \cos \omega t \cos^2 3\omega t + A_3^3 \cos^3 3\omega t \right) = 0$$

$$\cos^3 \omega t = \frac{3\cos \omega t + \cos 3\omega t}{4}$$

$$\cos^2 \omega t \cos 3\omega t = \frac{\cos 3\omega t}{2} + \frac{(\cos \omega t + \cos 5\omega t)}{4}$$

$$\cos \omega t \cos^2 3\omega t = \frac{\cos \omega t}{2} + \frac{(\cos 5\omega t + \cos 7\omega t)}{4}$$

$$\cos^3 3\omega t = \frac{3\cos 3\omega t + \cos 9\omega t}{4}$$

Even more harmonics !

A 2-term HB approximation

Neglecting harmonics 5, 7 and 9:

$$
\begin{aligned}
-\omega^2 + \omega_0^2 + \frac{3\alpha_3}{4}\left(A_1^2 + A_1 A_3 + 2A_3^2\right) &= 0 \\
(-9\omega^2 + \omega_0^2)A_3 + \frac{\alpha_3}{4}\left(A_1^3 + 6A_1^2 A_3 + 3A_3^3\right) &= 0 \\
y(0) = A_1 + A_3 &= y_0
\end{aligned}
$$

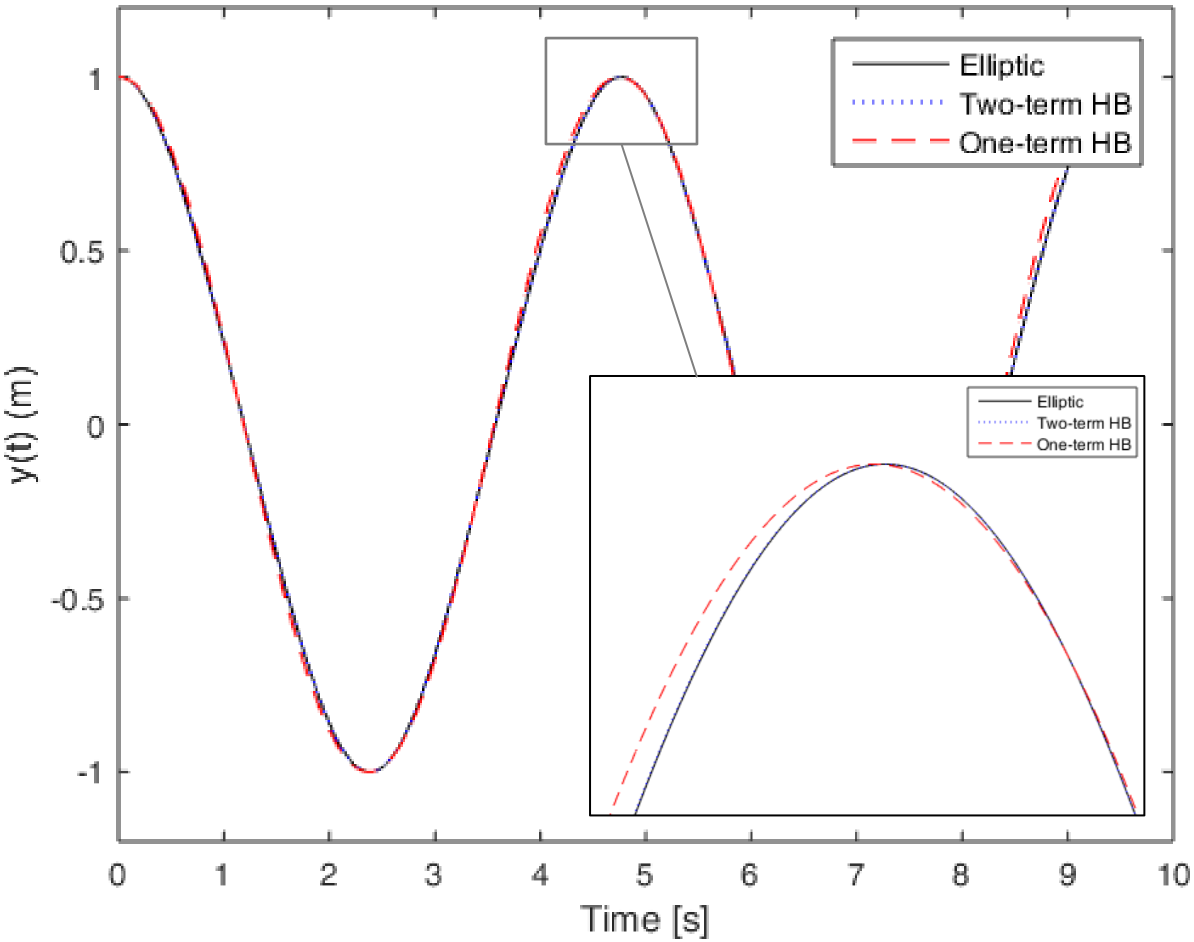
3 unknowns & 3 equations

$$
\frac{23\alpha_3}{2}A_1^3 - 30\alpha_3 y_0 A_1^2 + \left(8\omega_0^2 + \frac{63\alpha_3 y_0^2}{2}\right)A_1 - 8\omega_0^2 y_0 - \frac{51\alpha_3}{4}y_0^3 = 0
$$

This third-order polynomial can be solved in closed form but the analytical expression is lengthy.

Eventually…

$$\ddot{y}(t) + y(t) + y^3(t) = 0 \qquad y_0 = 1$$



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Enrich our assumption: an endless process

$$\ddot{y}(t) + \omega_0^2 y(t) + \alpha_3 y^3(t) = 0$$

$$y(t) = \sum_{k=1}^{\infty} A_{2k-1} \cos\left((2k-1)\omega t\right)$$

