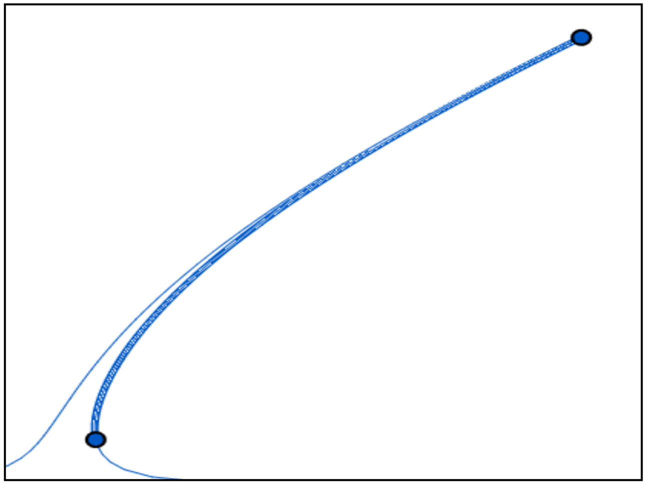
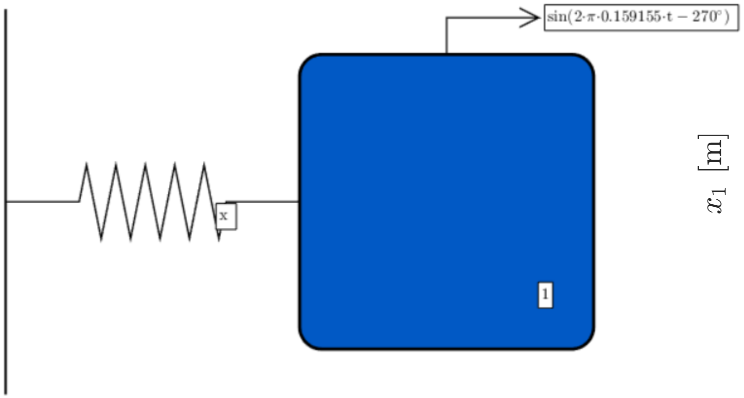
Formation sur les non-linéarités en dynamique des structures

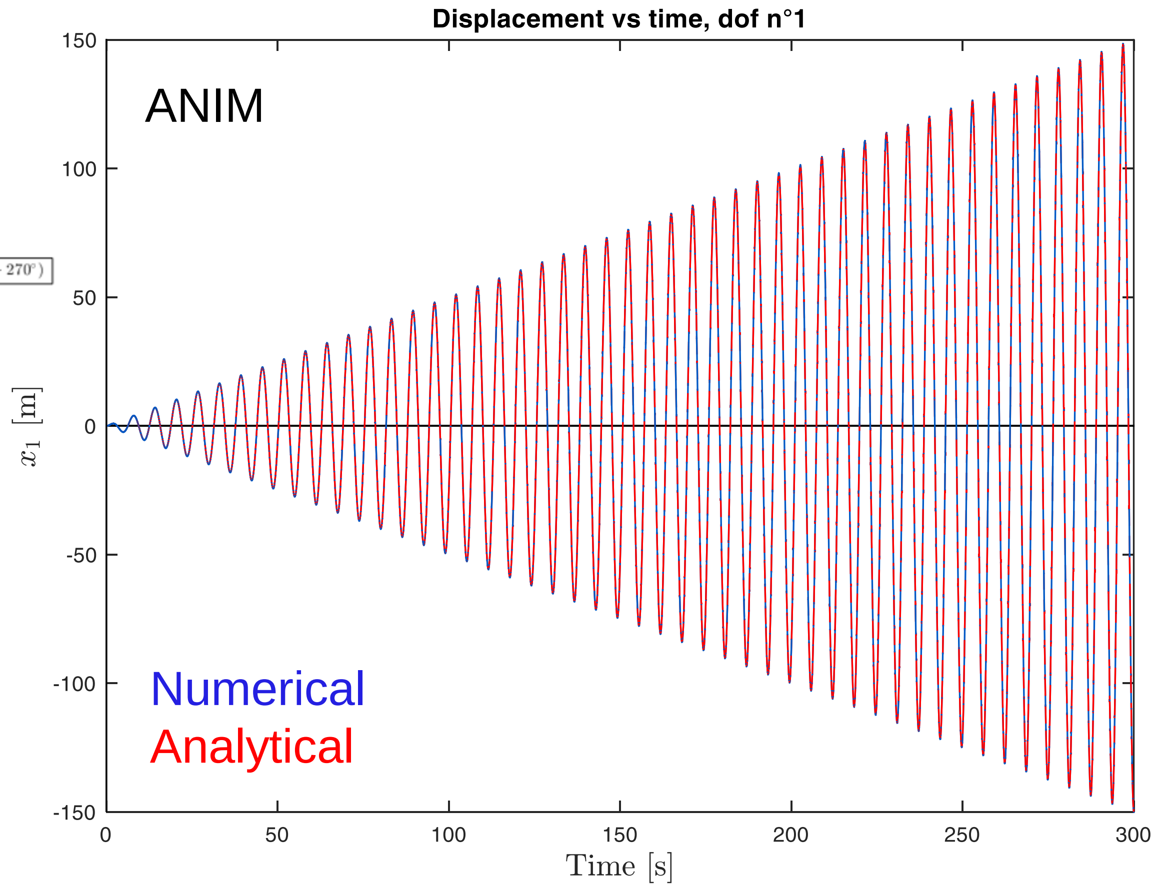
L02 Fundamental properties

- Review of linear dynamics
- Breakdown of linear properties & new phenomena
- Nonlinear FRCs



Resonance, a key concept in vibration theory

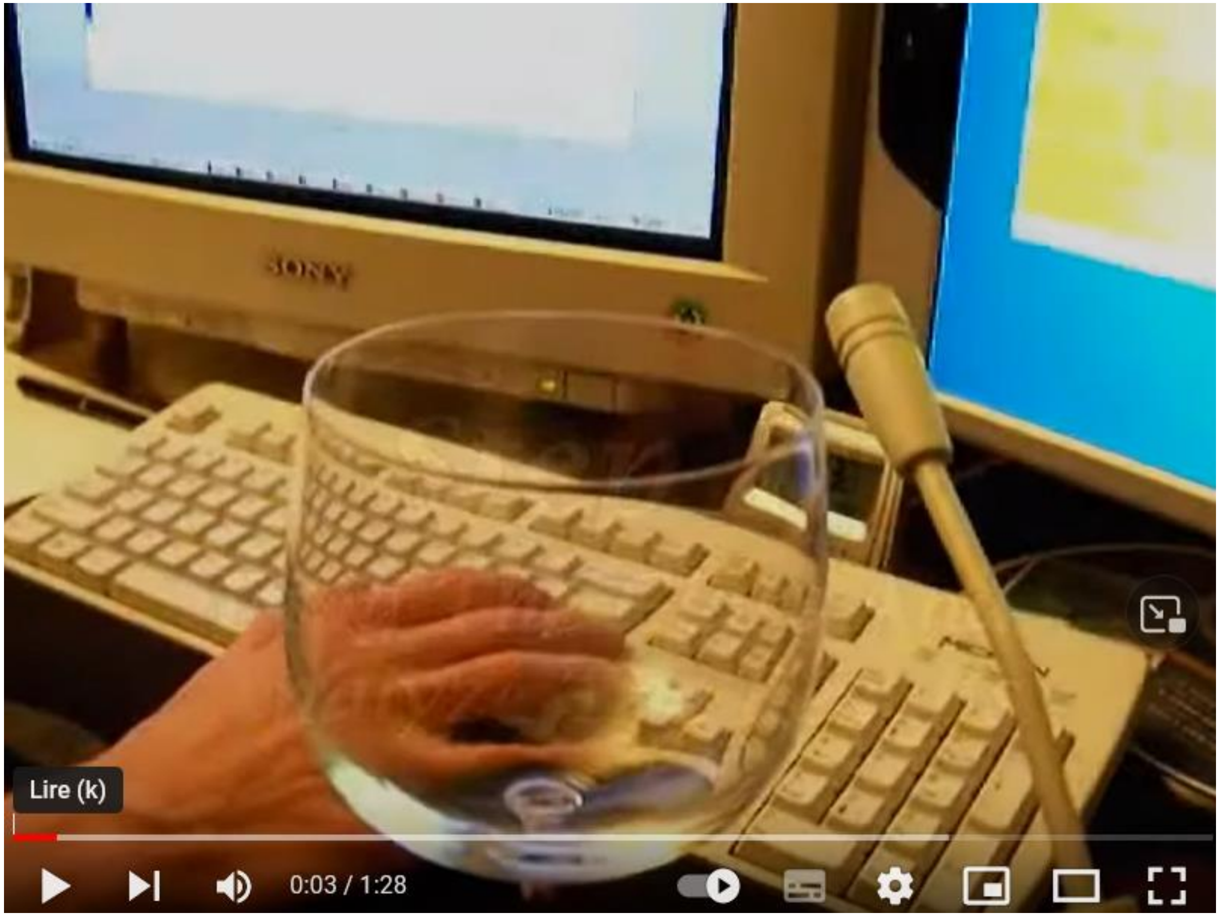




Resonance, a key concept in vibration theory

https://www.youtube.com/watch?v=10lWpHyN0Ok

Resonance, a key concept in vibration theory



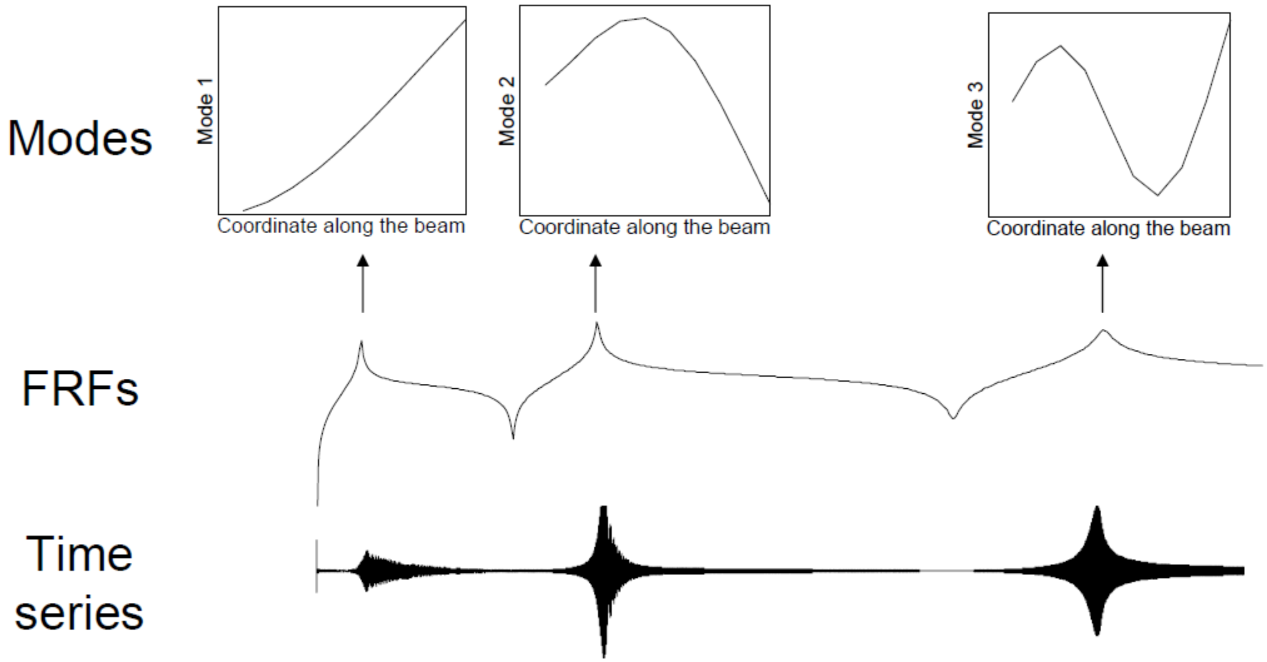
https://www.youtube.com/watch?v=JiM6AtNLXX4

The concept of a FRF

- Voir DSM
- Input/output

FRF: important properties

- The FRF is a constant system properties for a linear system
- FRF can be easily estimated from measured data
- Very convenient way of locating resonance frequencies



Outline

Focus on a 1DOF oscillator

Linear vs. nonlinear

Undamped, unforced dynamics: linear vs. nonlinear

Damped, unforced dynamics: linear vs. nonlinear

Undamped/damped, harmonic forcing: linear vs. nonlinear

Going beyond…

Linear system: damped, harmonic forcing

$$\ddot{y}(t) + 2\xi\omega_0\dot{y}(t) + \omega_0^2 y(t) = f \sin \omega t, \quad \dot{y}(0) = \dot{y}_0, \ y(0) = y_0$$

Superposition principle

$$y(t) = y_h(t) + y_p(t)$$

Interest in the steady state response

$$y_p(t) = \frac{f}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\xi\omega_0\omega)^2}} \sin\left(\omega t - \tan^{-1}\left(\frac{2\xi\omega_0\omega}{\omega_0^2 - \omega^2}\right)\right)$$

1. *As $\omega \rightarrow \omega_0$, the steady-state response amplitude gets very large (or infinite) even for small forcing amplitudes. This phenomenon is known as resonance.*

2. *The steady-state response does not depend on the initial conditions.*

3. *Evidence of the principle of superposition (y_p scales linearly with f)*

The FRF concept

$$\ddot{y}(t) + 2\xi\omega_0\dot{y}(t) + \omega_0^2 y(t) = f e^{i\omega t}$$

$$y_p(t) = Y_p e^{i\omega t}$$

$$Y_p = \frac{f}{\omega_0^2 - \omega^2 + 2i\xi\omega_0\omega}$$

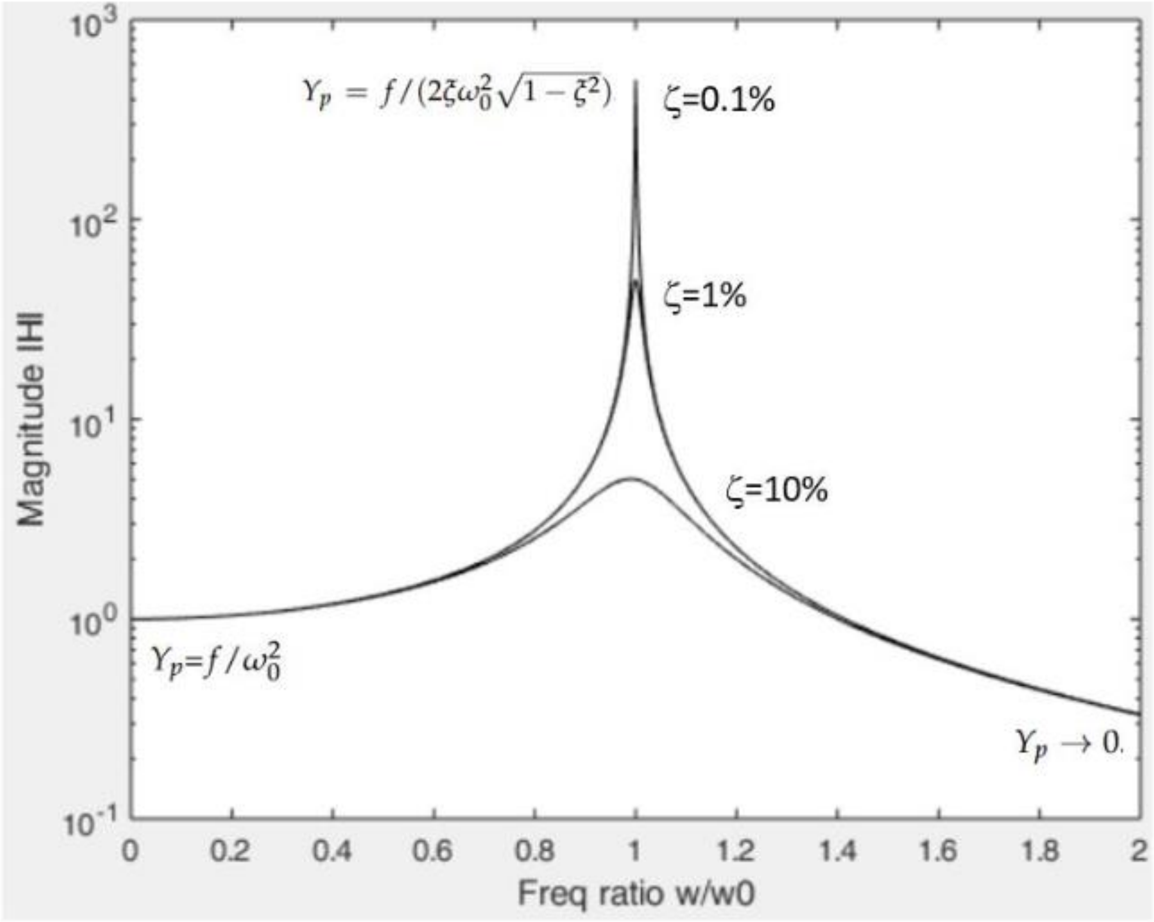
$$H(\omega) = \frac{Y_p}{F} = \frac{Y_p}{fm} = \frac{1}{-m\omega^2 + ic\omega + k}$$

FRF

$$= \frac{1}{k}\frac{1}{1 - \omega^2/\omega_0^2 + 2i\xi\omega/\omega_0}$$

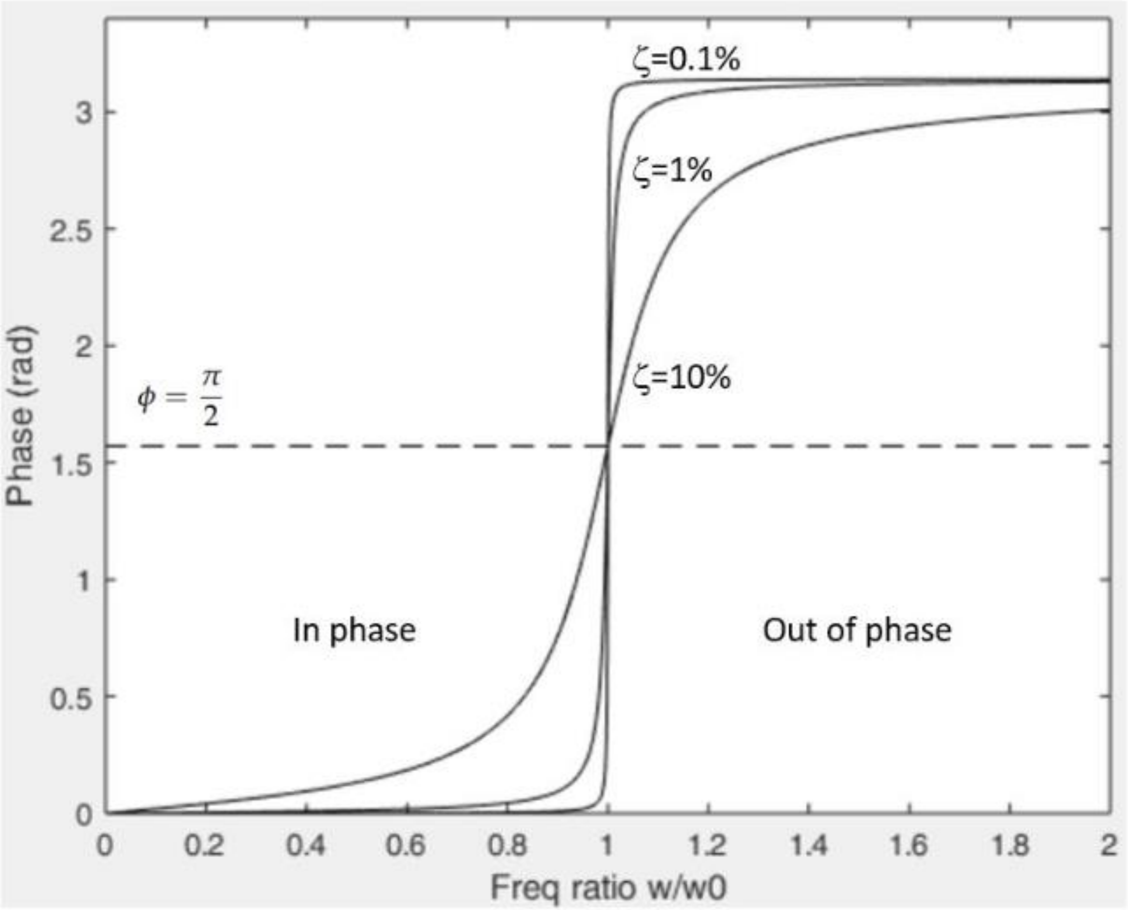
The Bode plots: amplitude

$$|H(\omega)| = \frac{1}{k}\frac{1}{\sqrt{\left(1-\omega^2/\omega_0^2\right)^2 + \left(2\xi\omega/\omega_0\right)^2}}$$



The Bode plots: phase

$$\phi = \tan^{-1} \frac{2\xi\omega/\omega_0}{1-\omega^2/\omega_0^2}$$



Illustration

https://www.youtube.com/watch?v=cfKwnTfNhog

The undamped, harmonically-forced Duffing oscillator

$$\ddot{y}(t) + \omega_0^2 y(t) + \alpha_3 y^3(t) = f \sin \omega t$$

Absence of damping (phase is trivial)

$$y(t) = A \sin \omega t$$

To be determined

$$(\omega_0^2 - \omega^2) A \sin \omega t + \frac{\alpha_3}{4} A^3 (3 \sin \omega t - \sin 3\omega t) = f \sin \omega t$$

Nonlinear systems generate harmonics

What are our
2 options at this stage ?

