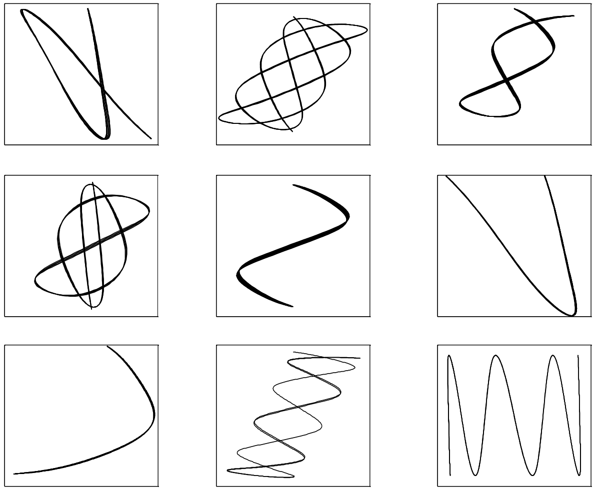
Nonlinear Vibrations of Aerospace Structures

L04 Modal analysis

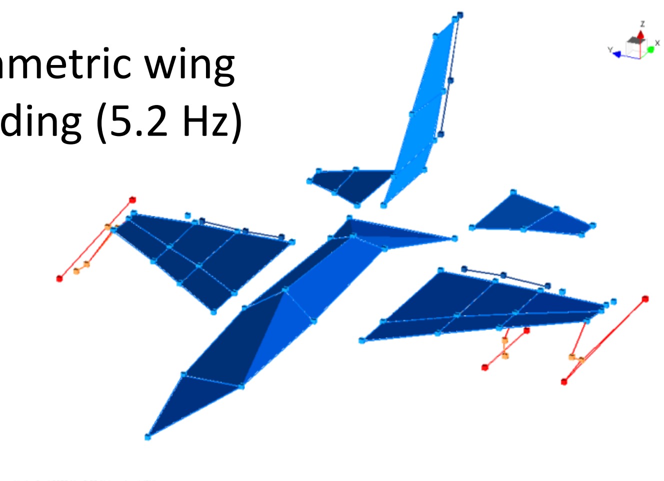
Unforced dynamics
(2DOF)



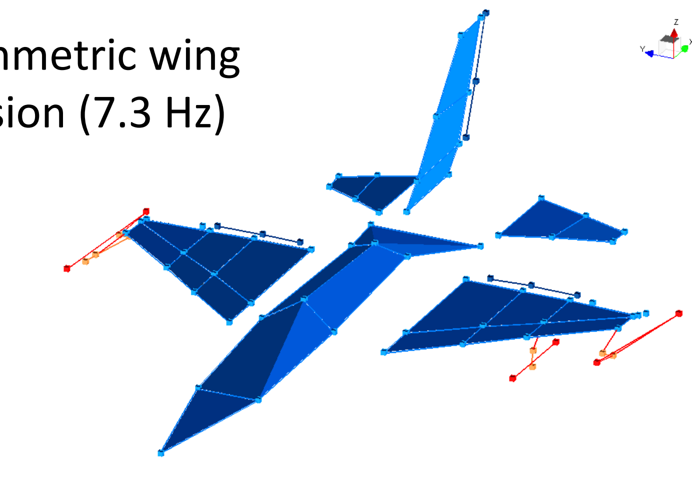
Modes correspond to the deformation at resonance

Modal analysis provides key information

Symmetric wing bending (5.2 Hz)



Symmetric wing torsion (7.3 Hz)



What is a linear normal mode ?

2.2. Modes normaux de vibration

Pour résoudre les équations des petites oscillations libres (2.1.12)

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = 0$$

cherchons une solution particulière dans laquelle toutes les coordonnées généralisées suivent, à un facteur près, la même loi temporelle

$$\mathbf{q} = \mathbf{x}\ \phi(t) \tag{2.2.1}$$

où $\mathbf{x}$ est un vecteur de constantes constituant *la forme propre du mouvement*, propre dans ce sens que le rapport de deux coordonnées est indépendant du temps et est toujours égal au rapport des éléments correspondants de $\mathbf{x}$. L'essai d'une solution de ce type fournit

$$\ddot{\phi}(t)\mathbf{M}\mathbf{x} + \phi(t)\mathbf{K}\mathbf{x} = 0 \tag{2.2.2}$$

Cours de théorie des vibrations

How do we calculate linear normal modes ?

$$\ddot{q}_1 + (2q_1 - q_2) = 0$$
$$\ddot{q}_2 + (2q_2 - q_1) = 0$$

How do we calculate linear normal modes ?