The response of a nonlinear system should be expressed as an infinite series of harmonics, highlighting the very rich frequency content of nonlinear oscillations.

16.23. Series Expansions in Terms of the Nome $q=e^{-\pi K'/K}$ and the Argument $v=\pi u/(2K)$

16.23.1 $$\operatorname{sn}(u|m)=\frac{2\pi}{m^{1/2}K}\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1-q^{2n+1}}\sin(2n+1)v$$

16.23.2 $$\operatorname{cn}(u|m)=\frac{2\pi}{m^{1/2}K}\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1+q^{2n+1}}\cos(2n+1)v$$

Indeed…

Advantages and drawbacks of HB

- Conceptual simplicity and physical insight
- Usually accurate with a few harmonics
- Efficient computational implementation of HBM (tomorrow)
- Filtering property and focus on steady-state response

- Analytical solutions only for very simple problems
- Transient and nonperiodic responses
- Less efficient for nonsmooth nonlinearies

Outline

Focus on a 1DOF oscillator

Linear vs. nonlinear

Undamped, unforced dynamics: linear vs. nonlinear

Damped, unforced dynamics: linear vs. nonlinear

Undamped/damped, harmonic forcing: linear vs. nonlinear

Going beyond…

Linear system: damped, unforced case

LINEAR DAMPING ?

$$\ddot{y}(t) + 2\xi\omega_0\dot{y}(t) + \omega_0^2 y(t) = 0, \quad \dot{y}(0) = \dot{y}_0,\ y(0) = y_0$$

$$\tan^{-1}\frac{\frac{\dot{y}}{y} + \xi\omega_0}{\omega_d} + \omega_d t = \phi$$