A one-term harmonic balance approximation

$$\frac{3\alpha_3}{4}A^3 + (\omega_0^2 - \omega^2)A - f = 0$$

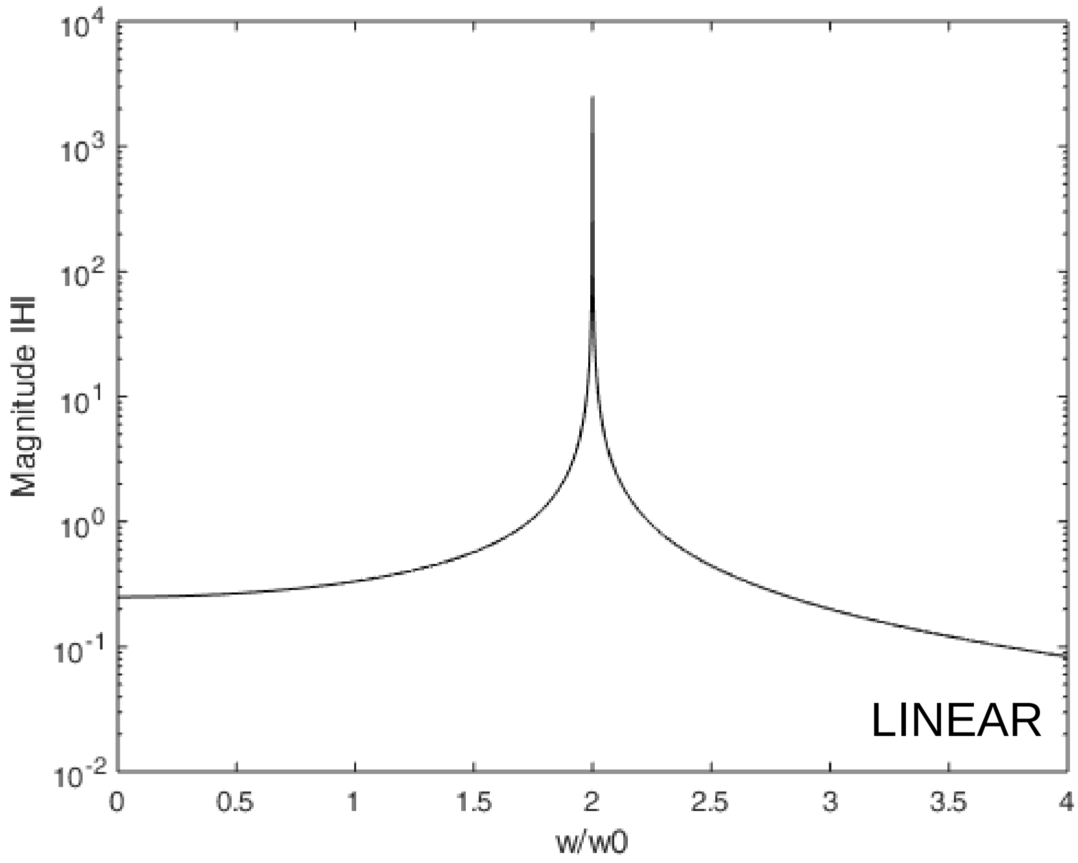
What are the possible roots for a 3rd order polynomial?

One real root
2 complex conjugates

Three real roots

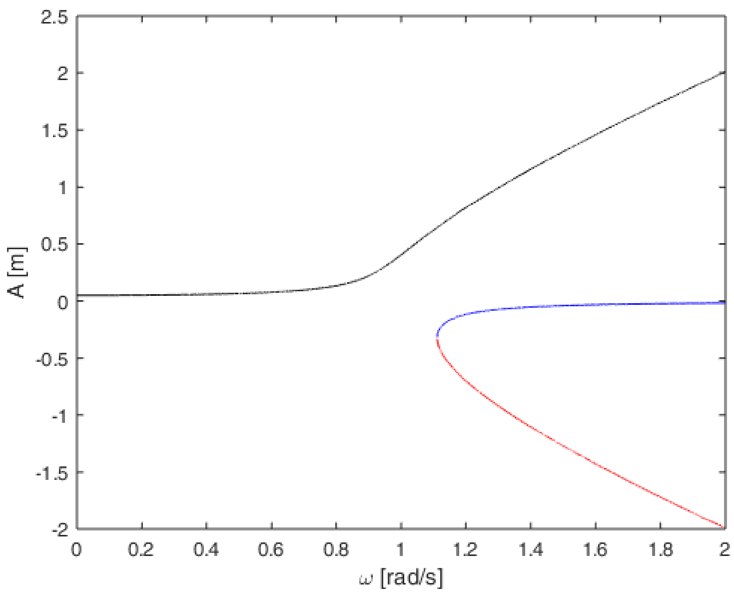
Nonlinear systems undergo bifurcations

Can you guess the nonlinear FRF ? Draw it !

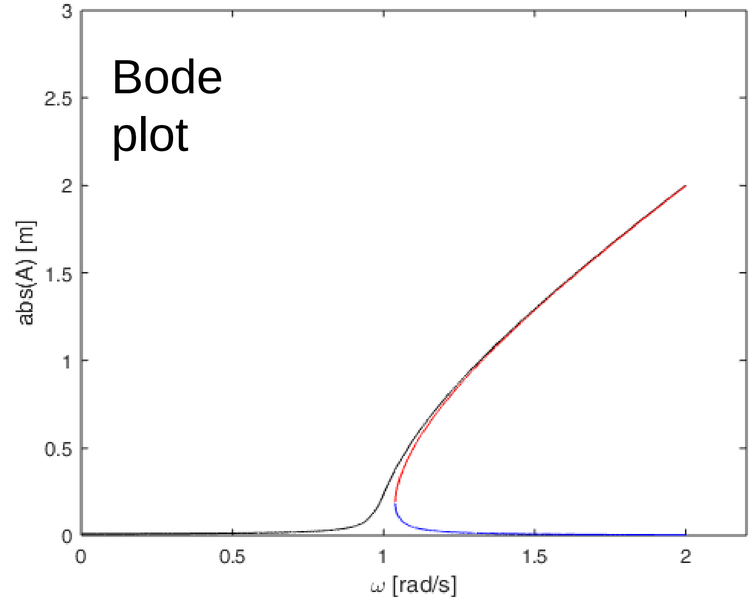


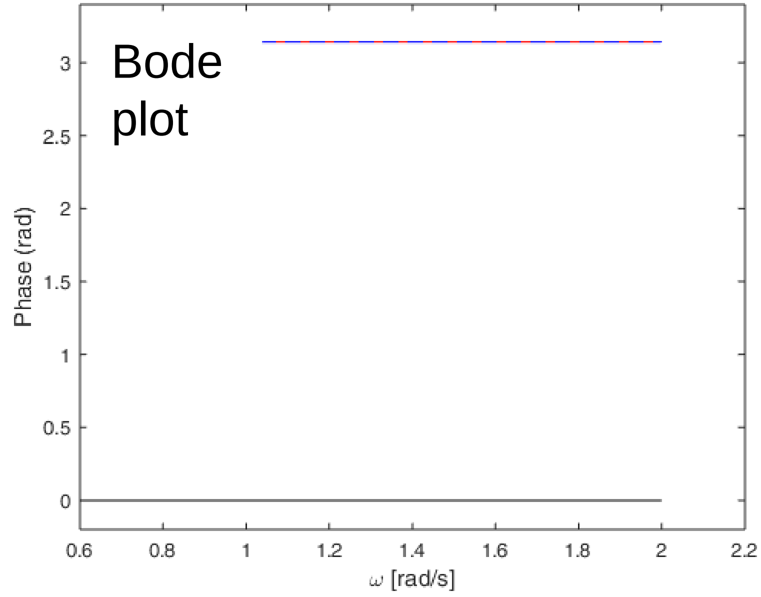
The nonlinear frequency response: hardening

$$\ddot{y}(t) + y(t) + y^3(t) = 0.01 \sin \omega t$$



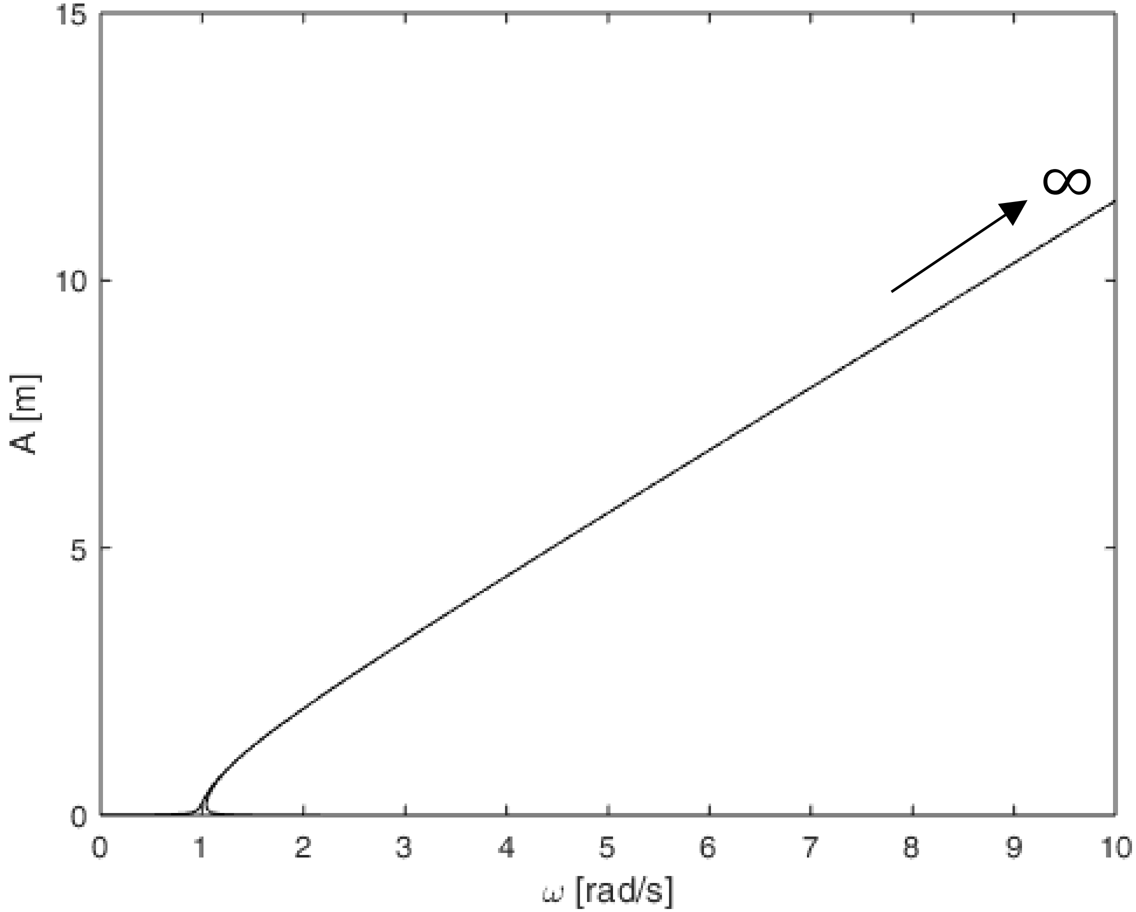
Roots of the third-order polynomial



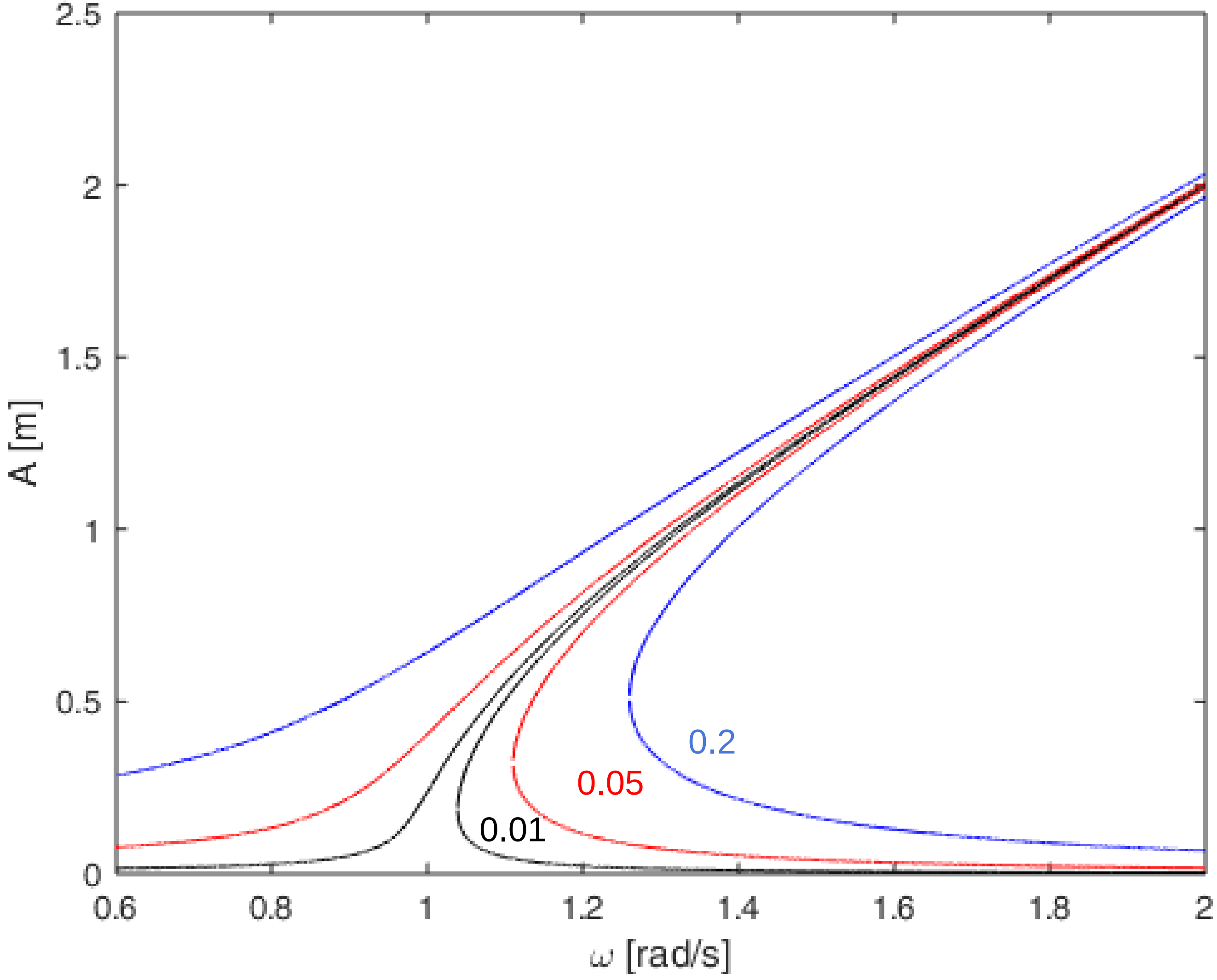


The resonance frequency really goes to infinity

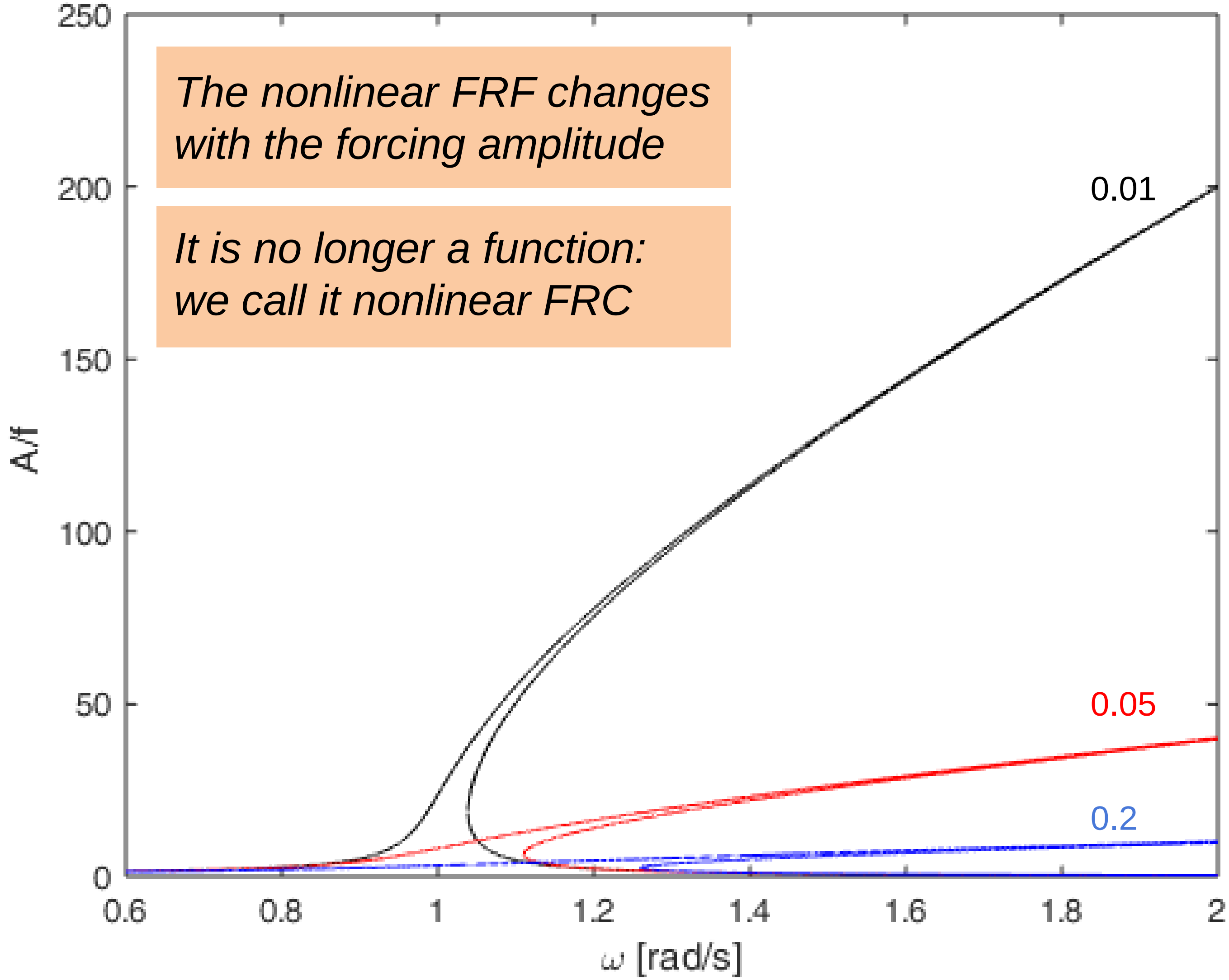
$$\ddot{y}(t) + y(t) + y^3(t) = 0.01 \sin \omega t$$