$q_{1,2} = A, B cos\omega t$

$$\ddot{q}_1 + (2q_1 - q_2) = 0$$
$$\ddot{q}_2 + (2q_2 - q_1) = 0$$

$$-\omega^2 A + 2A - B = 0$$
$$-\omega^2 B + 2B - A = 0$$

$$B = A(2 - \omega^2)$$
$$-\omega^2 B + 2B - A = 0$$

$$-\omega^2 A(2 - \omega^2)$$
$$+ 2A(2 - \omega^2) - A = 0$$

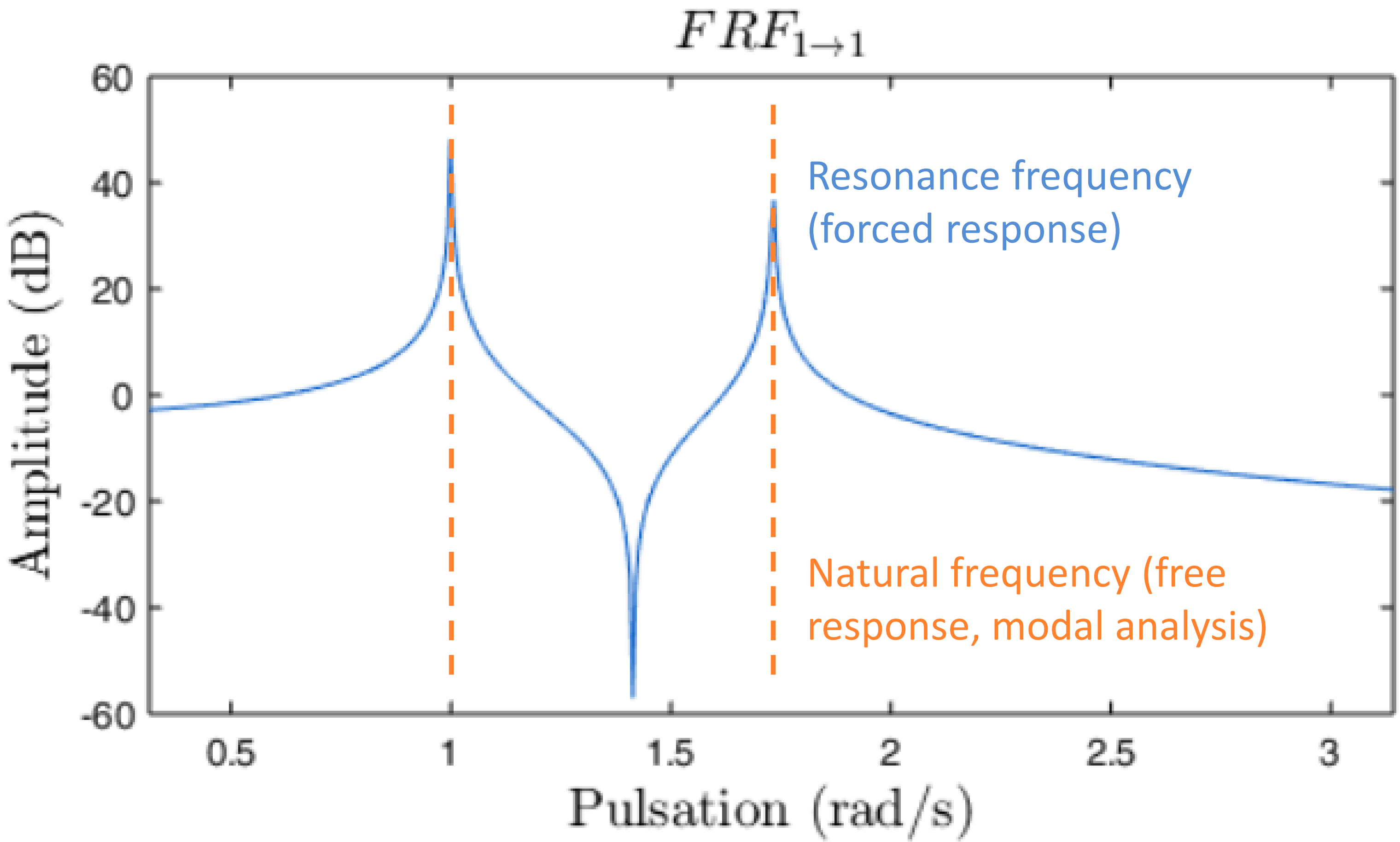
$$\omega^4 - 4\omega^2 + 3 = 0$$

$\omega_1 = 1$ rad/s with $A = B$,

$\omega_2 = \sqrt{3}$ rad/s with $A = -B$,

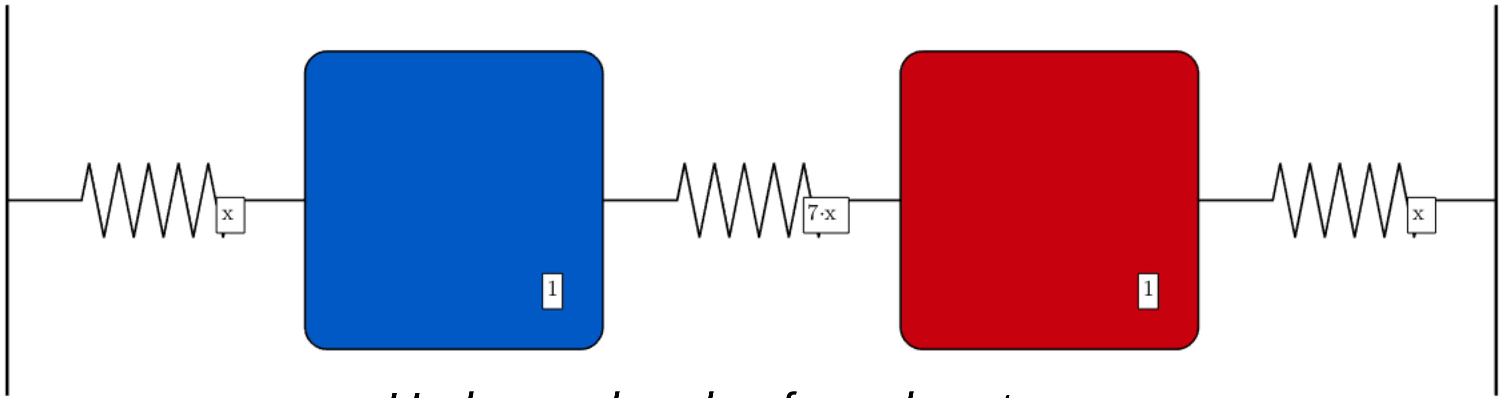