$$\omega_d = \omega_0\sqrt{1-\xi^2}$$

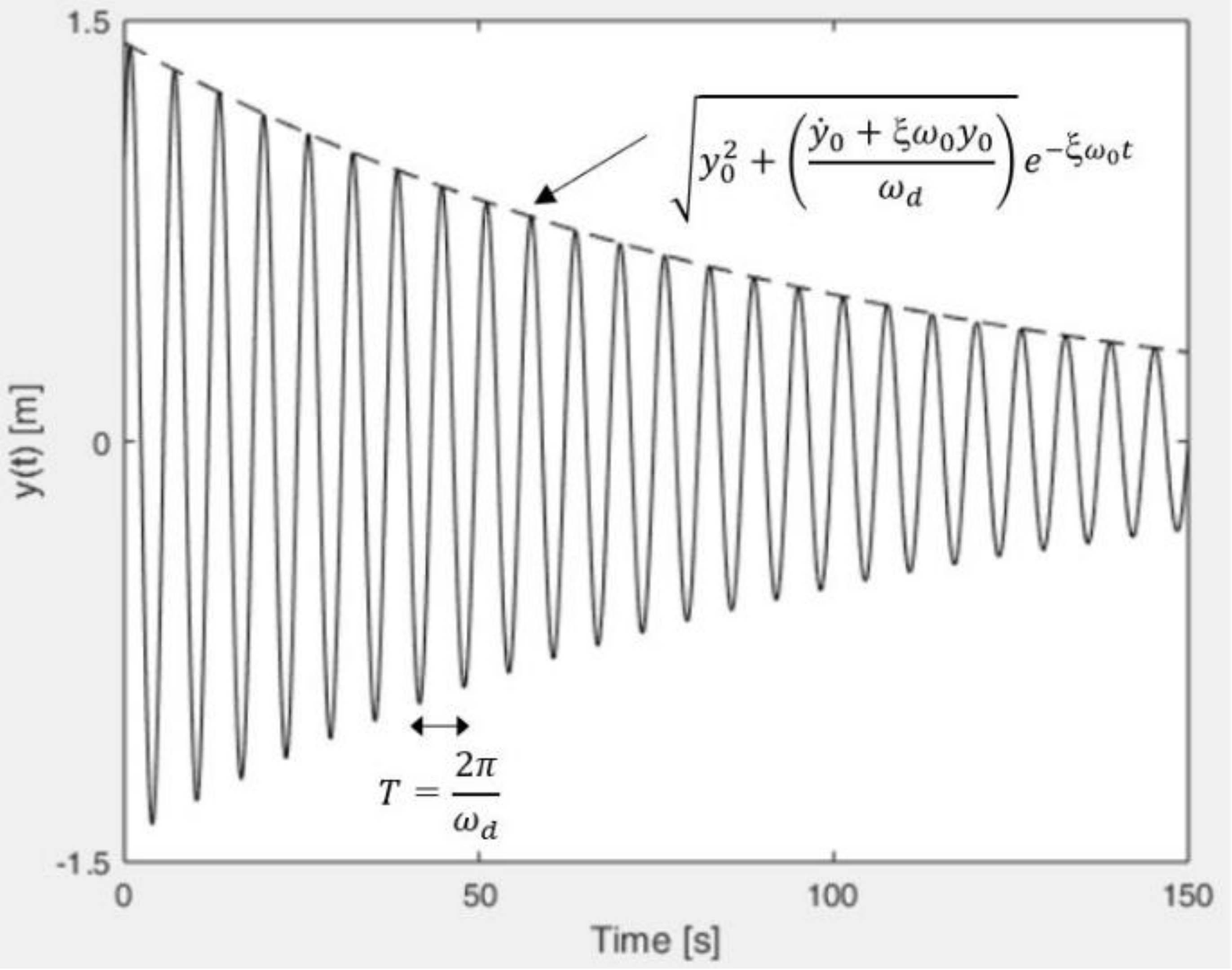
$$y(t) = Ye^{-\xi\omega_0 t}\cos\left(\omega_d t - \phi\right)$$

Find again a constant of motion called ϕ (no longer the energy !)

$$y(t) = \sqrt{y_0^2 + \left(\frac{\dot{y}_0 + \xi\omega_0 y_0}{\omega_d}\right)^2} e^{-\xi\omega_0 t} \sin\left(\omega_d t + \tan^{-1}\frac{y_0\omega_d}{\dot{y}_0 + \xi\omega_0 y_0}\right)$$