Increasing forcing amplitudes

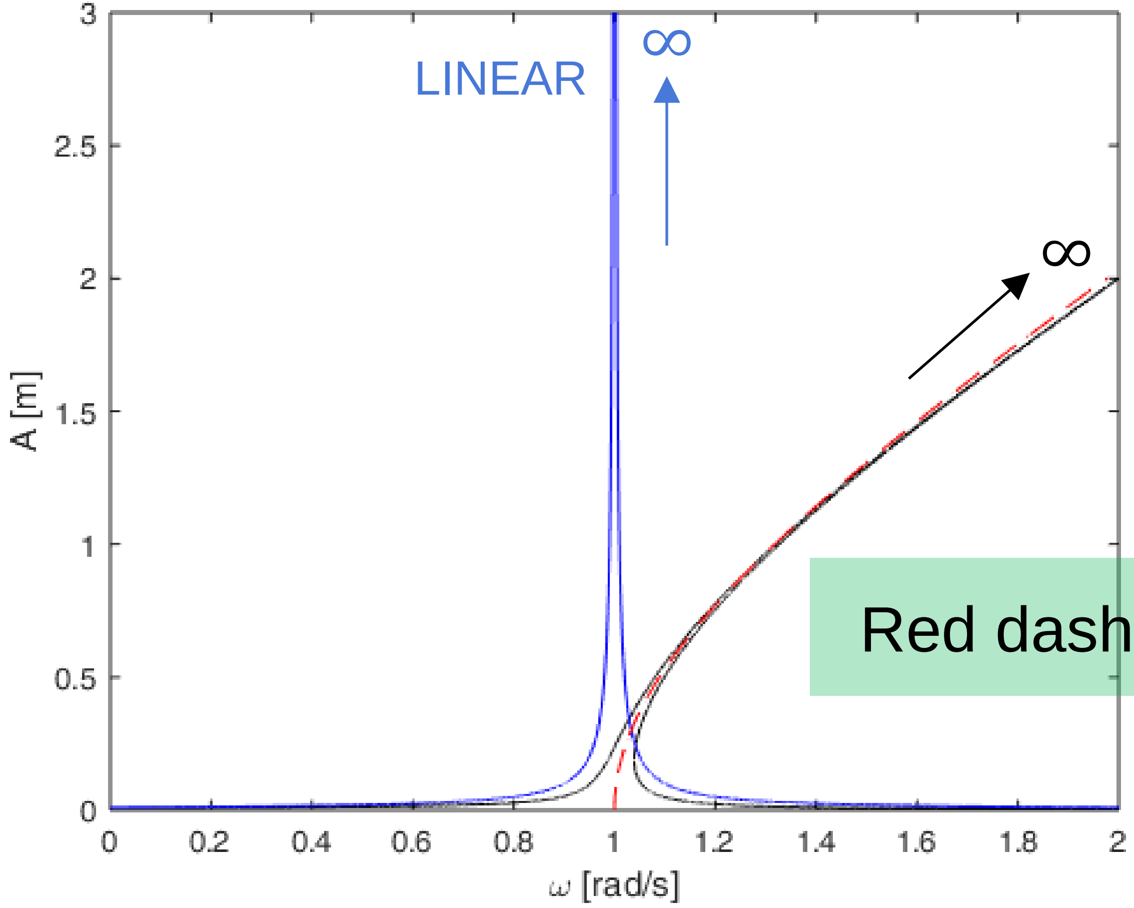


Nonlinear FRF ? Divide by the forcing amplitude



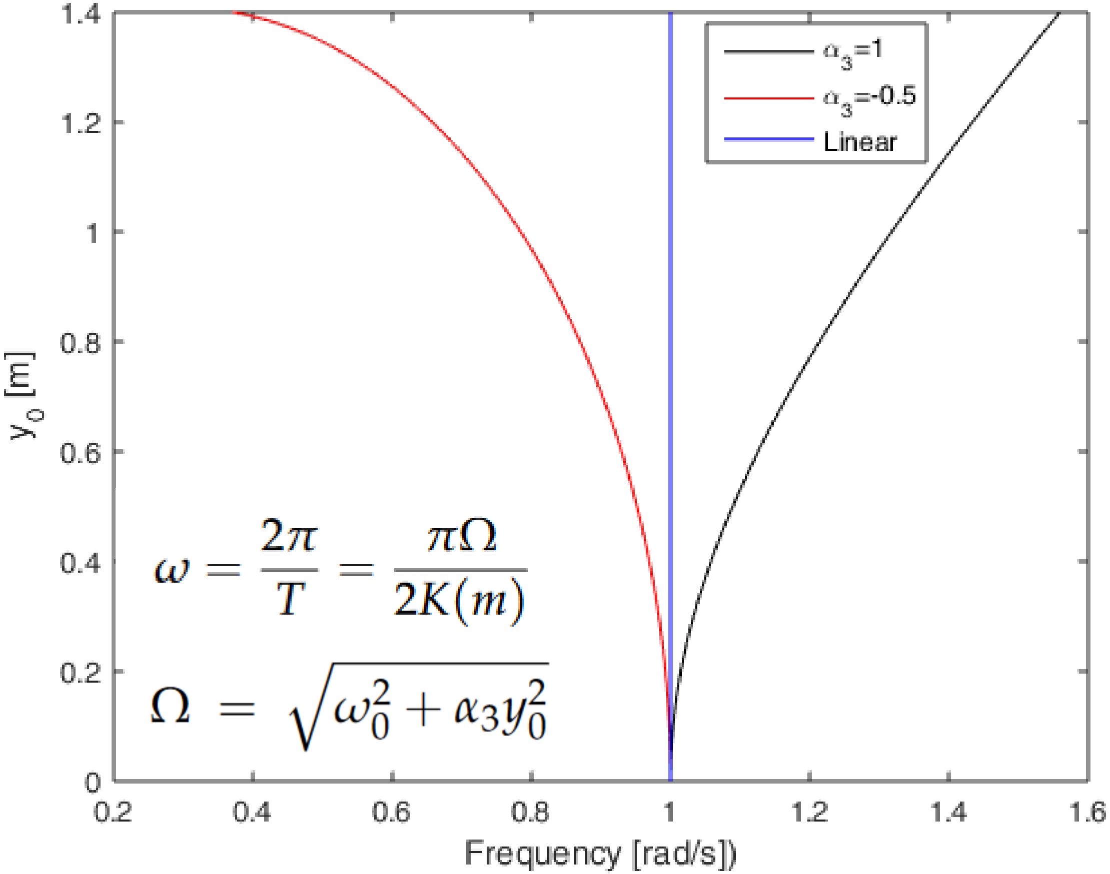
Comparison with the linear case

$$\ddot{y}(t) + y(t) + y^3(t) = 0.01 \sin \omega t$$



Red dashed curve ?

The natural frequency of the Duffing oscillator



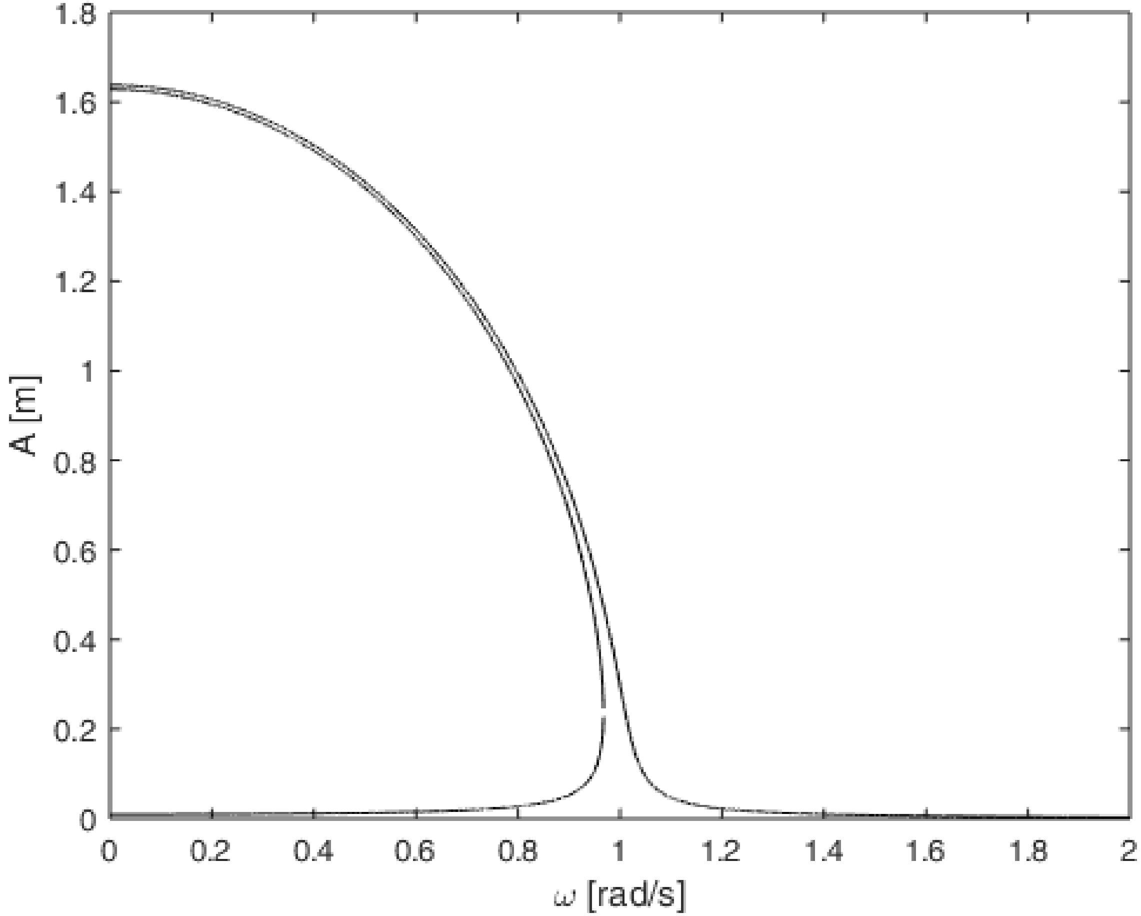
$$\omega = \frac{2\pi}{T} = \frac{\pi\Omega}{2K(m)}$$

$$\Omega = \sqrt{\omega_0^2 + \alpha_3 y_0^2}$$

The backbone curve is the backbone of the nonlinear FRCs (hence its name !)

The nonlinear frequency response: softening

$$\ddot{y}(t) + y(t) - y^3(t) = 0.01 \sin \omega t$$



The damped, harmonically-forced Duffing oscillator

$$\ddot{y}(t) + 2\xi\omega_0\dot{y}(t) + \omega_0^2 y(t) + \alpha_3 y^3(t) = f \sin\omega t$$

$$y(t) = A \sin\omega t + B \cos\omega t$$

To be determined

...

$$\ddot{q} + \delta\dot{q} + q + \gamma q^3 = P\cos(\Omega t) \qquad q(t) \approx q_\mathrm{h}(t) = Q_\mathrm{c}\cos(\Omega t) + Q_\mathrm{s}\sin(\Omega t)$$

MALTE KRACK (STUTTGART)

Time derivatives of ansatz:

$$\begin{aligned} q_\mathrm{h} &= +Q_\mathrm{c}\cos(\Omega t) + Q_\mathrm{s}\sin(\Omega t) \\ \dot{q}_\mathrm{h} &= -Q_\mathrm{c}\Omega\sin(\Omega t) + Q_\mathrm{s}\Omega\cos(\Omega t) \\ \ddot{q}_\mathrm{h} &= -Q_\mathrm{c}\Omega^2\cos(\Omega t) - Q_\mathrm{s}\Omega^2\sin(\Omega t) \end{aligned}$$

With trigonometric identities

$$\begin{aligned} \cos^3 x &= \tfrac{3}{4}\cos x + \tfrac{1}{4}\cos 3x \\ \cos^2 x\ \sin x &= \tfrac{1}{4}\sin x + \tfrac{1}{4}\sin 3x \\ \cos x\ \sin^2 x &= \tfrac{1}{4}\cos x - \tfrac{1}{4}\cos 3x \\ \sin^3 x &= \tfrac{3}{4}\sin x - \tfrac{1}{4}\sin 3x \end{aligned}$$

Expansion of nonlinear term

$$\begin{aligned} q_\mathrm{h}^3 &= \left(Q_\mathrm{c}\cos(\Omega t) + Q_\mathrm{s}\sin(\Omega t)\right)^3 \\ &= Q_\mathrm{c}^3\cos^3(\Omega t) + 3Q_\mathrm{c}^2Q_\mathrm{s}\cos^2(\Omega t)\sin(\Omega t) + 3Q_\mathrm{c}Q_\mathrm{s}^2\cos(\Omega t)\sin^2(\Omega t) + Q_\mathrm{s}^3\sin^3(\Omega t) \\ &= \tfrac{3}{4}\left(Q_\mathrm{c}^3 + Q_\mathrm{c}Q_\mathrm{s}^2\right)\cos(\Omega t) + \tfrac{3}{4}\left(Q_\mathrm{s}^3 + Q_\mathrm{c}^3Q_\mathrm{s}\right)\sin(\Omega t) + (\ldots)\cos(3\Omega t) + (\ldots)\sin(3\Omega t) \end{aligned}$$

Substitute into Duffing equation and collect harmonics

$$\left[\left(1-\Omega^2\right)Q_\mathrm{c} + \delta\Omega Q_\mathrm{s} + \frac{3}{4}\gamma\left(Q_\mathrm{c}^3 + Q_\mathrm{c}Q_\mathrm{s}^2\right) - P\right]\cos(\Omega t)$$
$$+\left[\left(1-\Omega^2\right)Q_\mathrm{s} - \delta\Omega Q_\mathrm{c} + \frac{3}{4}\gamma\left(Q_\mathrm{s}^3 + Q_\mathrm{c}^2Q_\mathrm{s}\right)\right]\sin(\Omega t) + [\ldots]\cos(3\Omega t) + [\ldots]\sin(3\Omega t) = 0$$

We neglect harmonics with index higher than the ansatz (>1) and balance the harmonics:

$$\begin{aligned} R_\mathrm{c} &:= \left(1-\Omega^2\right)Q_\mathrm{c} + \delta\Omega Q_\mathrm{s} + \frac{3}{4}\gamma\left(Q_\mathrm{c}^3 + Q_\mathrm{c}Q_\mathrm{s}^2\right) - P = 0 \\ R_\mathrm{s} &:= \left(1-\Omega^2\right)Q_\mathrm{s} - \delta\Omega Q_\mathrm{c} + \frac{3}{4}\gamma\left(Q_\mathrm{s}^3 + Q_\mathrm{c}^2Q_\mathrm{s}\right) = 0 \end{aligned}$$

2 algebraic equations $R_\mathrm{c}, R_\mathrm{s}$ in 2 unknowns $Q_\mathrm{c}, Q_\mathrm{s}$

Transform to polar coordinates

$$Q_\mathrm{c} = a\cos\theta$$
$$Q_\mathrm{s} = a\sin\theta$$

$\Rightarrow$ $Q_\mathrm{c}^2 + Q_\mathrm{s}^2 = a^2$

Substitution into algebraic equations

$$\begin{aligned}
\left(1-\Omega^2\right) a\cos\theta + \delta\Omega a\sin\theta + \frac{3}{4}\gamma\left(a^3\cos^3\theta + a^3\cos\theta\sin^2\theta\right) &= P \qquad (1)\\
\left(1-\Omega^2\right) a\sin\theta - \delta\Omega a\cos\theta + \frac{3}{4}\gamma\left(a^3\sin^3\theta + a^3\cos^2\theta\sin\theta\right) &= 0 \qquad (2)
\end{aligned}$$

Algebraic manipulations of Eq. (1)-(2)

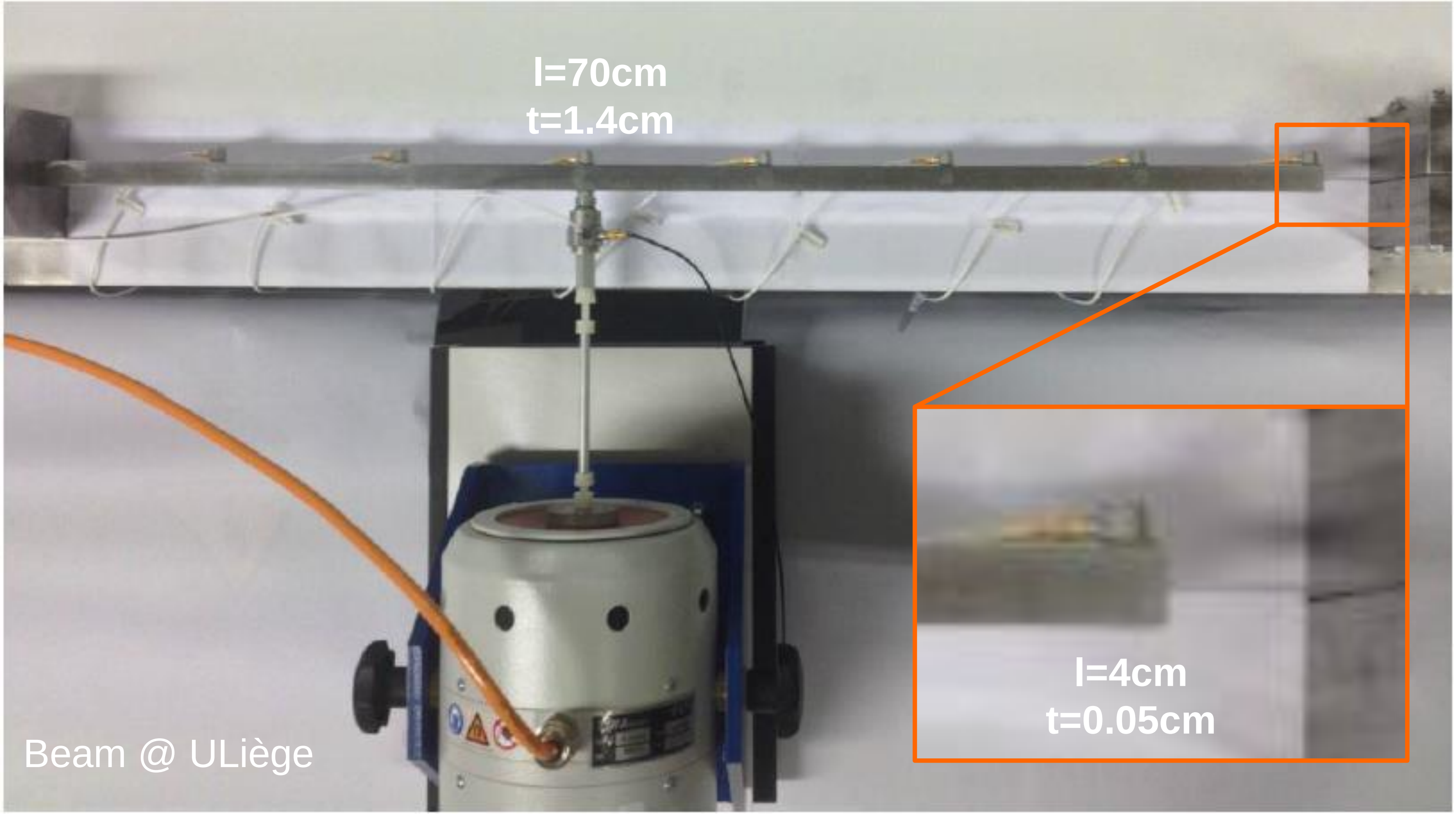
$$\begin{aligned}
\left(1-\Omega^2\right) a + \frac{3}{4}\gamma a^3 &= P\cos\theta \qquad (3)\\
\delta\Omega a &= P\sin\theta \qquad (4)\\
\left[1-\Omega^2+\frac{3}{4}\gamma a^2\right]^2 a^2 + \delta^2\Omega^2 a^2 &= P^2 \qquad (5)\\
\sin\theta &= \frac{\delta\Omega a}{P} \qquad (6)
\end{aligned}$$

It is easier to solve Eq. (5) for Ω:

$$\Omega_{1,2}^2 = 1 - \frac{\delta^2}{2} + \frac{3\gamma a^2}{4} \pm \sqrt{\frac{P^2}{a^2} + \frac{\delta^4}{4} - \delta^2 - \frac{3\delta^2\gamma a^2}{4}}$$

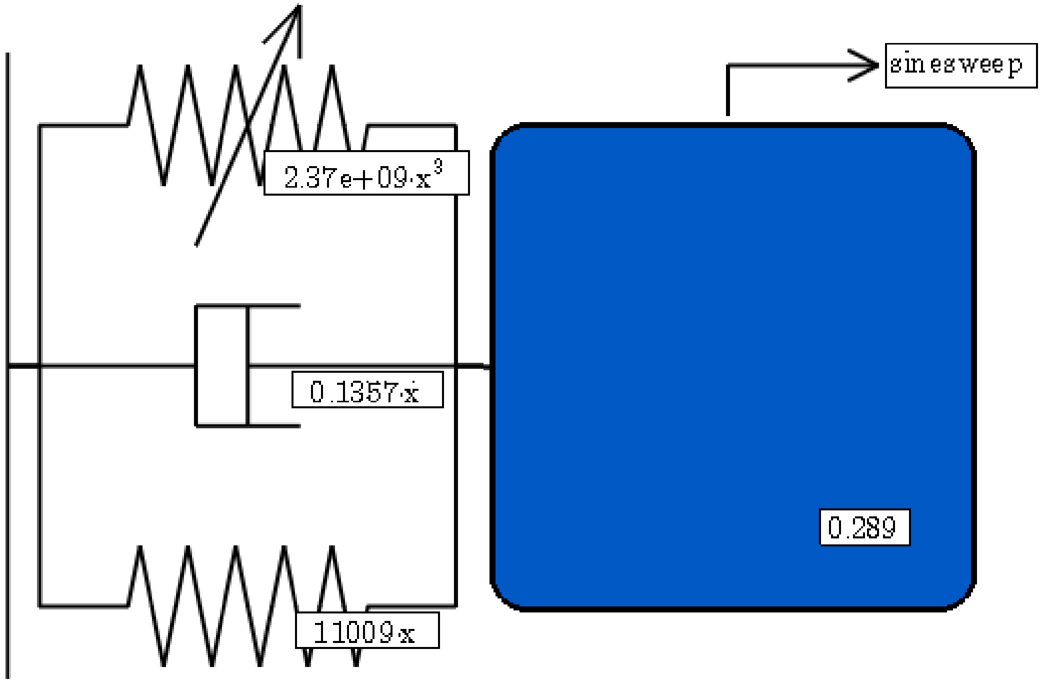
We can have zero, one or two real-valued solutions $\Omega_{1,2}^2$.