Linear modes are invariant

Link between natural and resonance frequencies

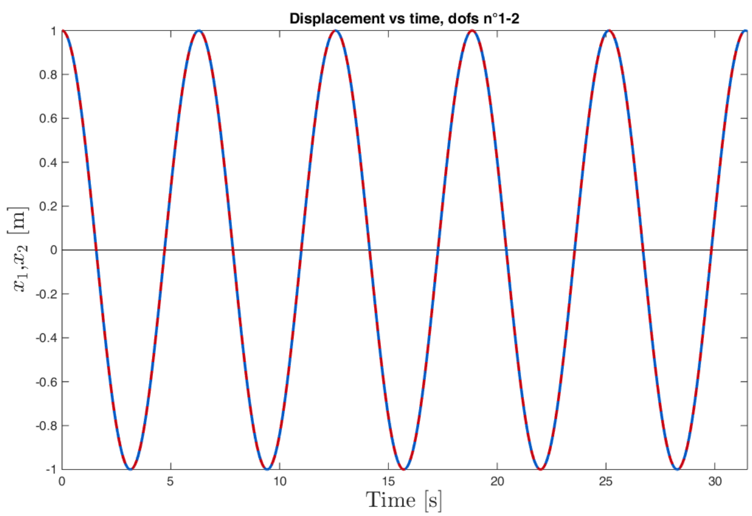


A linear mode is a time-periodic motion

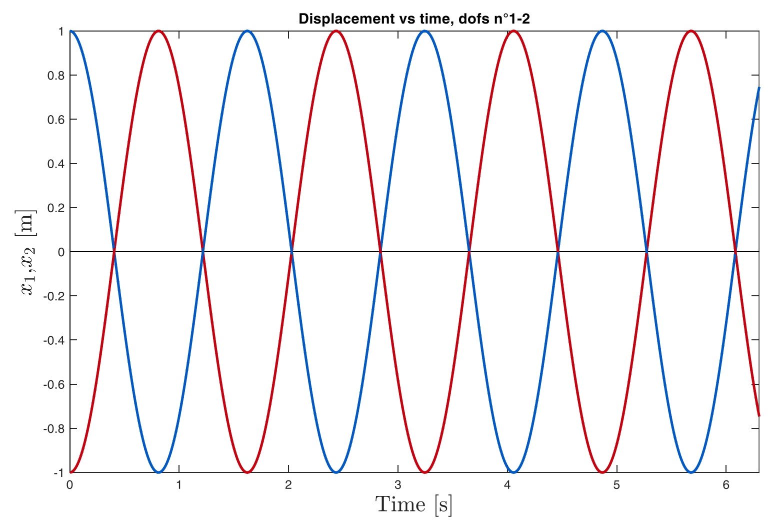
NI2D – 2DOF\Modes_Linear



Undamped and unforced system

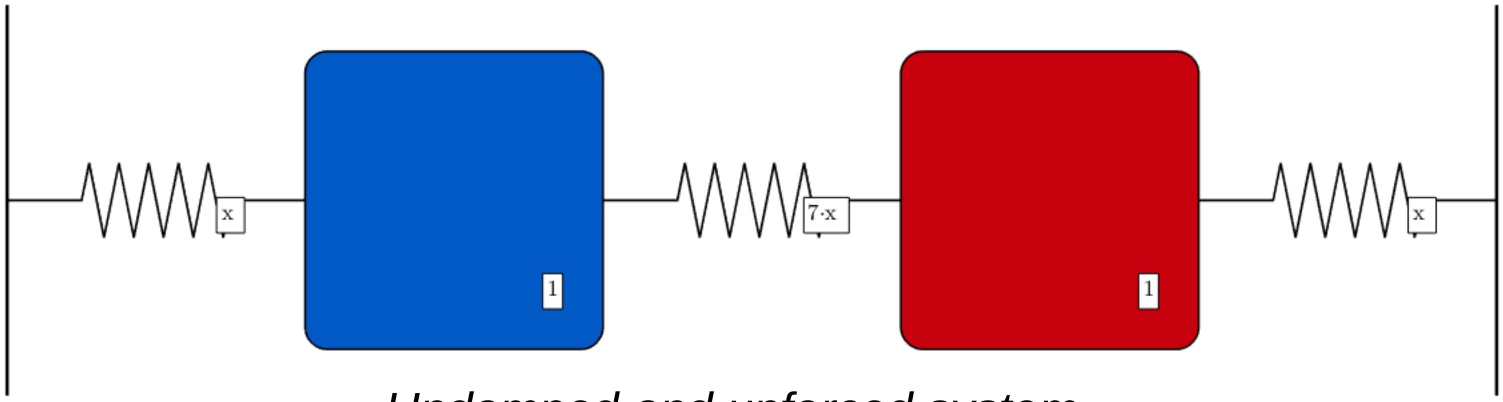


IN-PHASE MODE

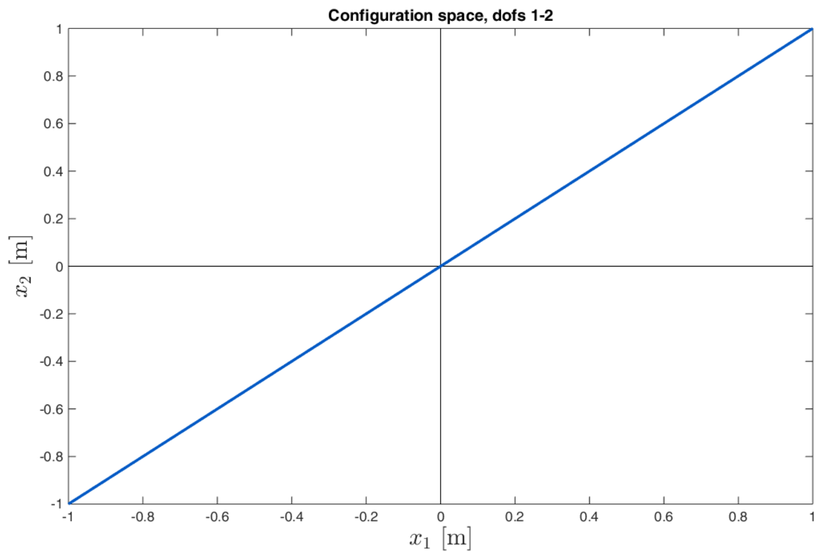


OUT-OF-PHASE MODE

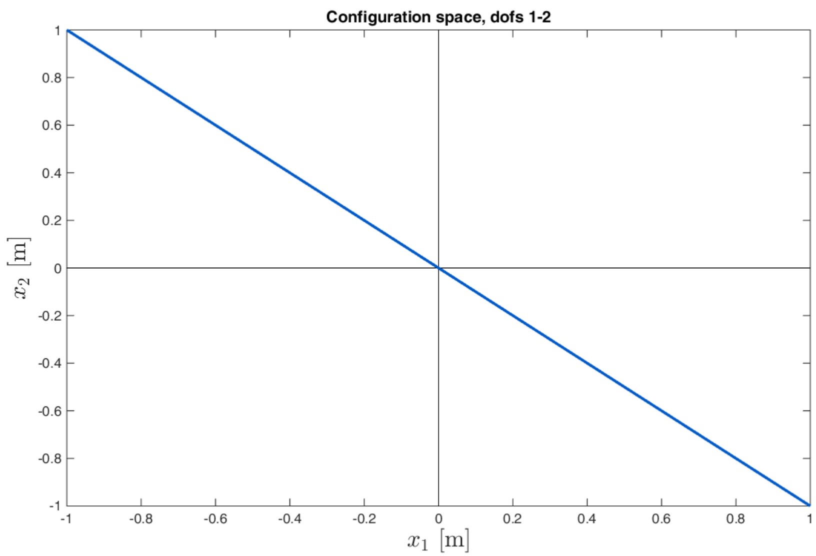
It is also a straight line in displacement space