The response of a damped linear oscillator is a damped sine wave

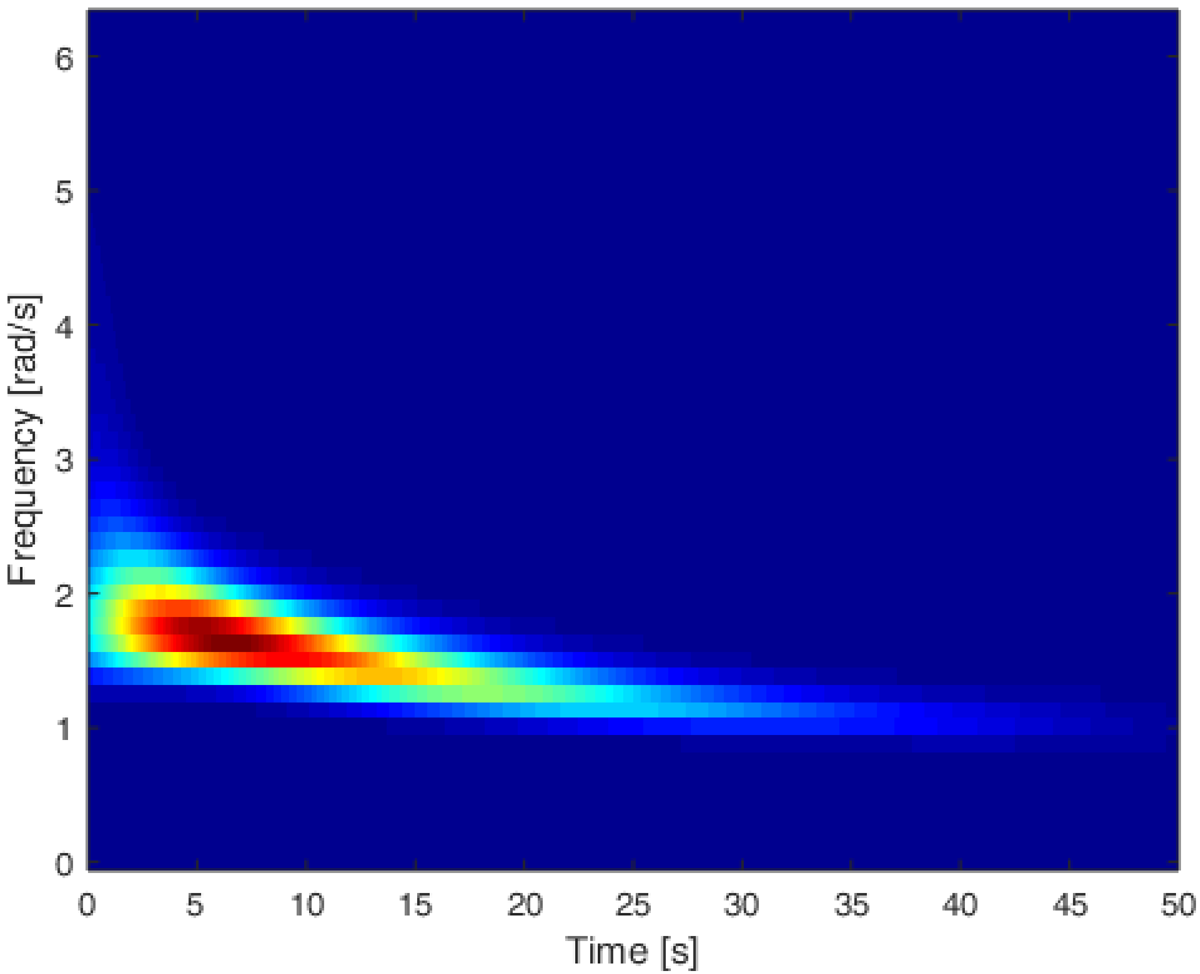
Damped, unforced case (linear system)



Can you guess the time series ?

$$\ddot{y} + 2\xi\omega_0\dot{y}(t) + \omega_0^2 y(t) + \alpha_3 y^3(t) = 0 \qquad \dot{y}(0) = 0\,,\ y(0) = y_0$$

Usefulness of time-frequency analysis



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A clear manifestation of the frequency-amplitude dependence of nonlinear ocillations