Cantilever beam with a very thin beam at the tip

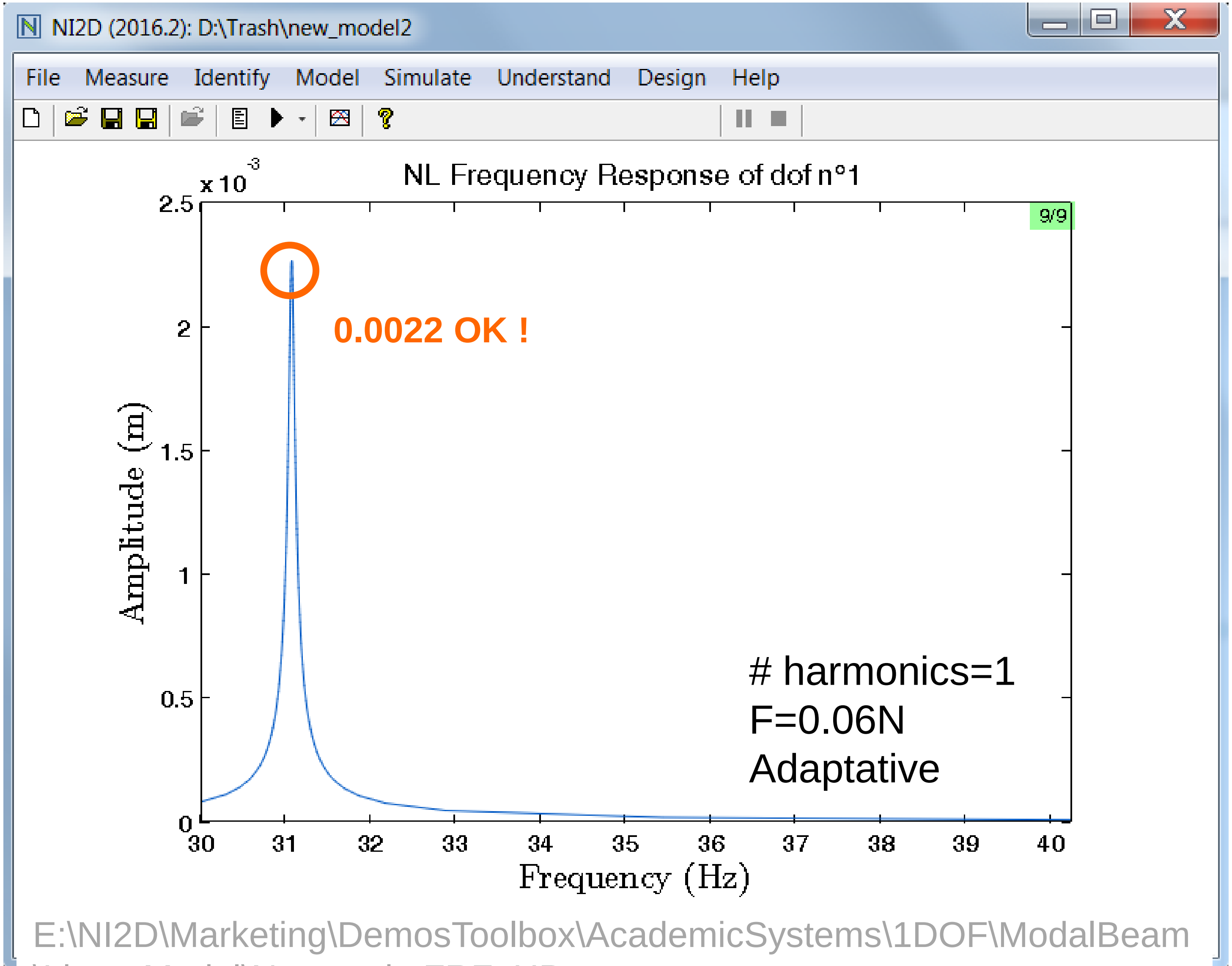


A 1DOF model of the first beam mode

$$0.289\ddot{x} + 0.1357\dot{x} + 11009x + 2.37.10^9\, x^3 = Fsin\omega t$$

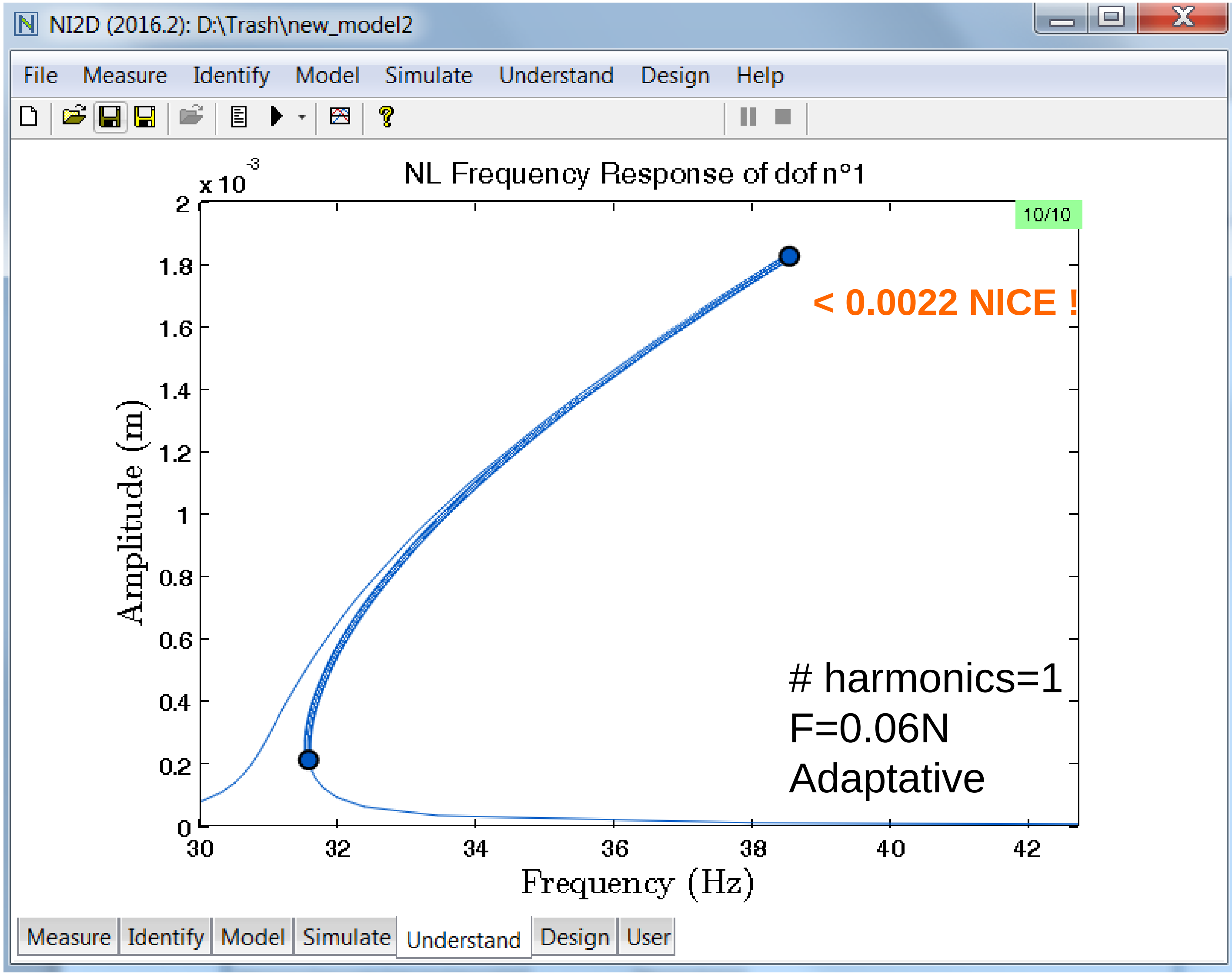


HBM for the linear system

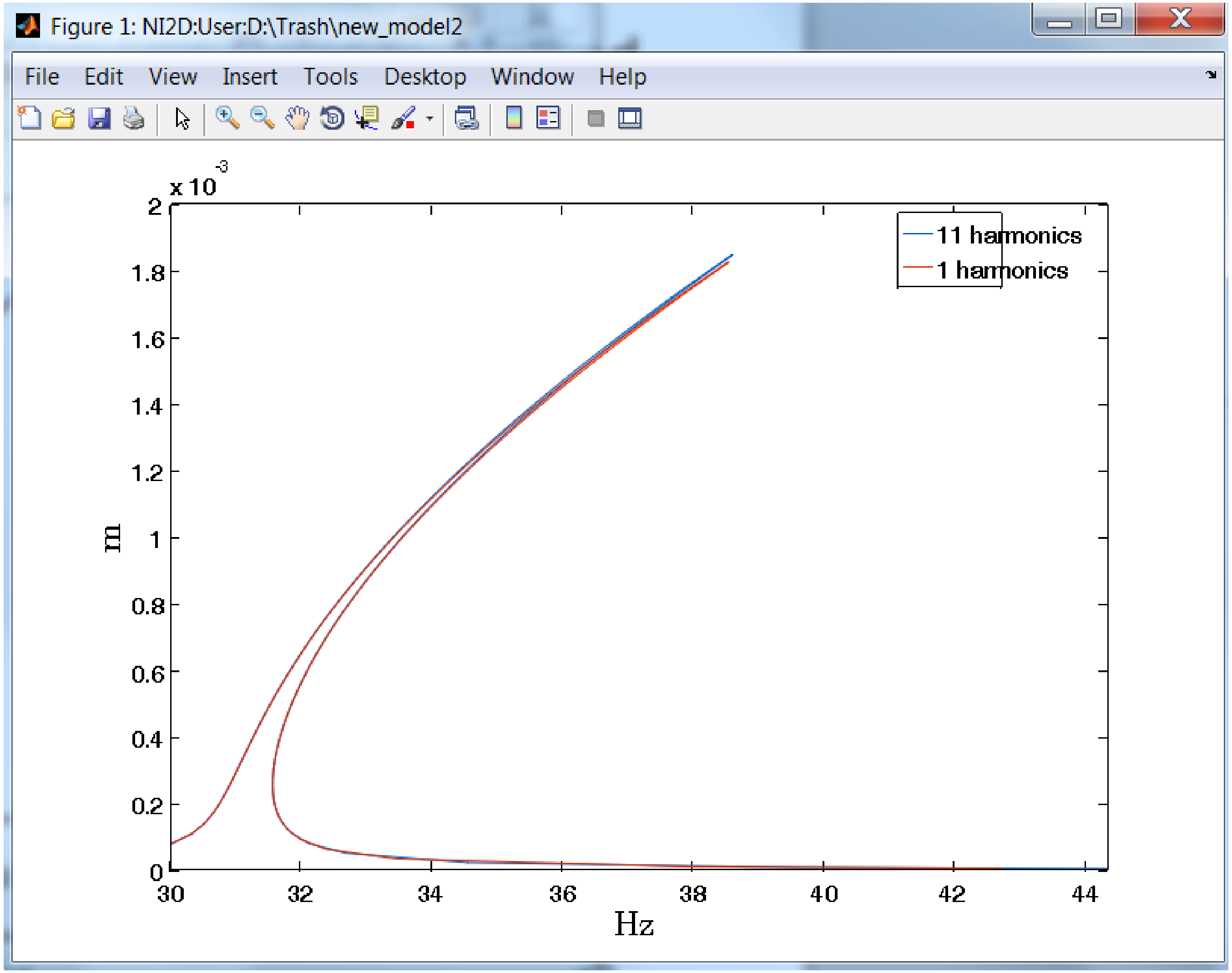


E:\NI2D\Marketing\DemosToolbox\AcademicSystems\1DOF\ModalBeam\LinearModel\Newmark_FRF_HB

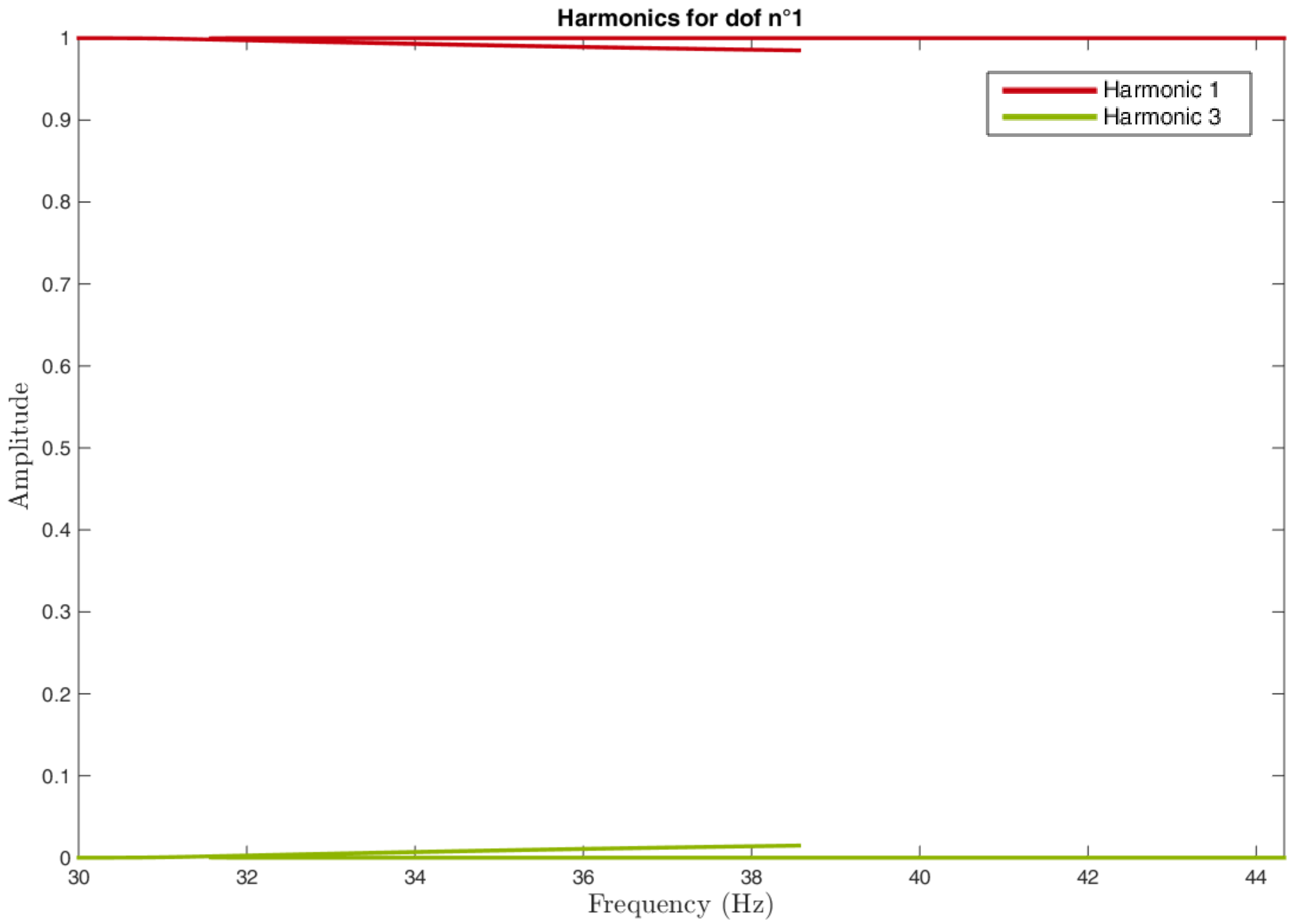
HBM for the nonlinear system



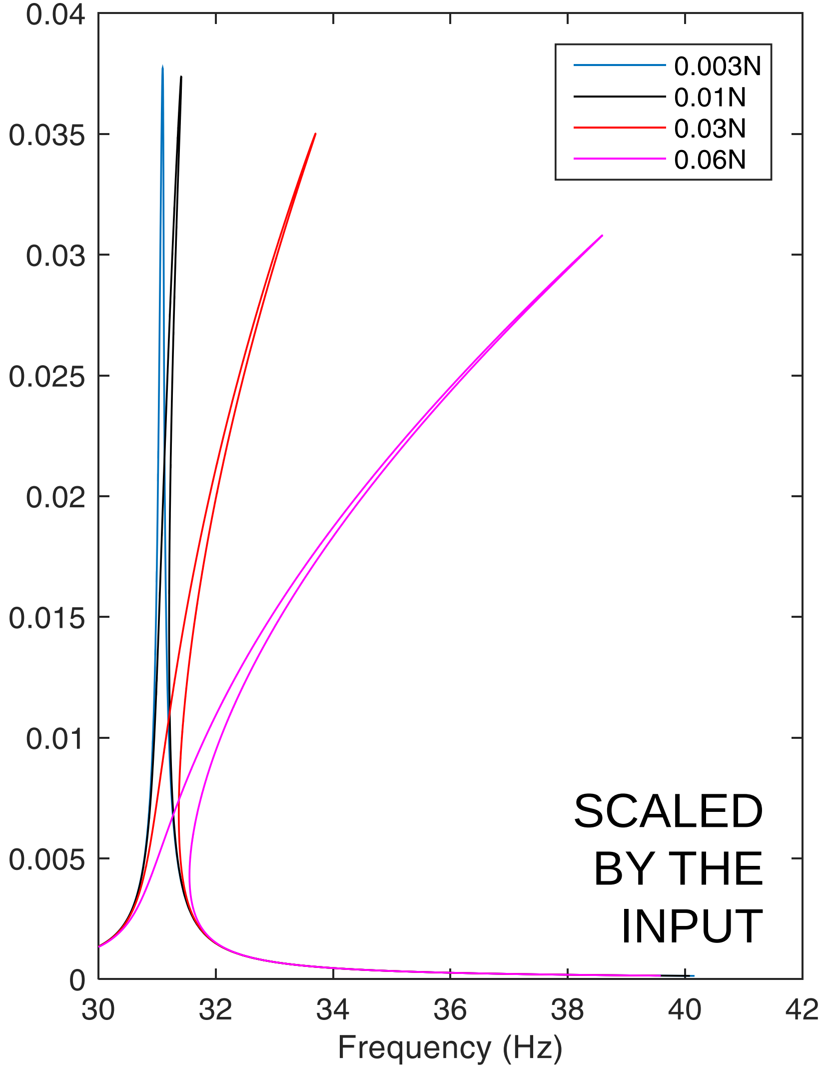
Fast convergence of HBM in this case

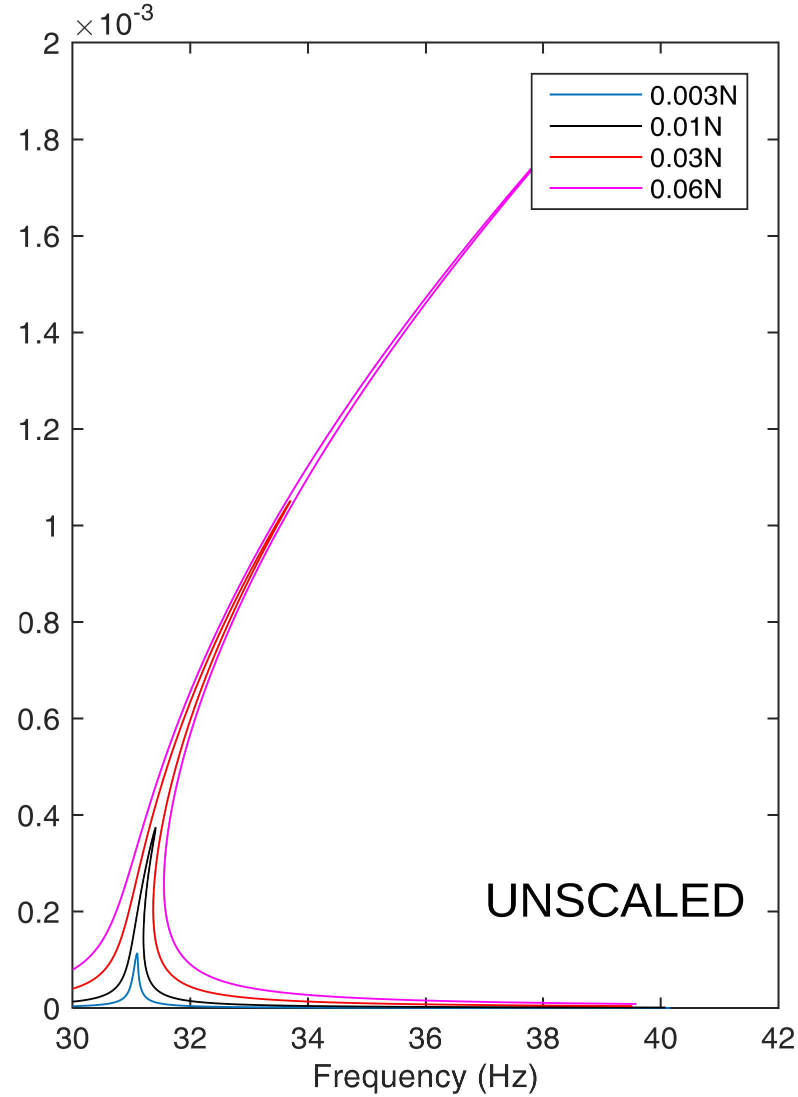


Importance of harmonics

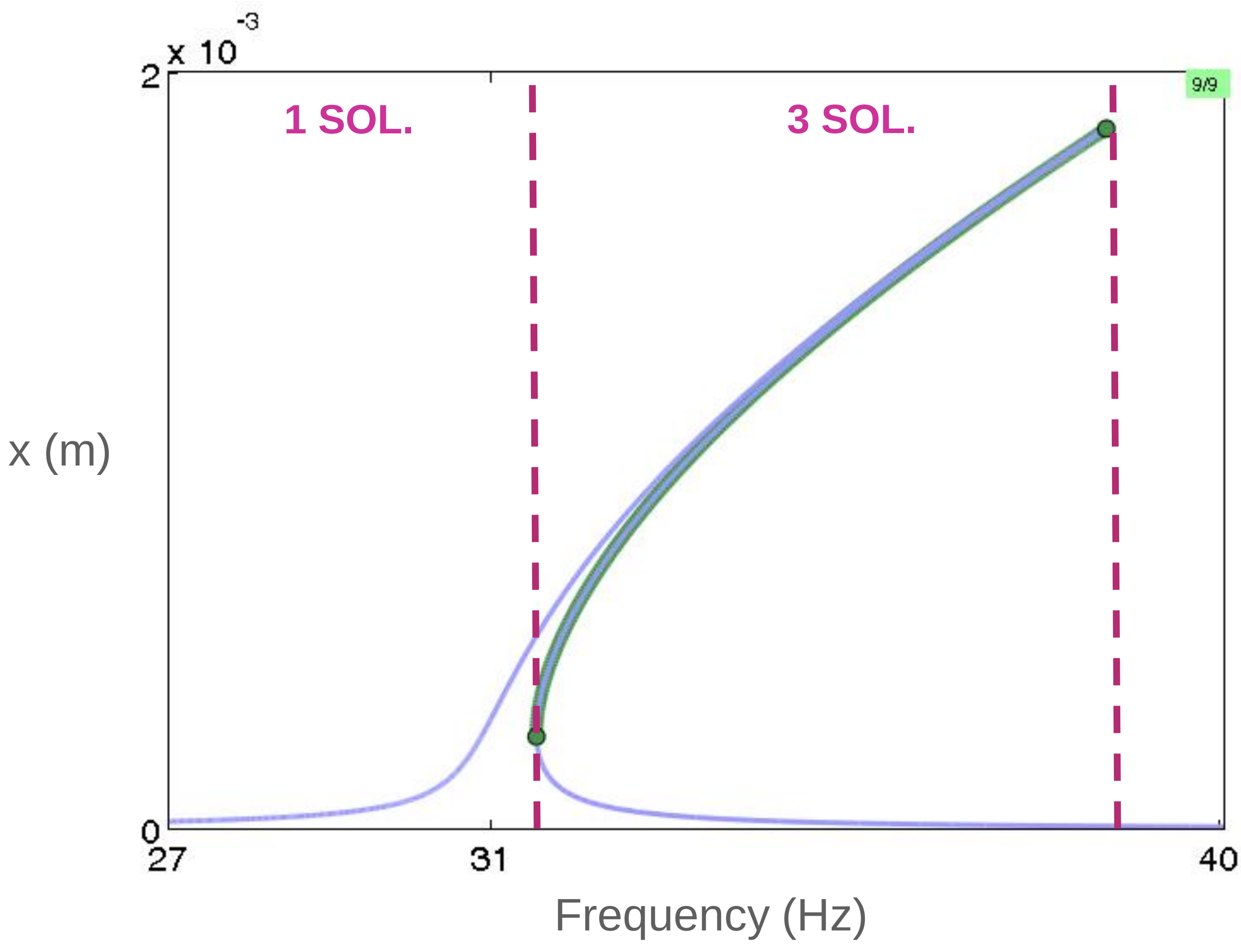


Nonlinear frequency response curves (FRCs)