Undamped and unforced system

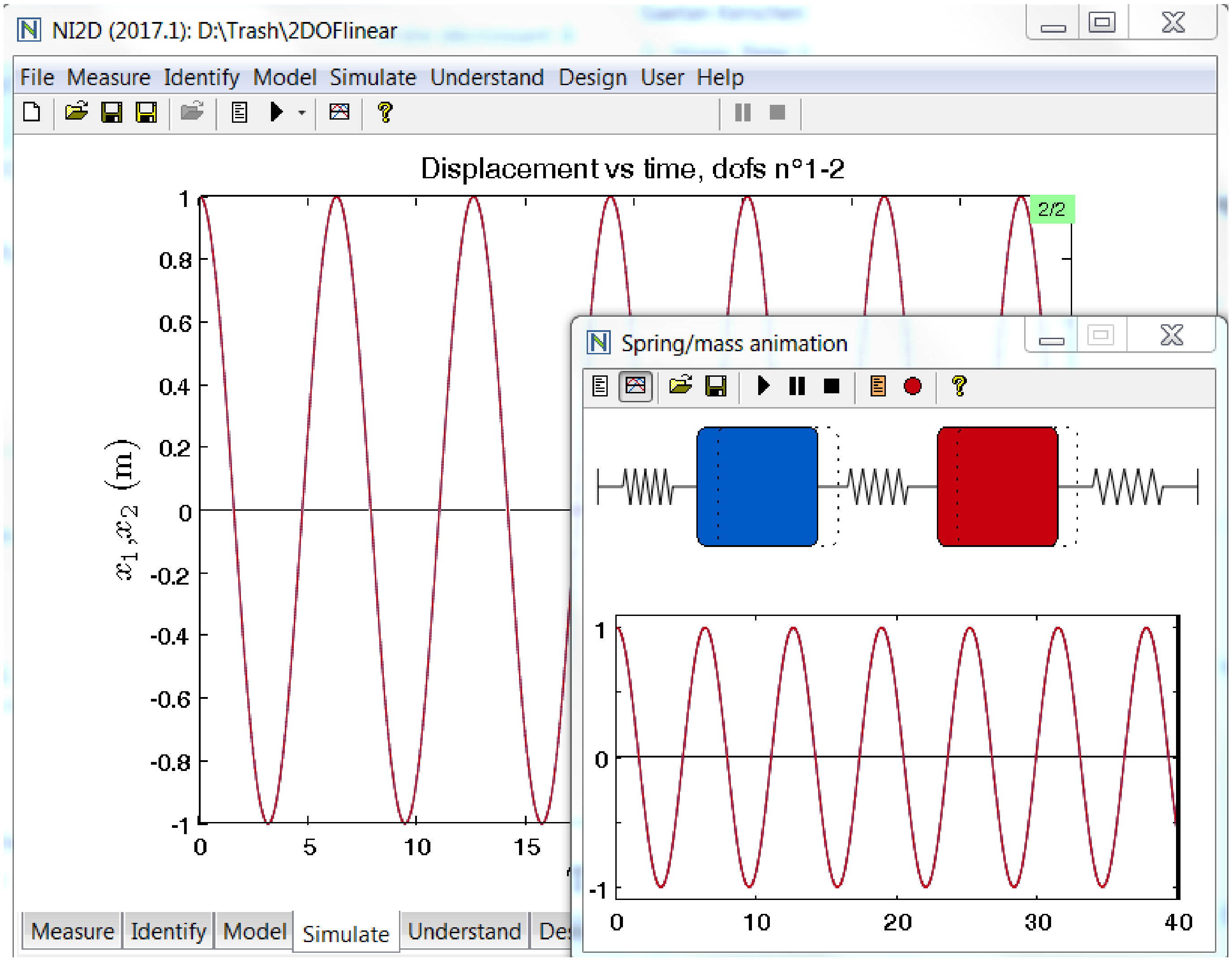


IN-PHASE MODE



OUT-OF-PHASE MODE

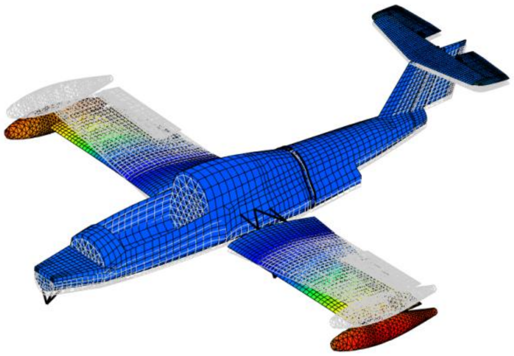
Animation



NI2D (2017.1): D:\Trash\2DOFlinear
File Measure Identify Model Simulate Understand Design User Help
Displacement vs time, dofs n°1-2
x_1, x_2 (m)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
5
10
15
2/2
Measure
Identify
Model
Simulate
Understand
Spring/mass animation
1
0
-1
0
10
20
30
40