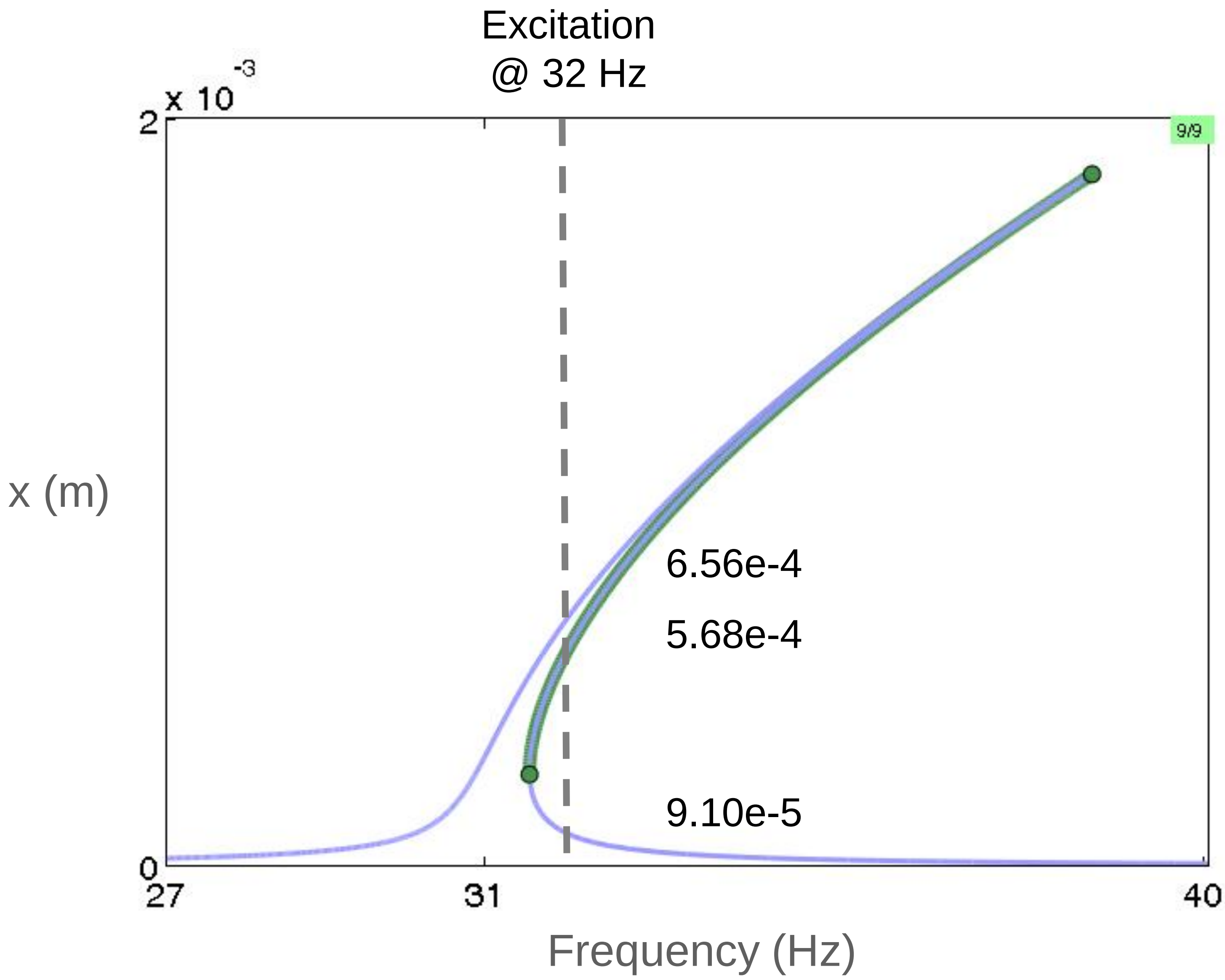




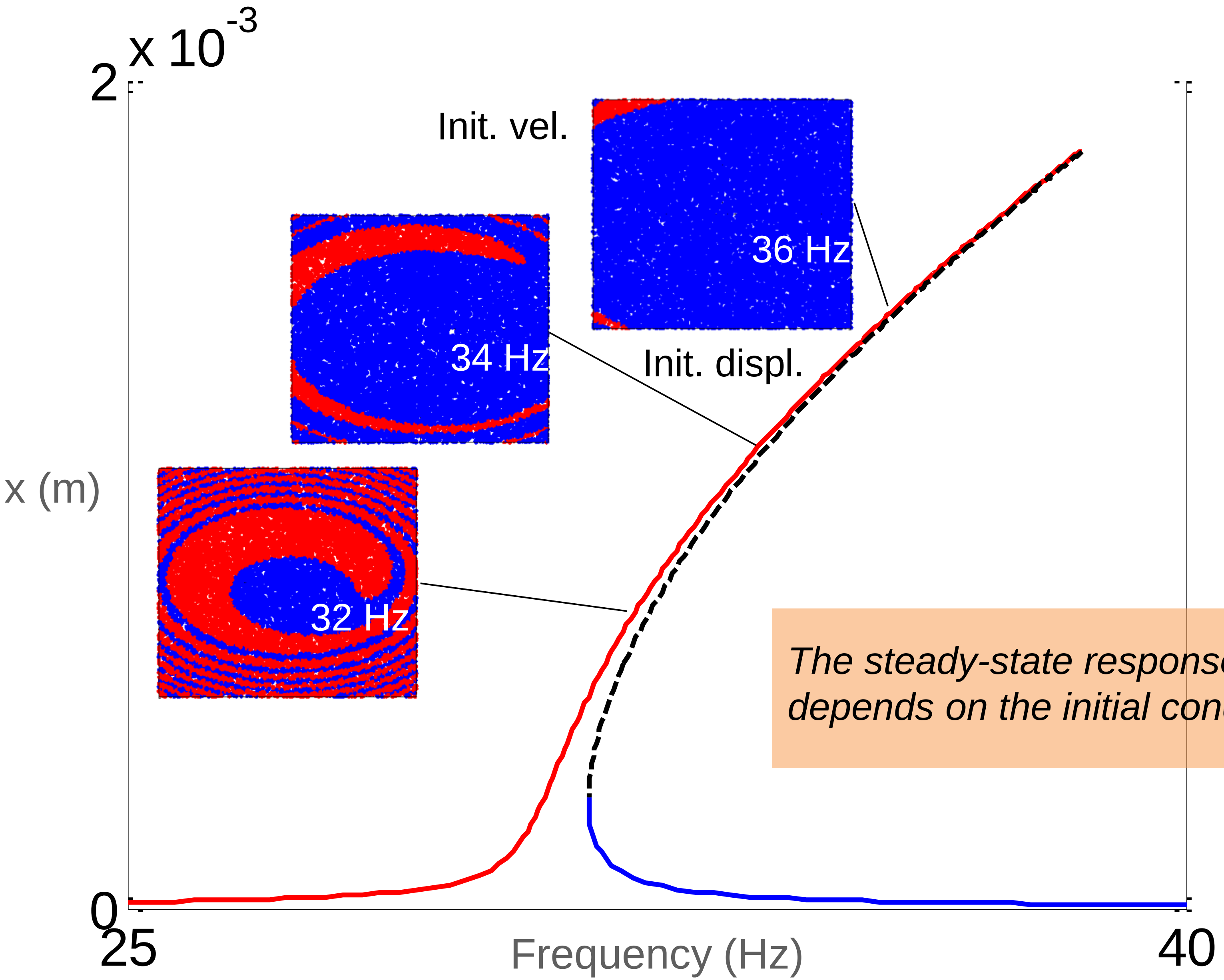
Bifurcations generate multi-valued response



How to know which solution will be excited ?

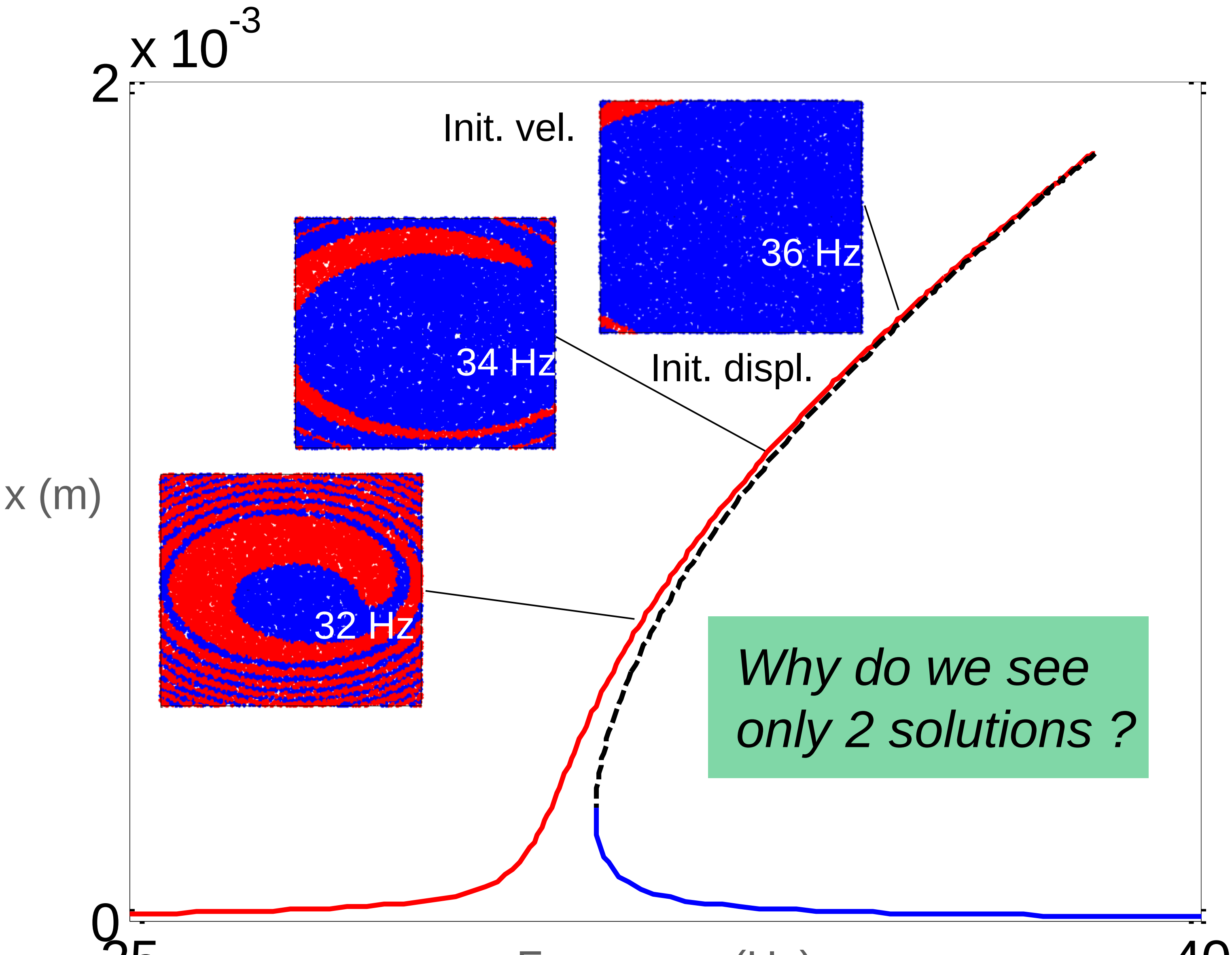


Dependence on initial conditions

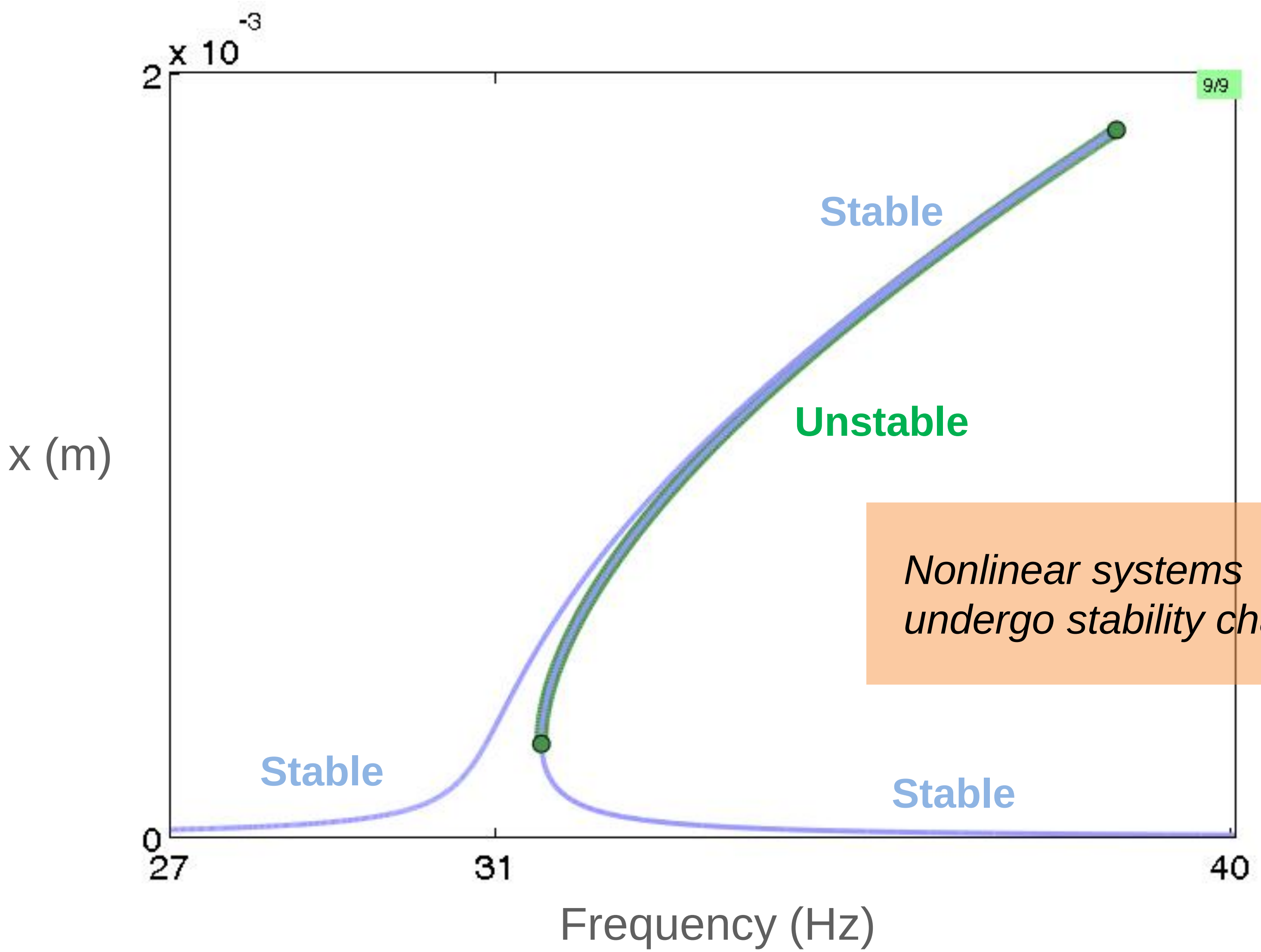


The steady-state response depends on the initial conditions.

Dependence on initial conditions



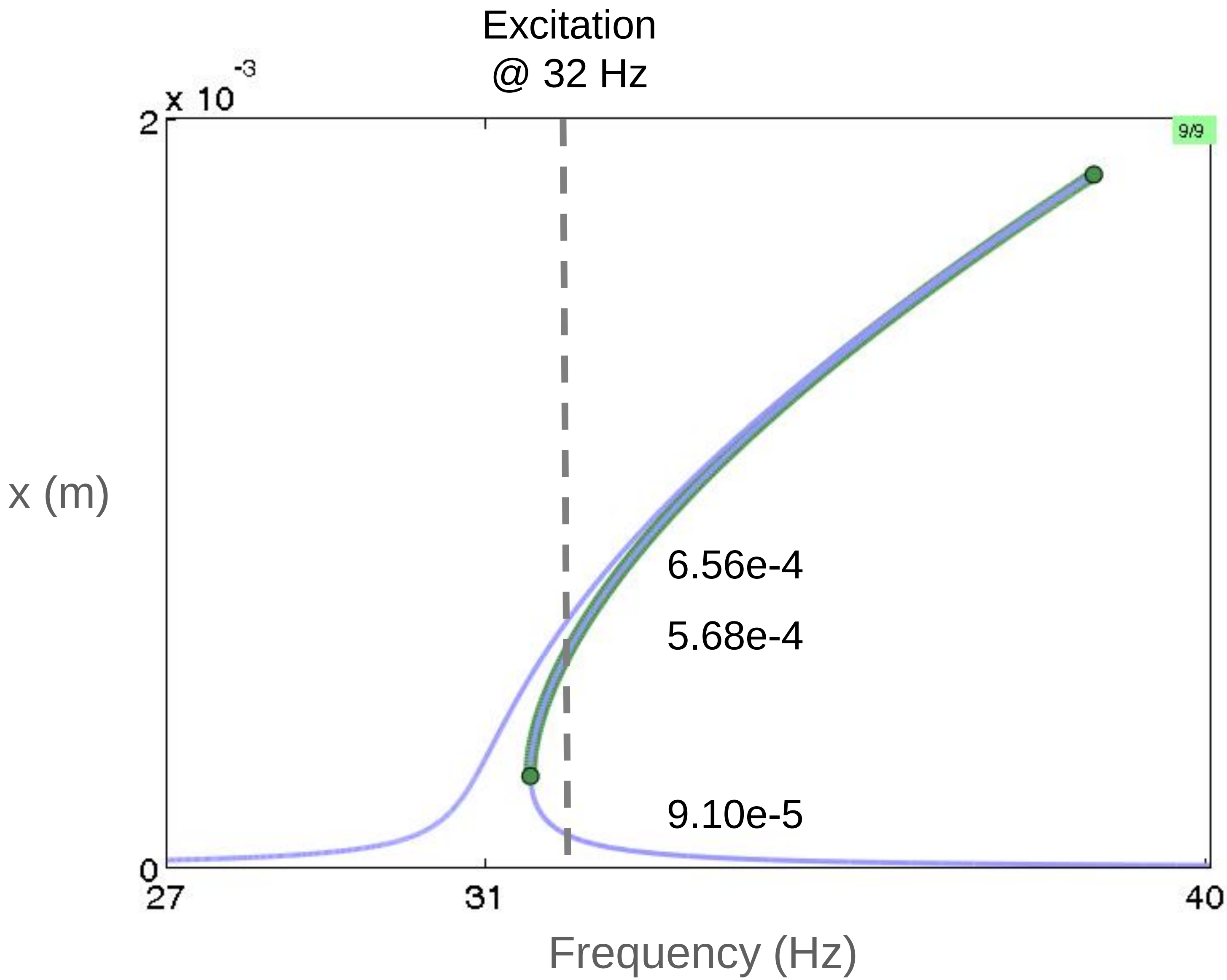
Bifurcations change stability



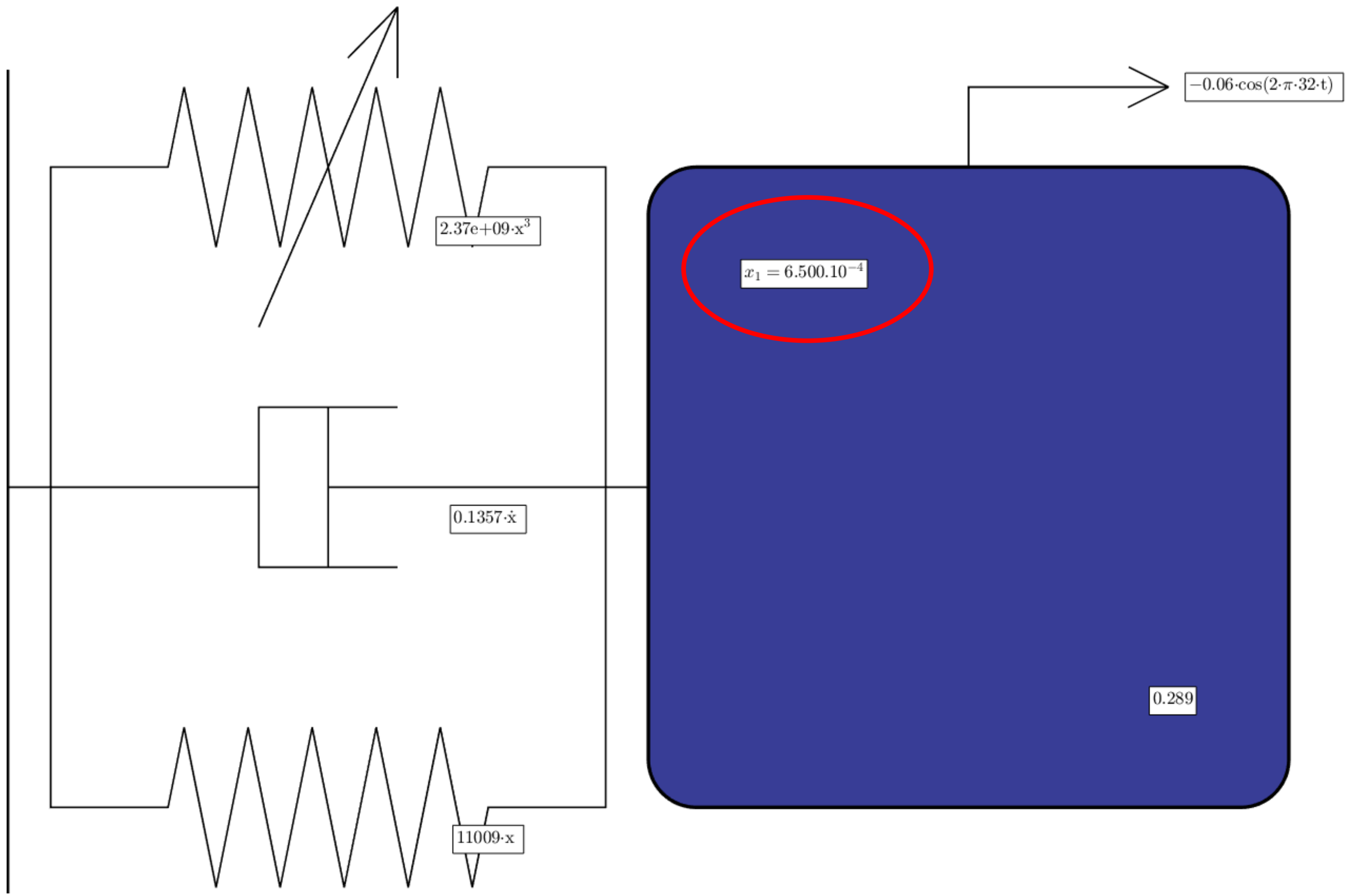
Nonlinear systems undergo stability changes

What is stability/instability ?

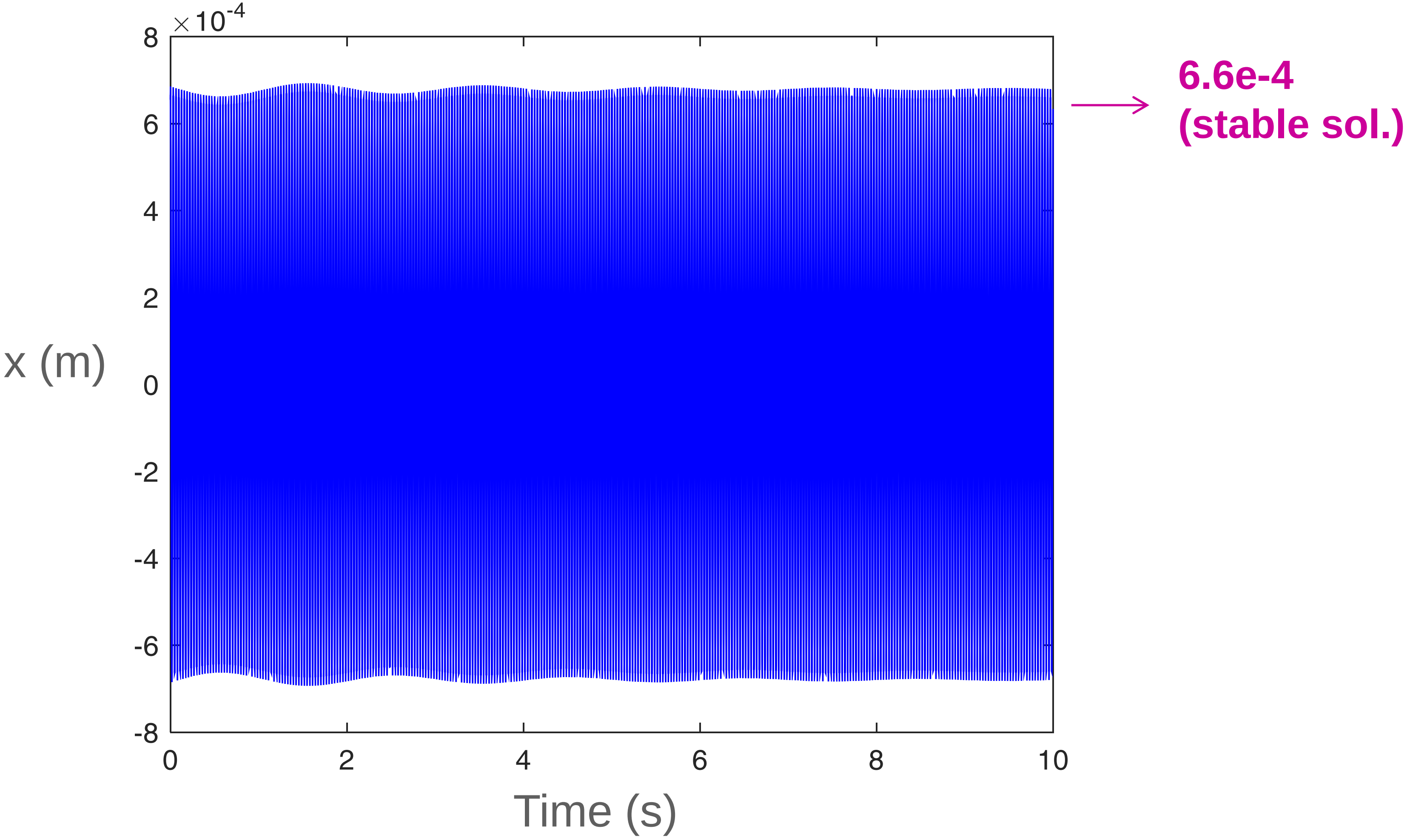
Starting from a stable/unstable solutions



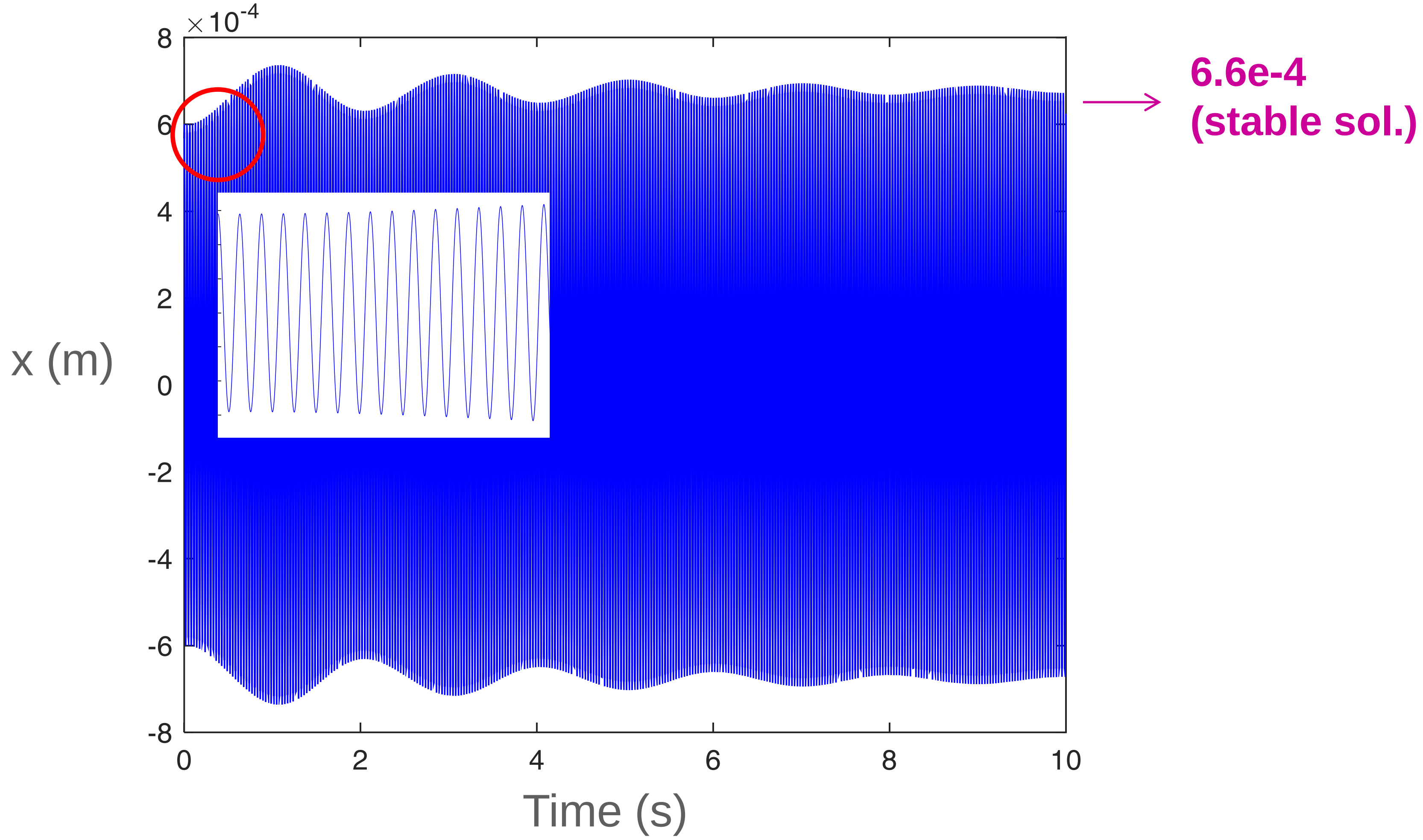
Starting from a stable/unstable solution



Starting from a perturbed stable solution



Starting from a perturbed unstable solution



Forced response: new lessons learned

Properties **Tools**

FREE

No superposition principle

Frequency-amplitude dependence: concept of **backbone curve**

Nonlinear systems generate **harmonics**: **harmonic balance**

+

Solutions of nonlinear systems may undergo **bifurcations**: concept of **nonlinear FRC** and its link with the backbone curve

The steady-state reponse depends on initial conditions: **basin of attraction**

The responses can be **stable/ unstable**: **nonlinear FRC**

FORCED

Outline

Focus on a 1DOF oscillator

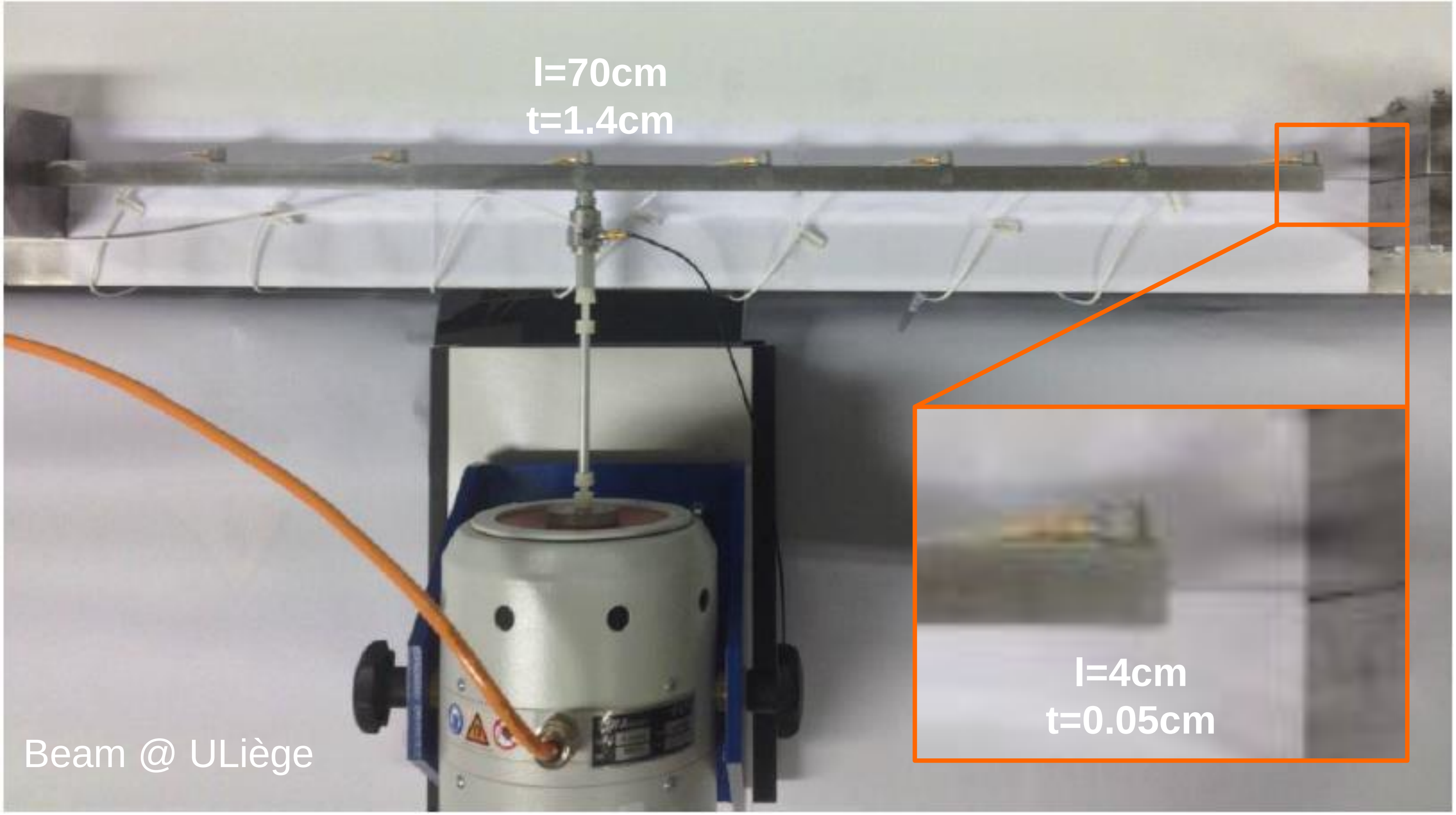
Undamped, unforced dynamics: linear vs. nonlinear

Damped, unforced dynamics: linear vs. nonlinear

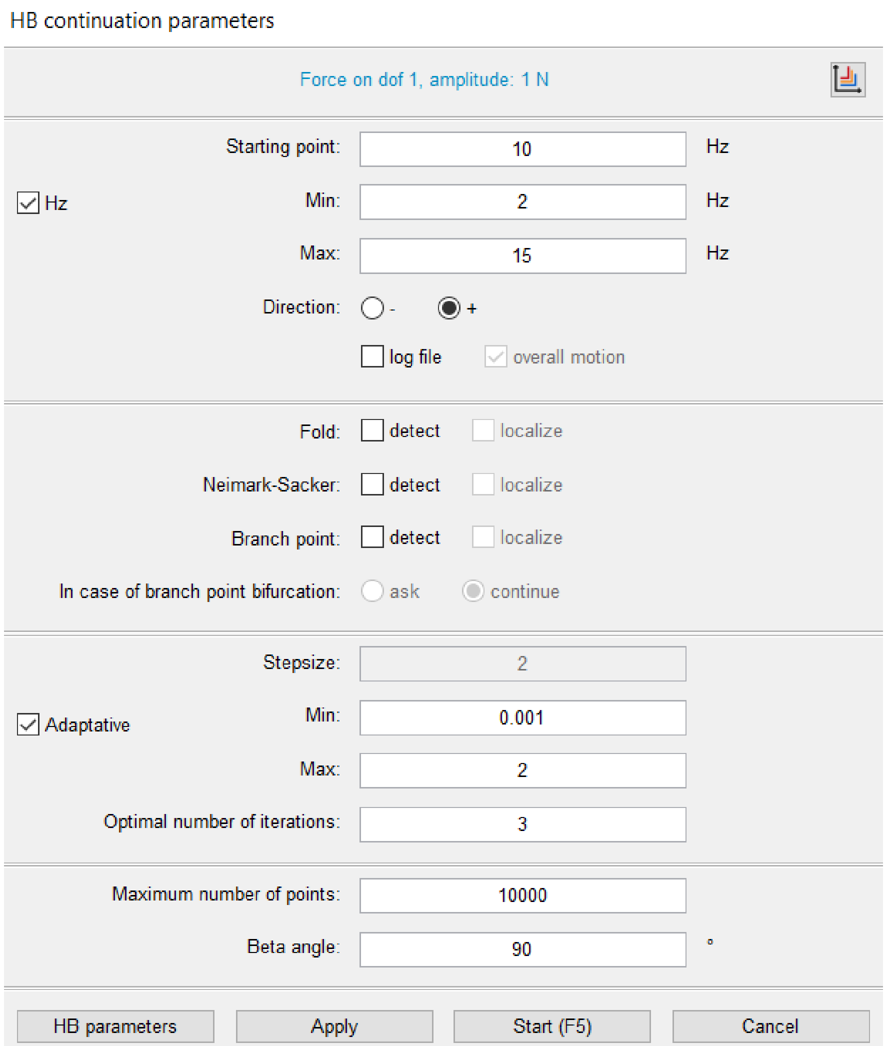
Undamped/damped, harmonic forcing: linear vs. nonlinear