In summary

Clear physical meaning:



- Structural deformation at resonance
- Synchronous vibration of the structure

Important mathematical properties:

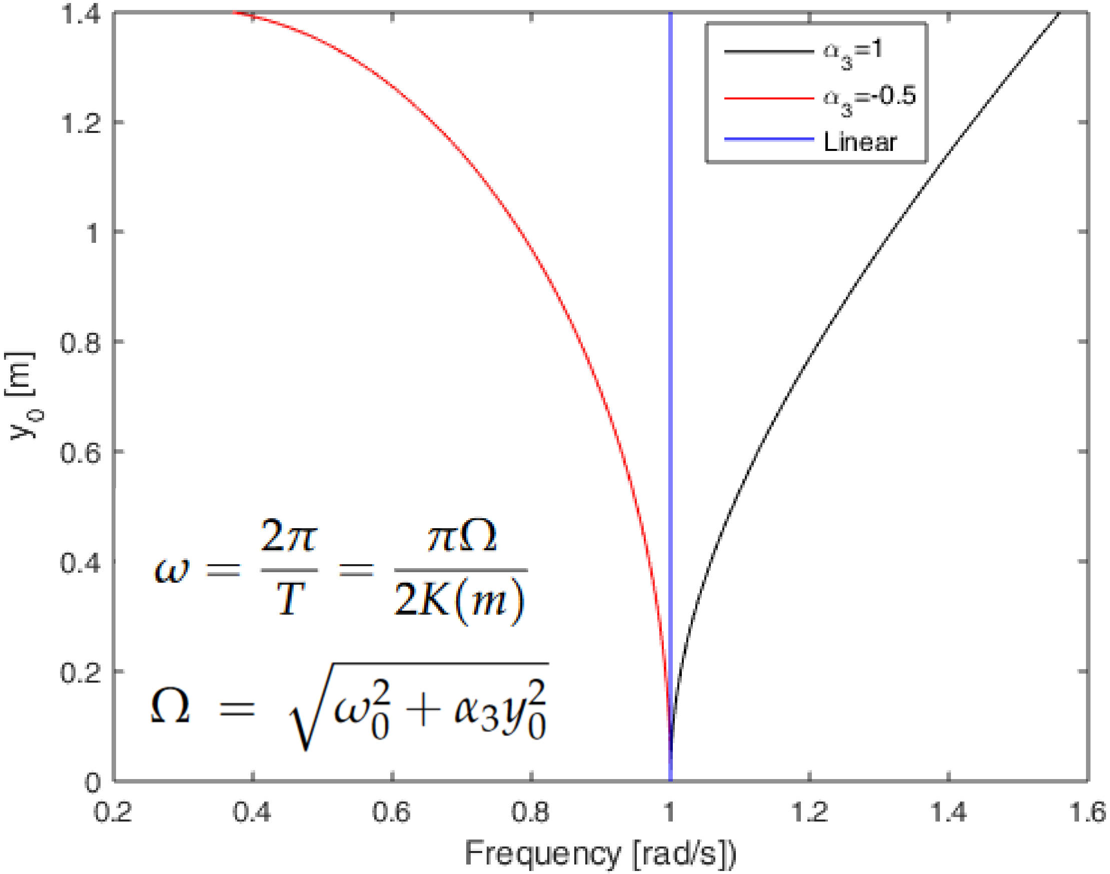
- Orthogonality
- Decoupling of the equations of motion (modal superposition)

Outline of this lecture

What are nonlinear modes ?

What are their fundamental properties ?

The 1DOF case



$$\omega = \frac{2\pi}{T} = \frac{\pi\Omega}{2K(m)}$$

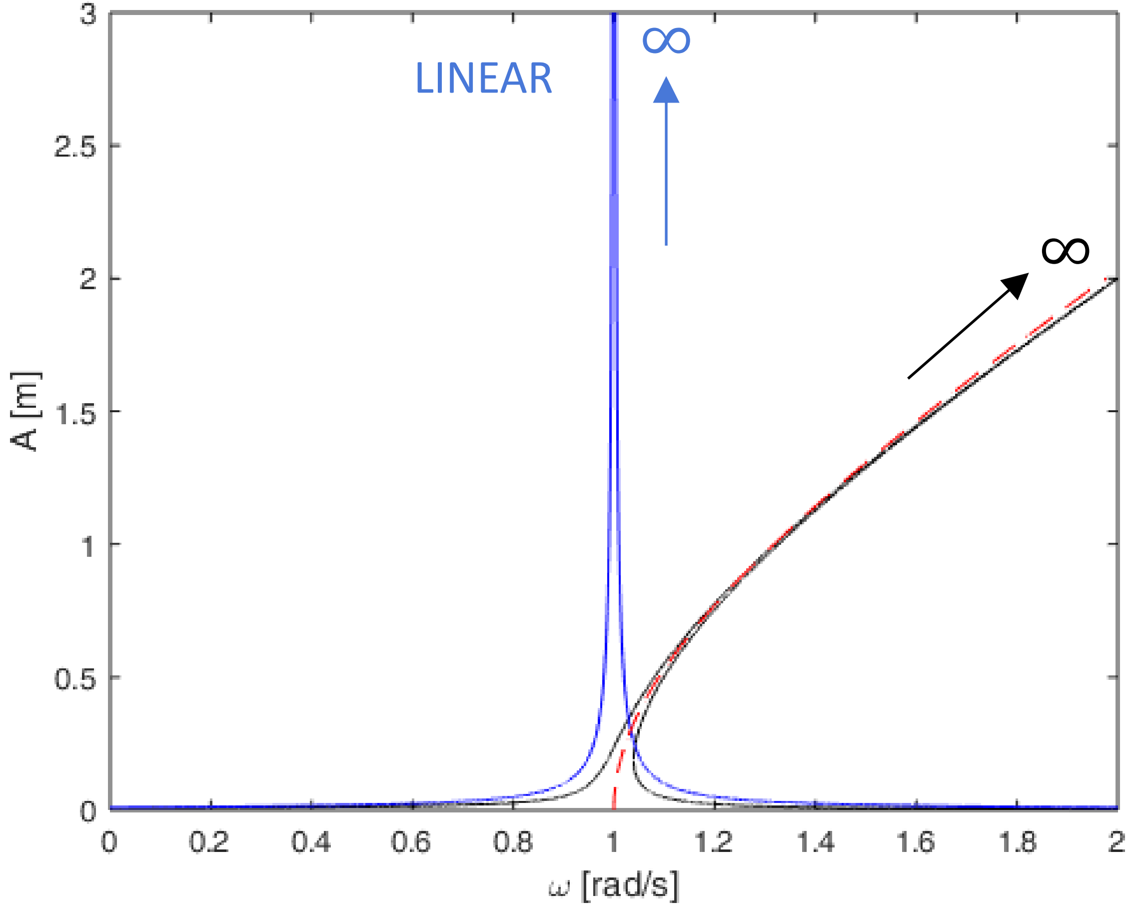
$$\Omega = \sqrt{\omega_0^2 + \alpha_3 y_0^2}$$