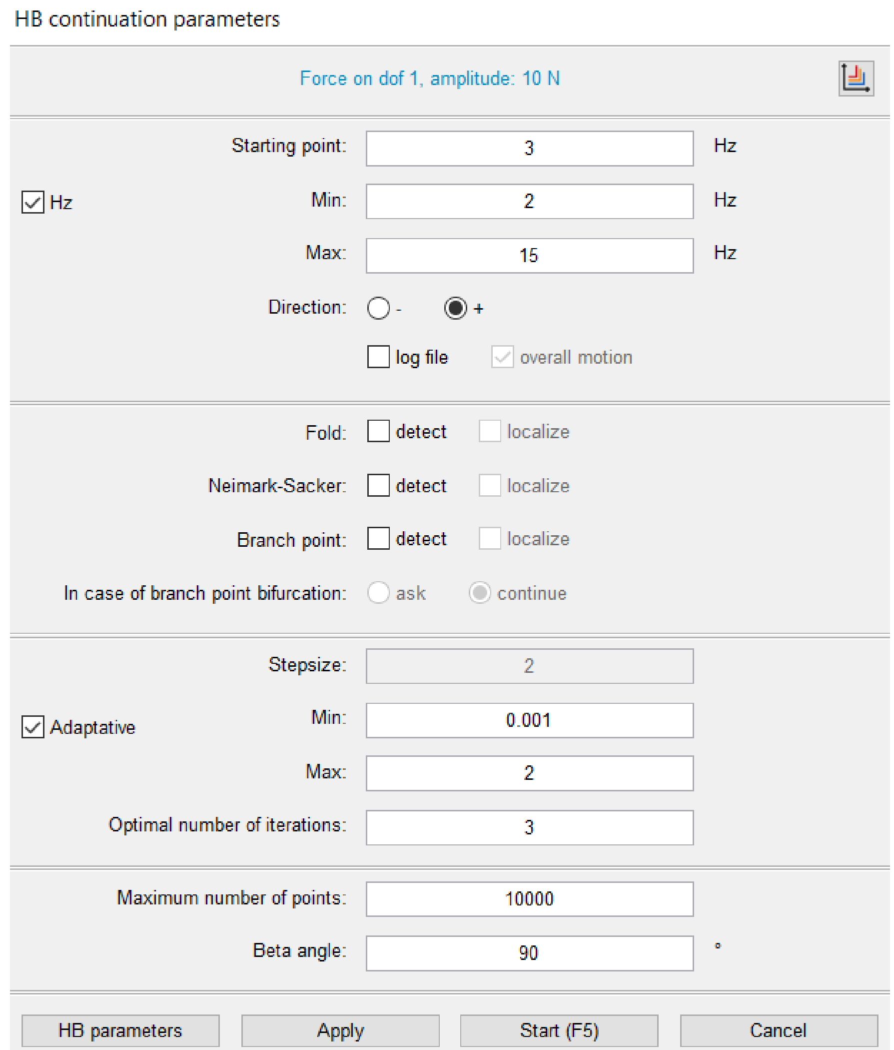
Going beyond…

Going back to the cantilever beam model

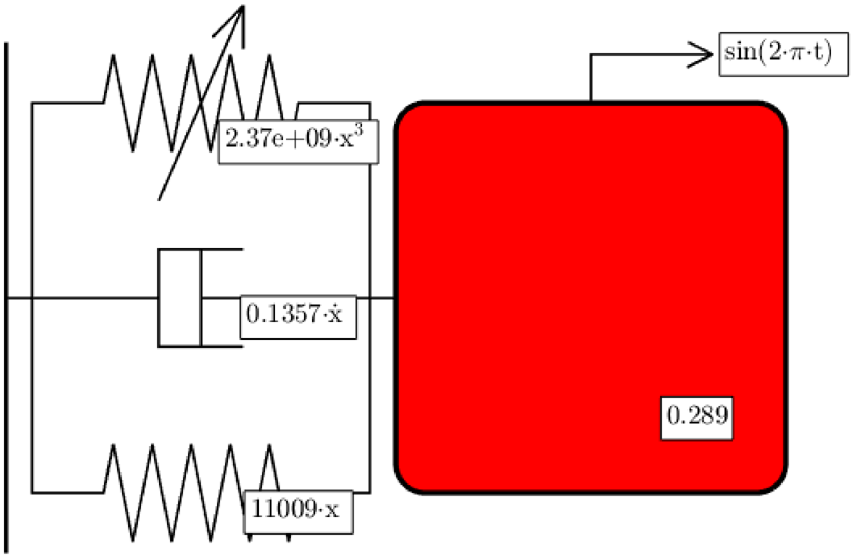


HB parameters

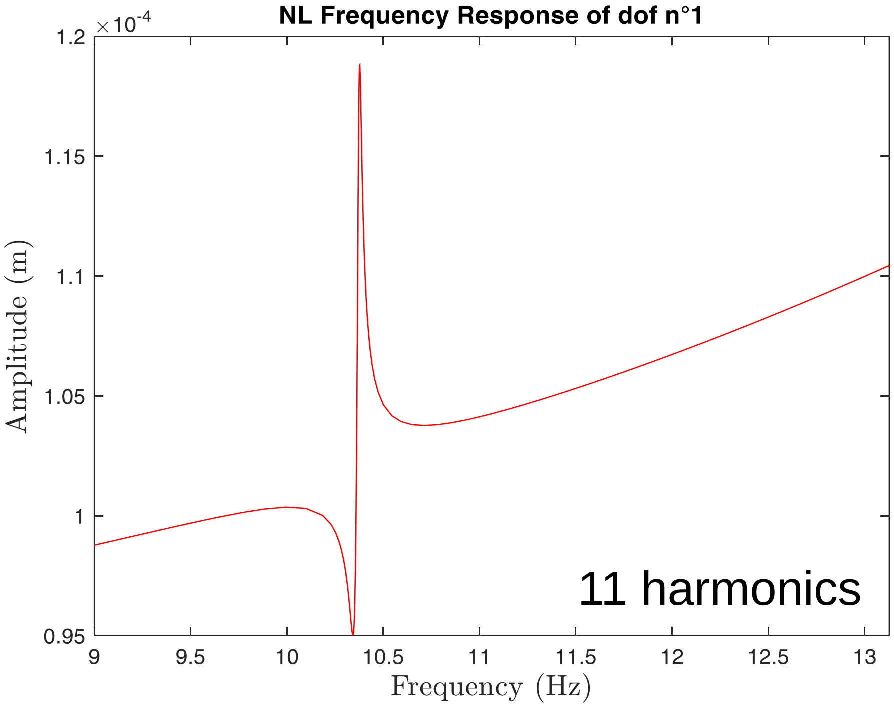




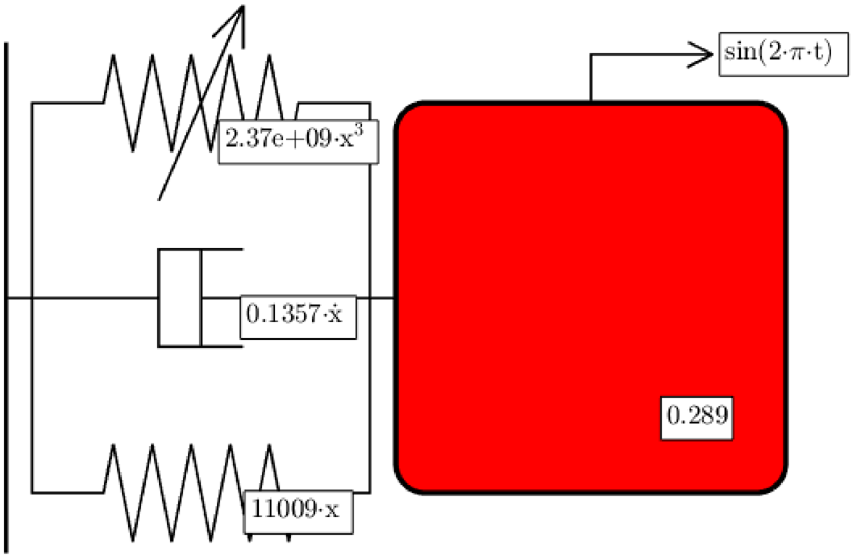
What's going on @ 1N (far from resonance) ?



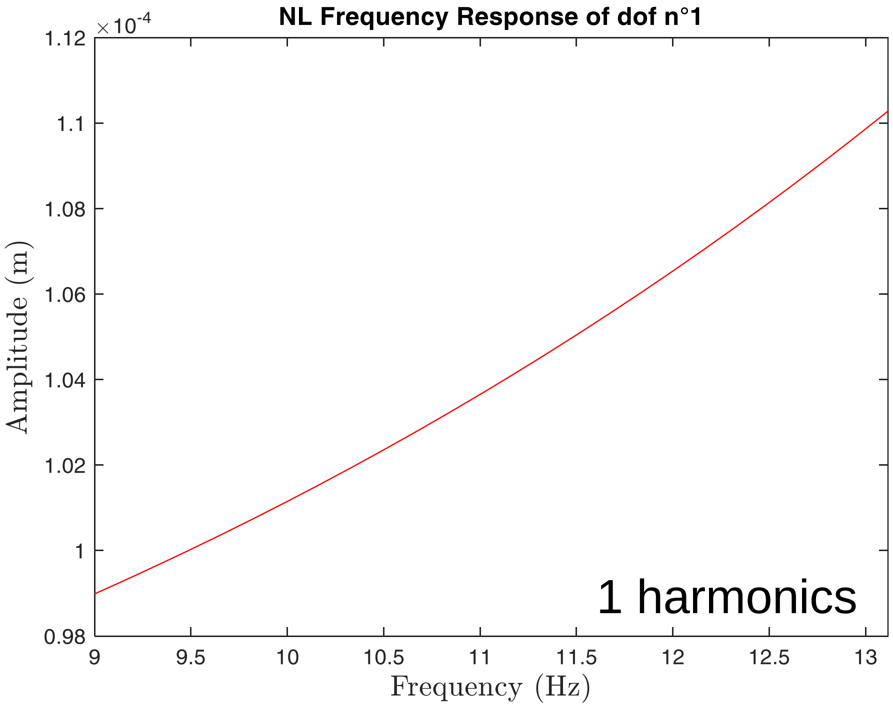
Adaptative stepsize



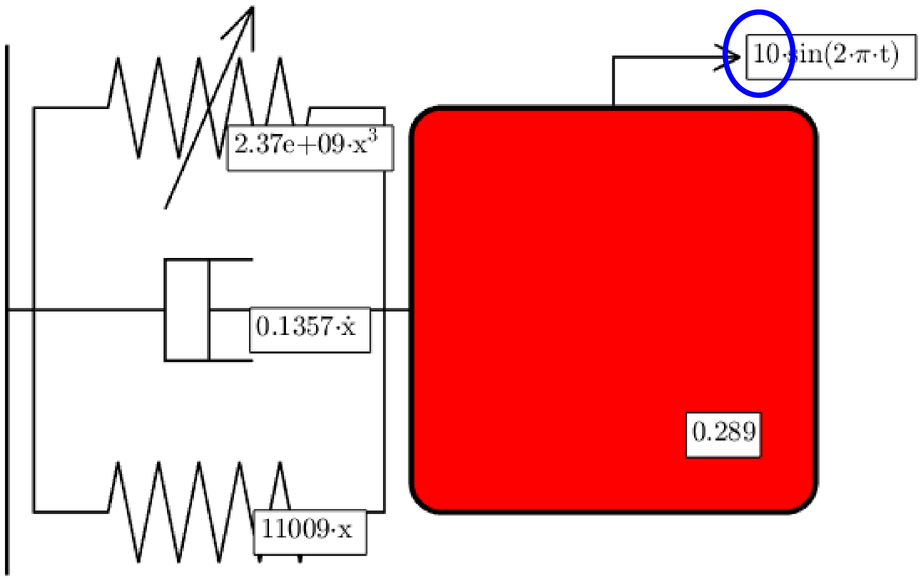
The new resonance has disappeared !



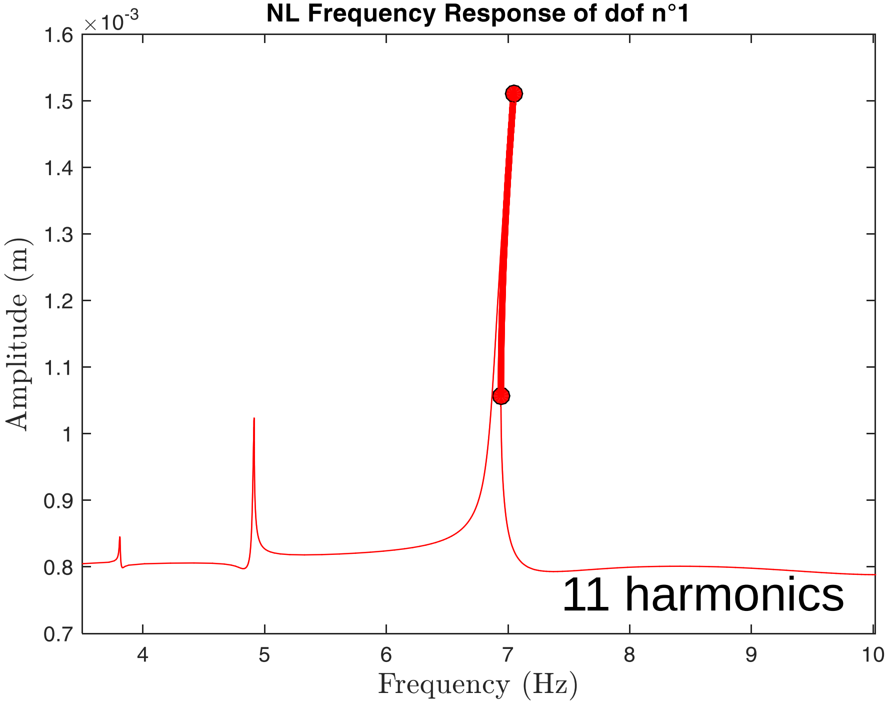
Adaptative stepsize



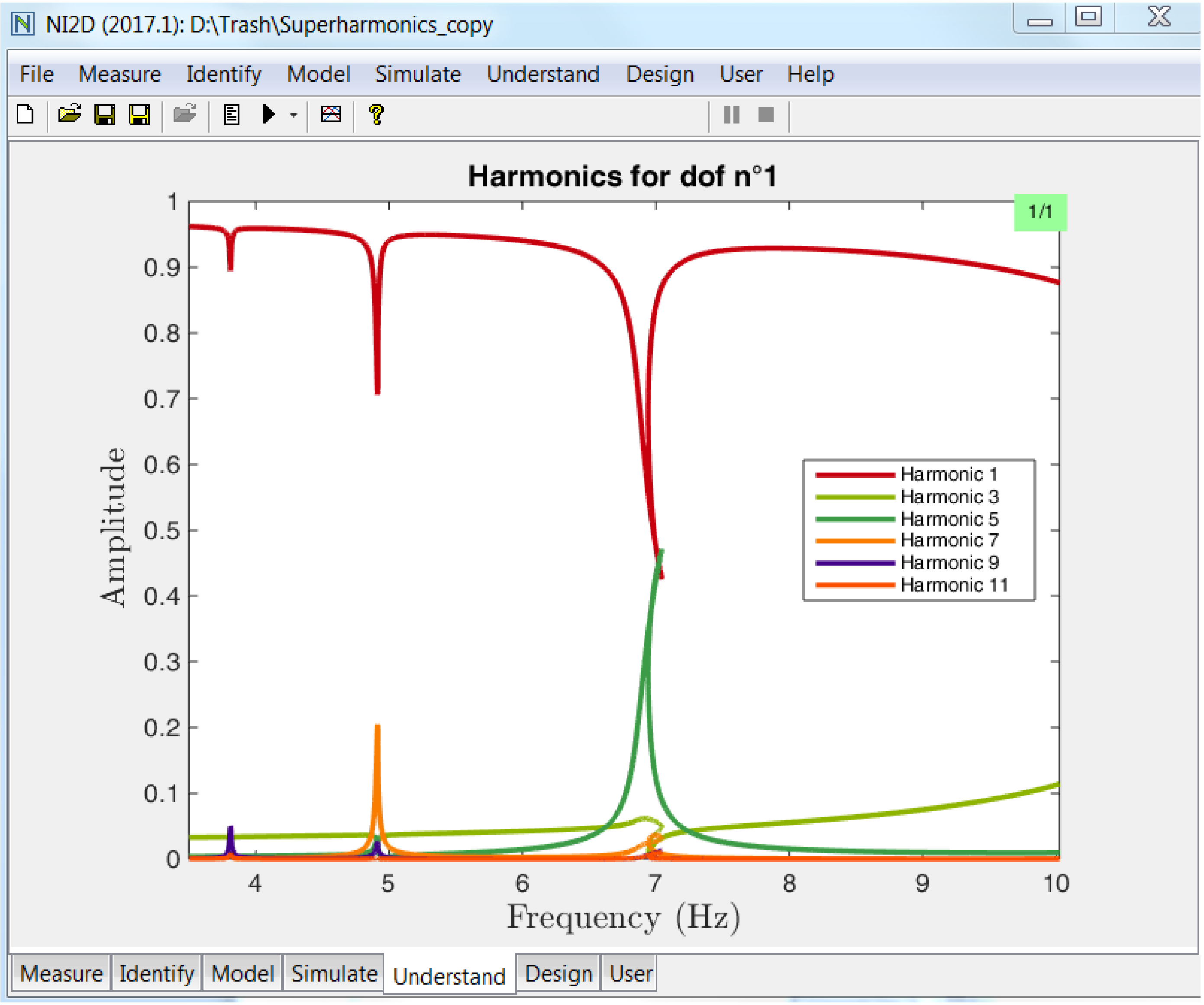
More resonances @ 10N



Stepsize=2



Superharmonic resonances



How to explain this result ?

$$\ddot{y}(t) + \omega_0^2 y(t) + \alpha_3 y^3(t) = f \sin \omega t$$

Absence of damping (phase is trivial)

$$y(t) = A \sin \omega t$$

To be determined

$$(\omega_0^2 - \omega^2) A \sin \omega t + \frac{\alpha_3}{4} A^3 (3 \sin \omega t - \sin 3\omega t) = f \sin \omega t$$

Nonlinear systems generate harmonics

What are our
2 options at this stage ?

Option 2: we enrich our assumption

$$\ddot{y}(t) + \omega_0^2 y(t) + \alpha_3 y^3(t) = f \sin \omega t$$

$$y(t) = A_1 \sin \omega t + A_3 \sin 3\omega t$$

$$-\omega^2 A_1 \sin \omega t - 9\omega^2 A_3 \sin 3\omega t + \omega_0^2 A_1 \sin \omega t + \omega_0^2 A_3 \sin 3\omega t +$$
$$\alpha_3 \left(A_1^3 \sin^3 \omega t + 3A_1^2 A_3 \sin^2 \omega t \sin 3\omega t + 3A_1 A_3^2 \sin \omega t \sin^2 3\omega t + A_3^3 \sin^3 3\omega t \right) = 0$$

$$\begin{aligned}
\sin^3 \omega t &= \frac{3 \sin \omega t - \sin 3\omega t}{4} \\
\sin^2 \omega t \sin 3\omega t &= \frac{\sin 3\omega t}{2} - \frac{(\sin \omega t + \sin 5\omega t)}{4} \\
\sin \omega t \sin^2 3\omega t &= \frac{\sin \omega t}{2} + \frac{(\sin 5\omega t - \sin 7\omega t)}{4} \\
\sin^3 3\omega t &= \frac{3 \sin 3\omega t - \sin 9\omega t}{4}
\end{aligned}$$

A nonlinear algebraic system to solve

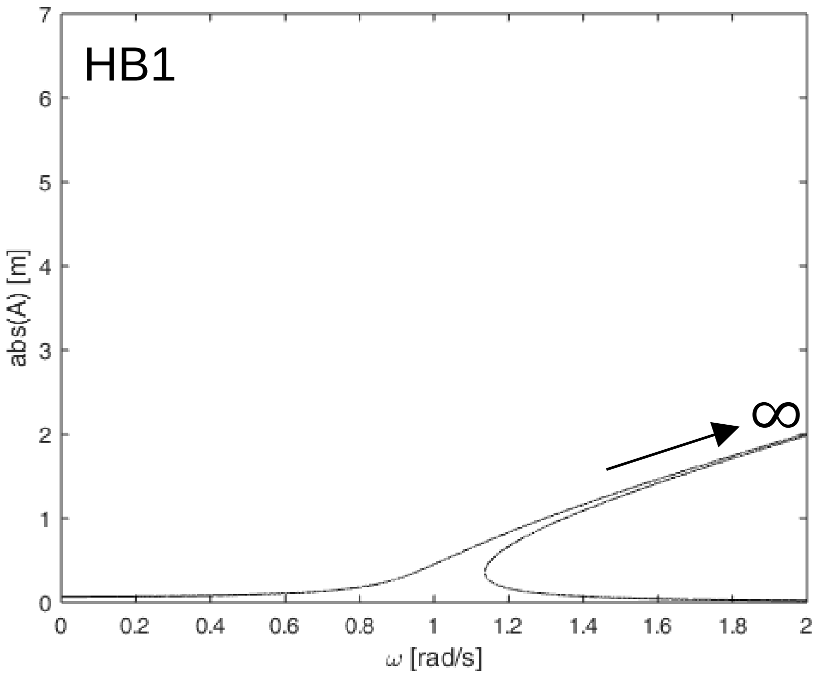
$$
\begin{aligned}
(-\omega^2 + \omega_0^2) A_1 + \frac{3\alpha_3}{4}\left(A_1^3 - A_1^2 A_3 + 2 A_1 A_3^2\right) &= f \\
(-9\omega^2 + \omega_0^2) A_3 + \frac{\alpha_3}{4}\left(-A_1^3 + 6 A_1^2 A_3 + 3 A_3^3\right) &= 0
\end{aligned}
$$

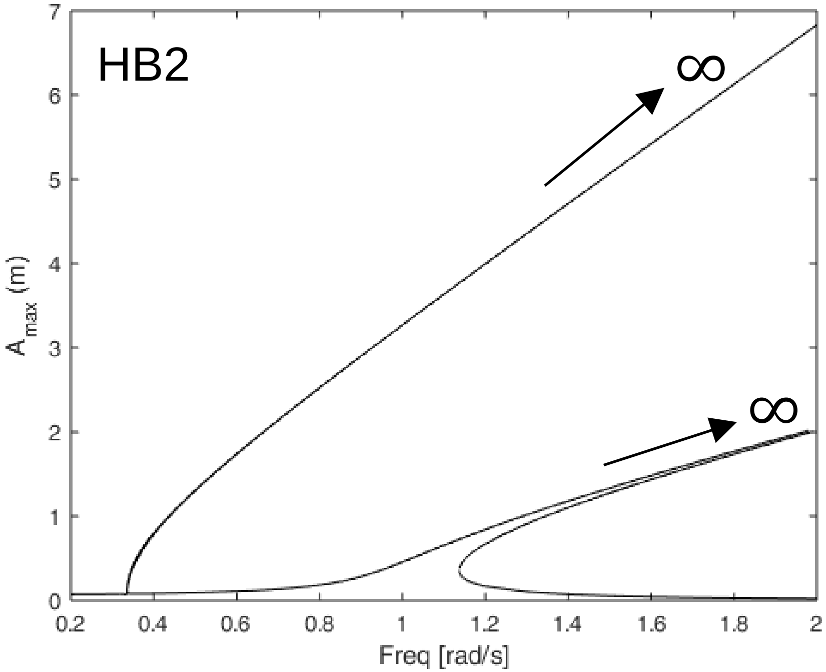
To be determined

```
SolutHB2=fmincon(@SolveTwoTermHB2,Init,[],[],[],[],[-10;-10],[10;10],[],options);


function Opt=SolveTwoTermHB2(y)

Term1=abs(-omegaa^2*A+omega0^2*A+alpha3*(3/4*A^3-3/4*A^2*B+3/2*A*B^2)-f);
Term2=abs(-9*omegaa^2*B +omega0^2*B+alpha3*(-A^3/4 +3/2*A^2*B+3/4*B^3));

Opt=Term1+Term2;
```

HB 1 term vs. HB 2 terms.