This highlights the frequency-amplitude dependence of nonlinear ocillations

Very important concept:
the backbone curve

The 1DOF case

$$\ddot{y}(t) + y(t) + y^3(t) = 0.01 \sin \omega t$$



The MDOF case: nonlinear normal modes

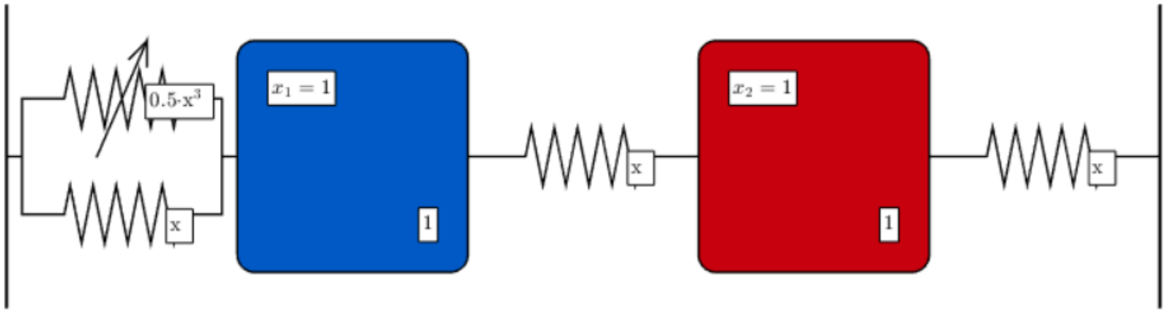
Definition due to Rosenberg (1960), couldn't be simpler !

$$M\ddot{x}(t) + Kx(t) = 0 \qquad\qquad M\ddot{x}(t) + Kx(t) + \boxed{f_{NL}[x(t)]} = 0$$

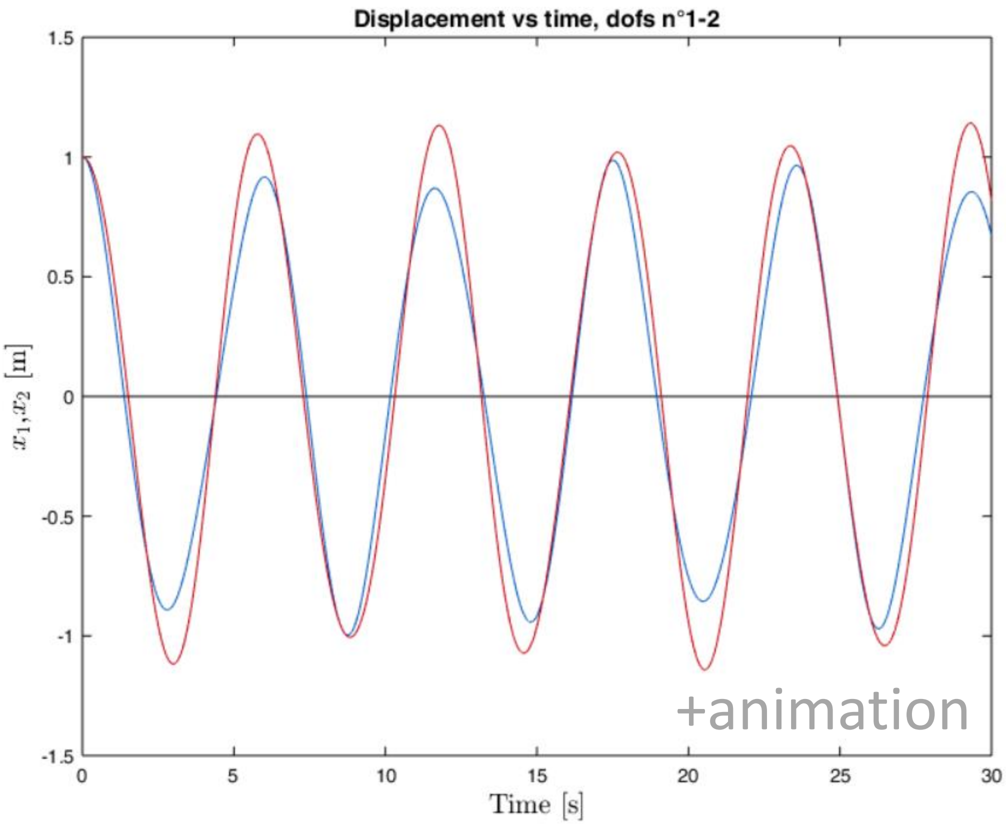
LNM: periodic motion.

NNM: periodic motion.

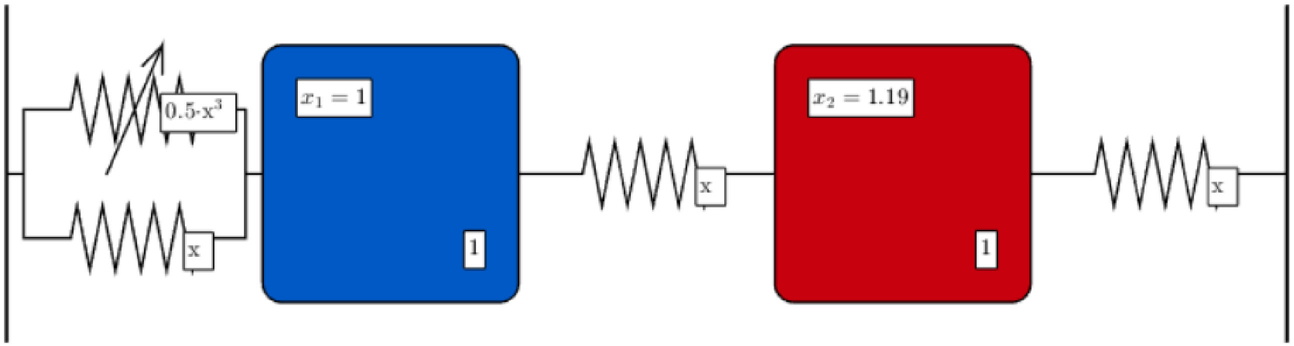
Is this a nonlinear mode ?