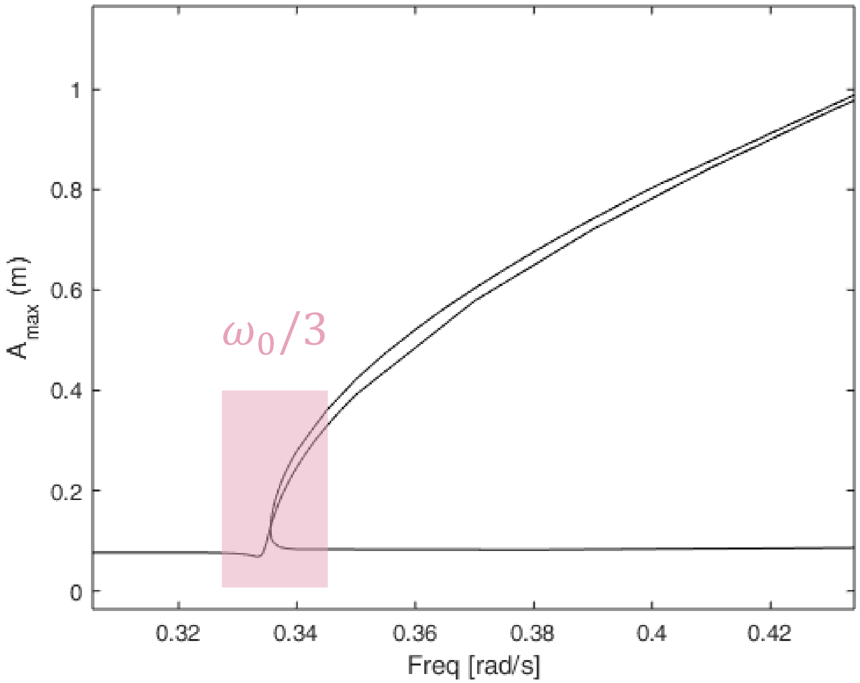
$$\ddot{y}(t) + y(t) + y^3(t) = 0.07 \sin \omega t$$

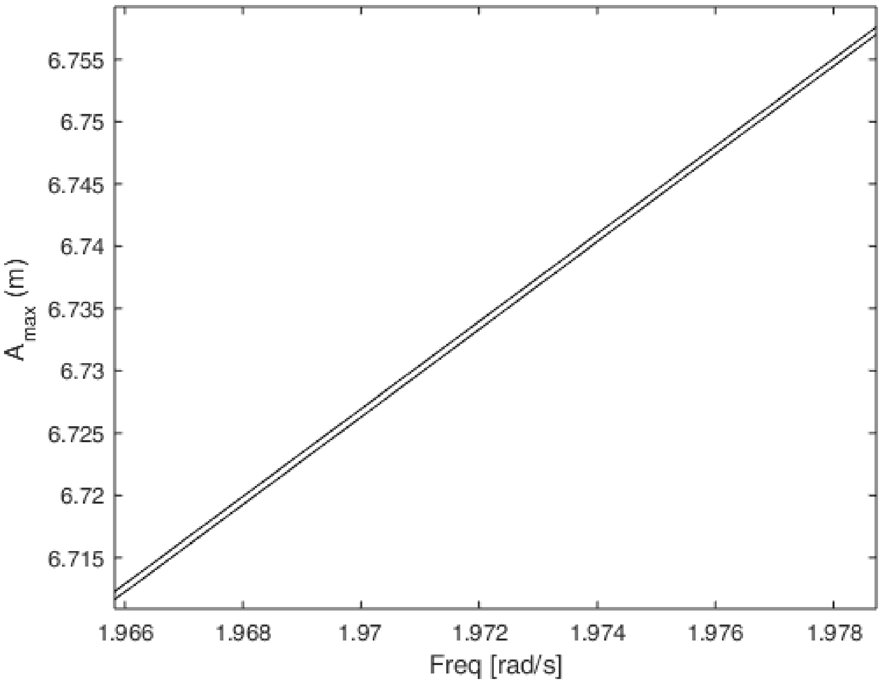




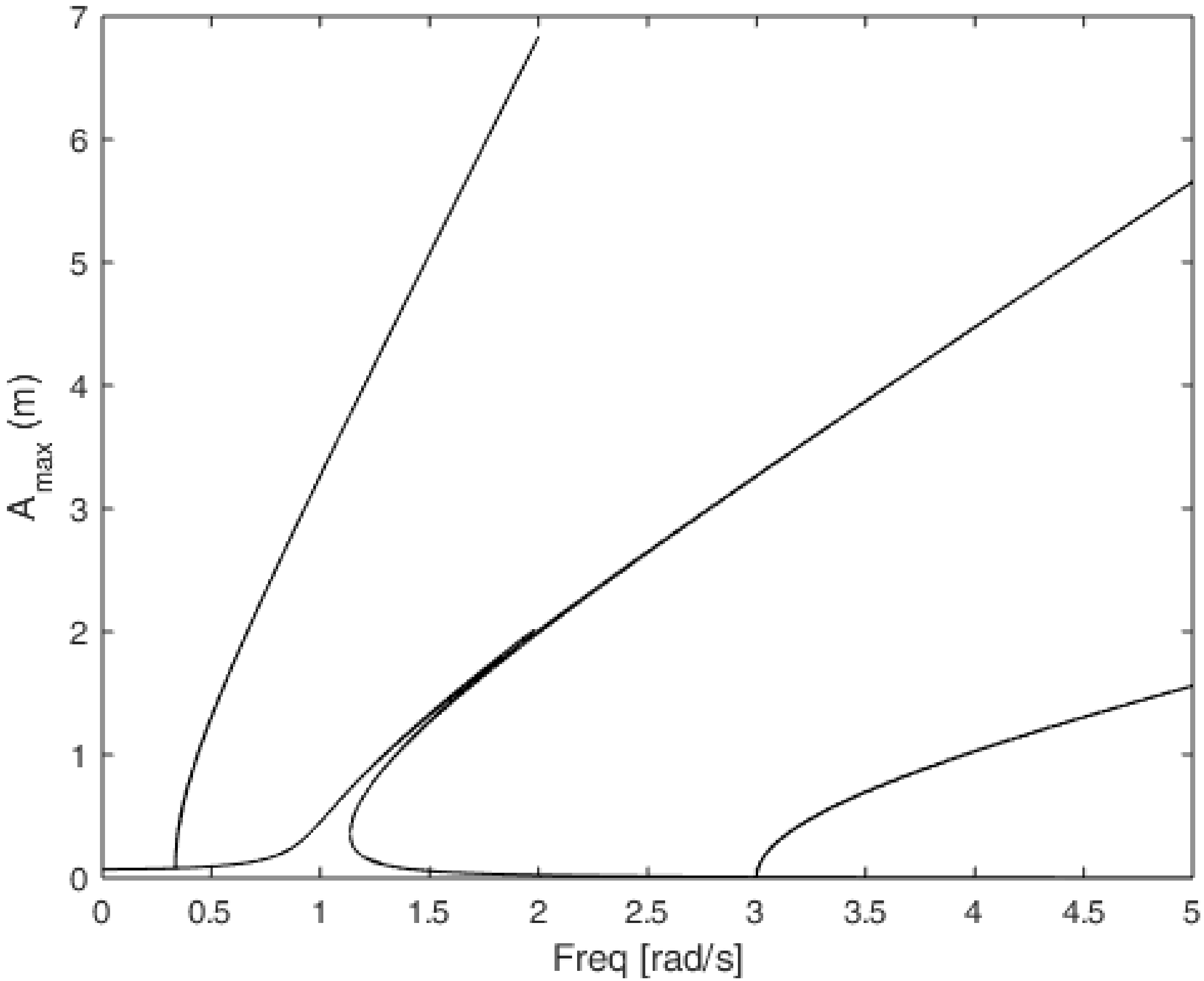
What's going on ?

This is a 3:1 superharmonic resonance





One more branch ???



E:\TheoryComputationTestingNlVib\Matlab\Chapter 3\DuffingForcedUndamped

How to capture it ?

$$\ddot{y}(t) + \omega_0^2 y(t) + \alpha_3 y^3(t) = f \sin \omega t$$

$$y(t) = A_1 \sin \omega t + A_3 \sin 3\omega t$$

What is a correct assumption ?

Option 2: we enrich our assumption

$$\ddot{y}(t) + \omega_0^2 y(t) + \alpha_3 y^3(t) = f \sin \omega t$$

$$y(t) = A_1 \sin \omega t + A_5 \sin 5\omega t$$

$$y(t) = A_1 \sin \omega t + A_7 \sin 7\omega t$$

Etc.

Superharmonic resonances

$$y(t) = A_1 \sin \omega t + A_3 \sin \frac{\omega}{3} t$$

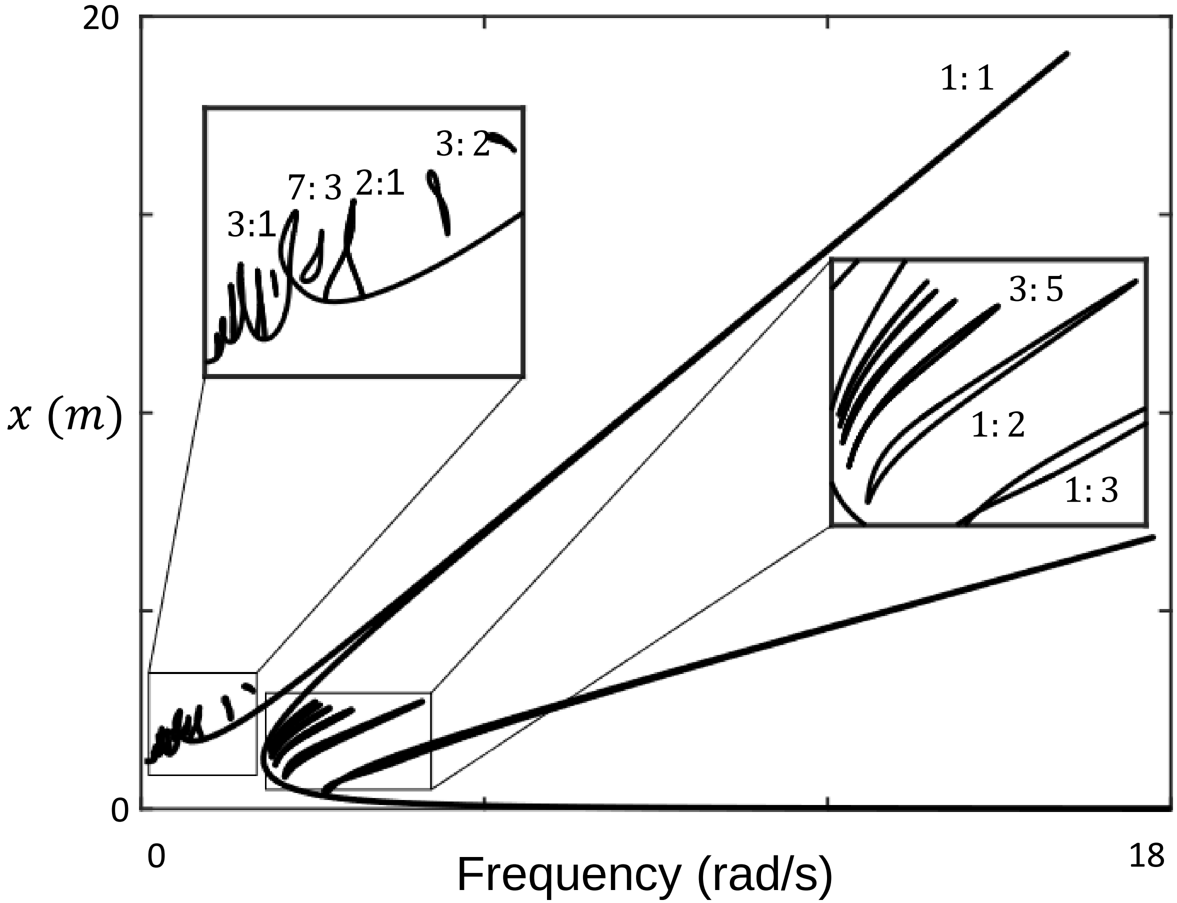
$$y(t) = A_1 \sin \omega t + A_5 \sin \frac{\omega}{5} t$$

Etc.

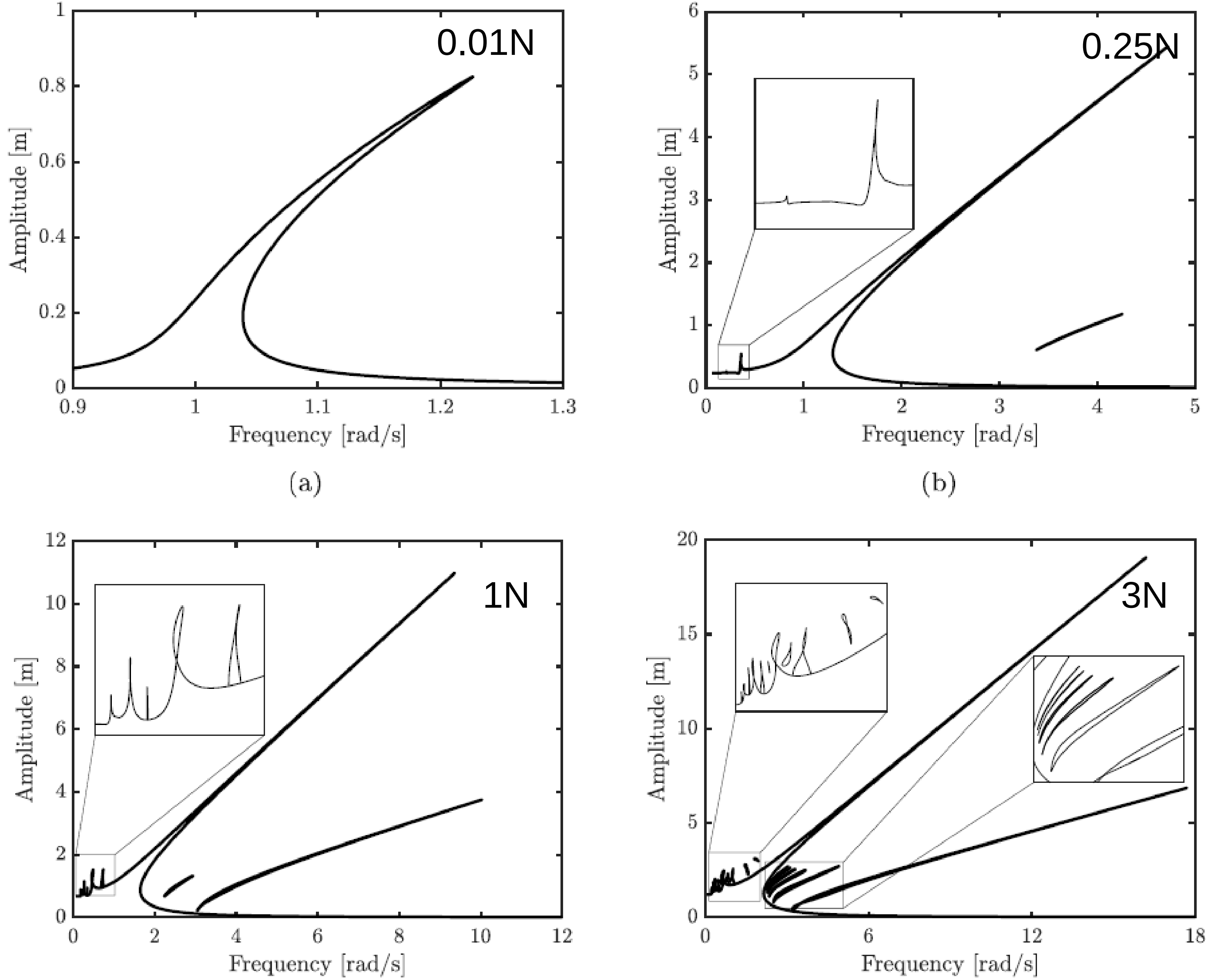
Subharmonic resonances

The complete picture

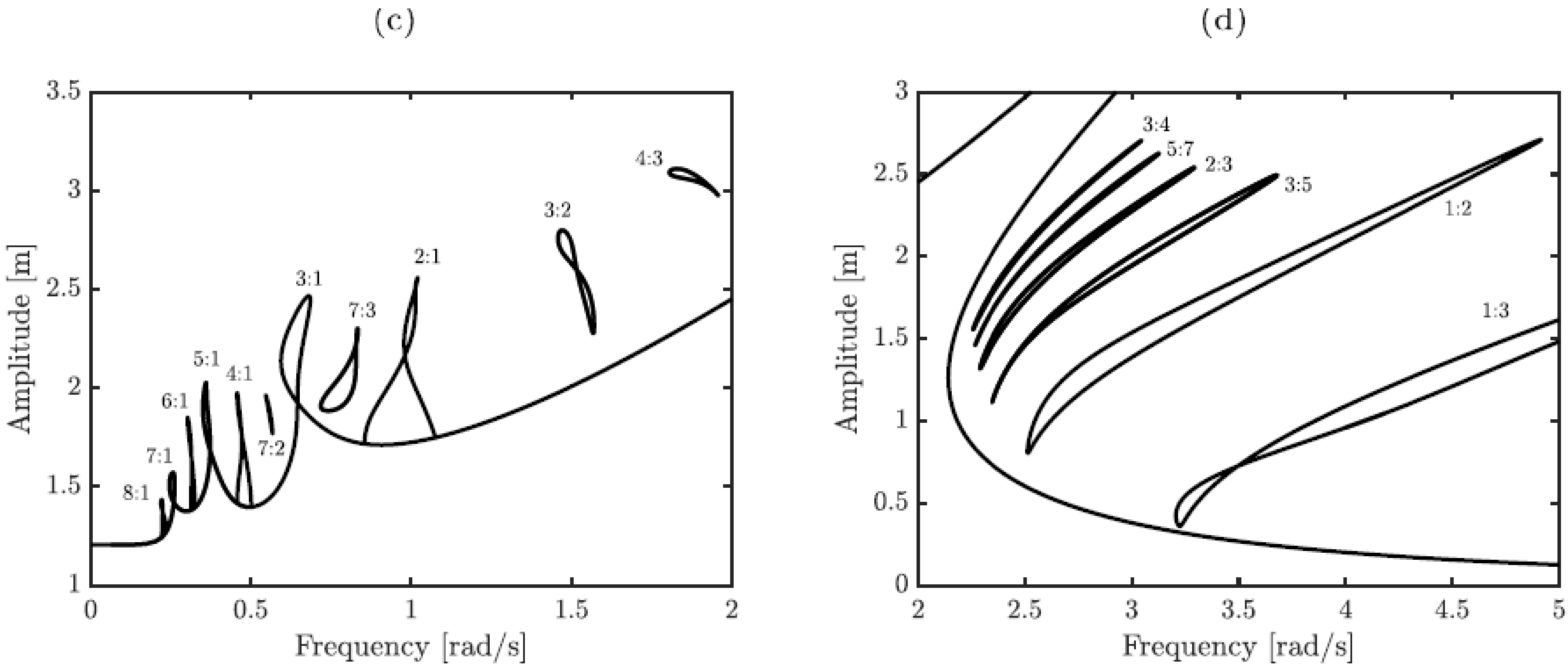
$$\ddot{y} + 0.01\dot{y} + y + y^3 = 3 sin\, \omega t$$



Gradual appearance of complexity



Close-ups: superharmonic/subharmonic resonances



Lessons learned

The response is no longer purely harmonic

FREE

No superposition principle

Frequency-amplitude dependence: concept of **backbone curve**

Nonlinear systems generate **harmonics** **(harmonic balance)**

+

Solutions of nonlinear systems may undergo **bifurcations:** concept of **nonlinear FRC** and its link with the backbone curve

The steady-state reponse depends on initial conditions (basin of attraction)

The responses can be **stable or unstable**

FORCED @ ω

+

Sub- or superharmonic resonances (even isolated !)

FORCED @ $\neq\omega$