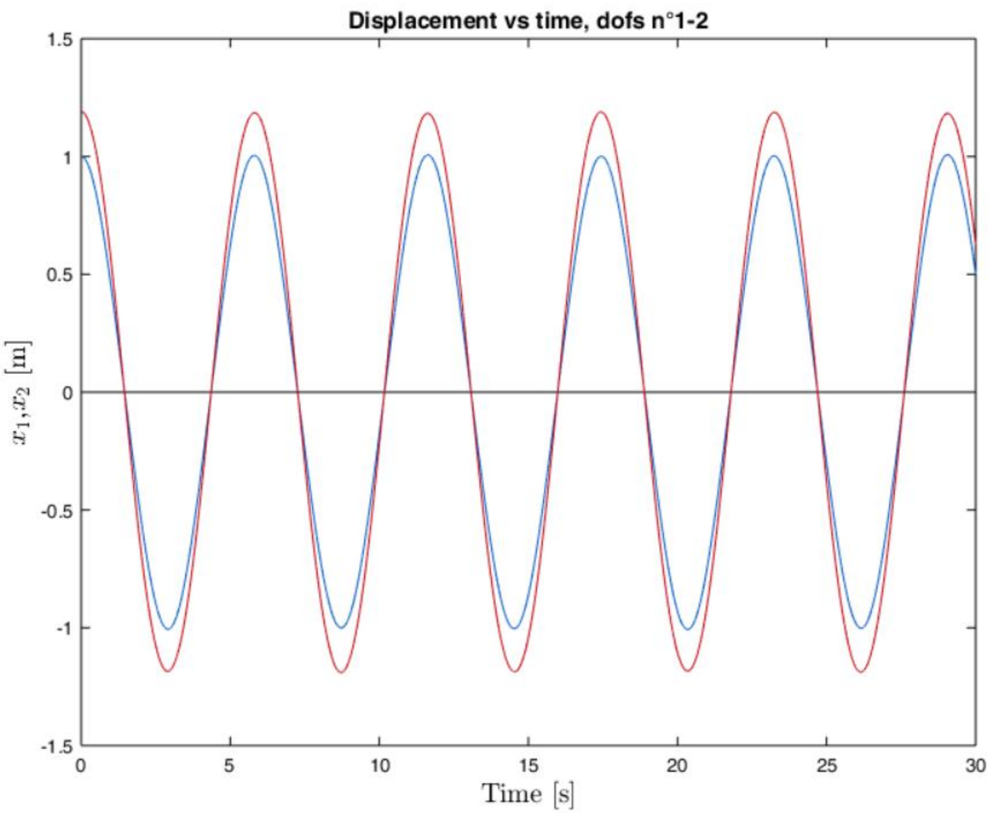


NI2D – 2DOF\NNM_ShawSystem

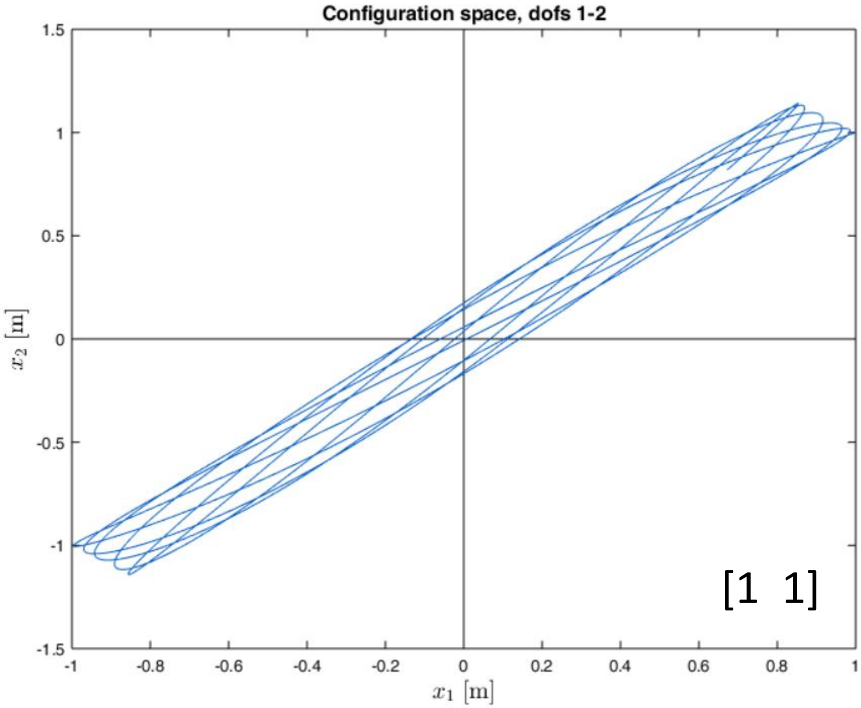


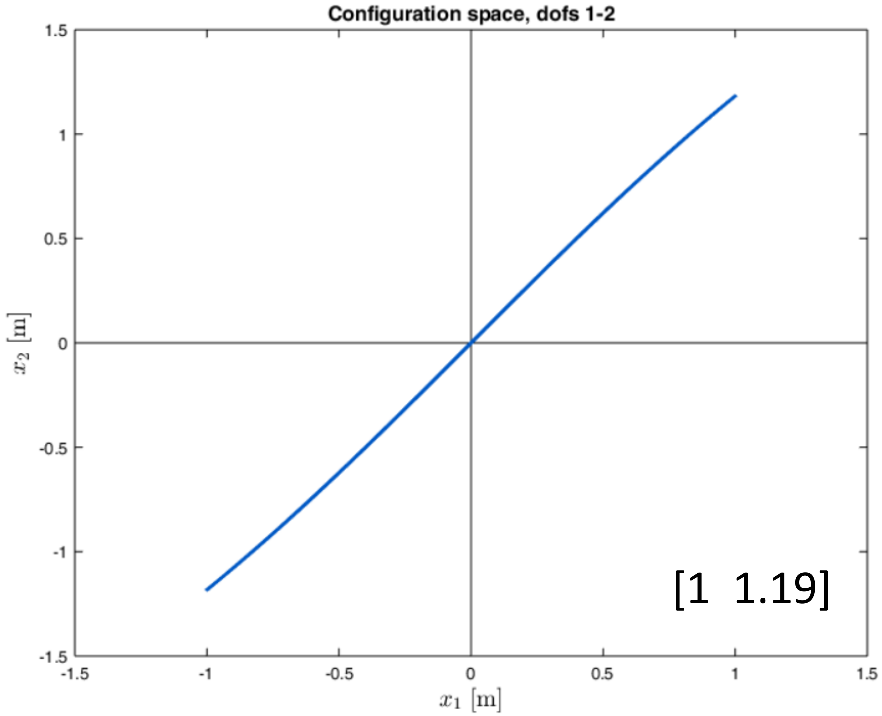
Is this a nonlinear mode ?

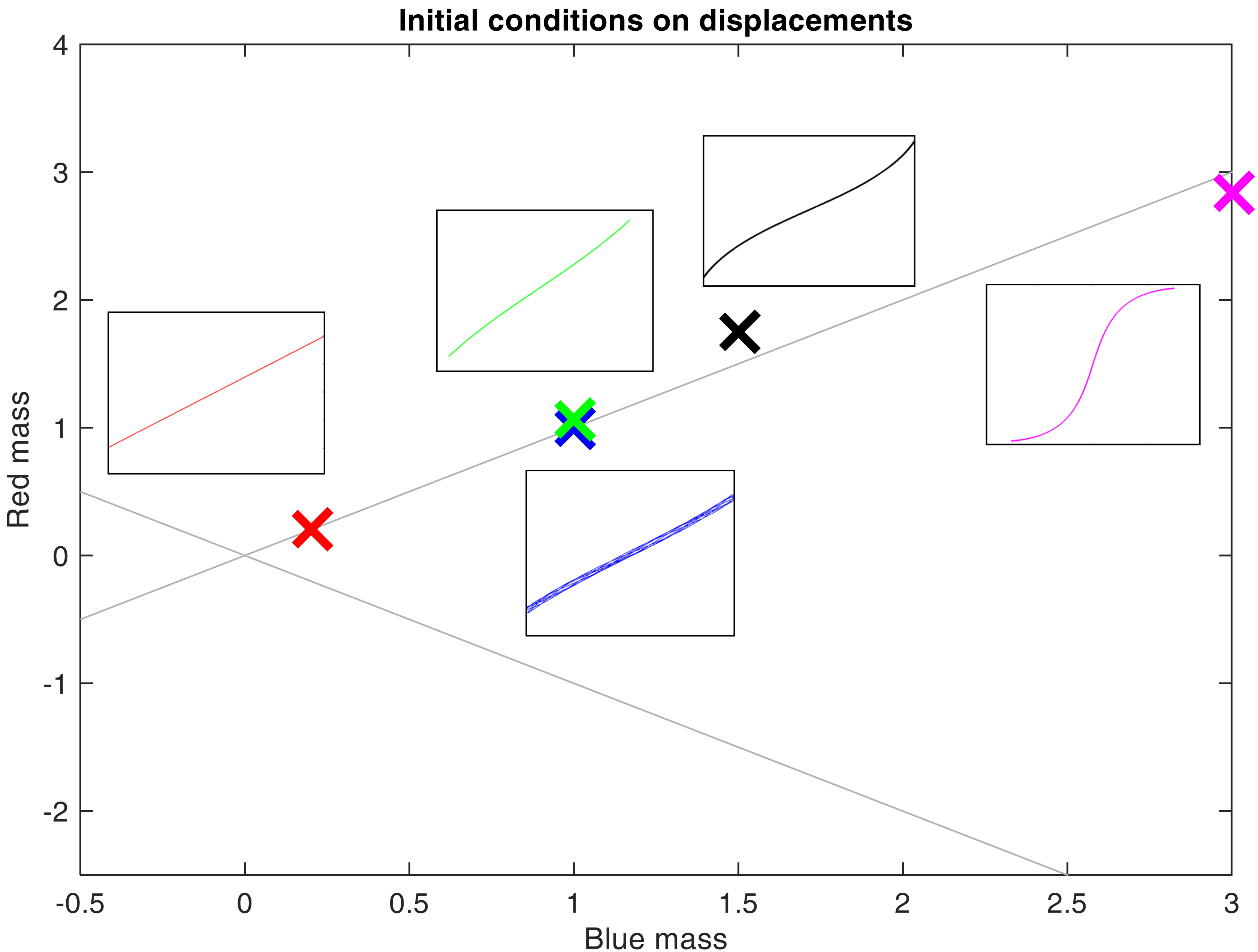




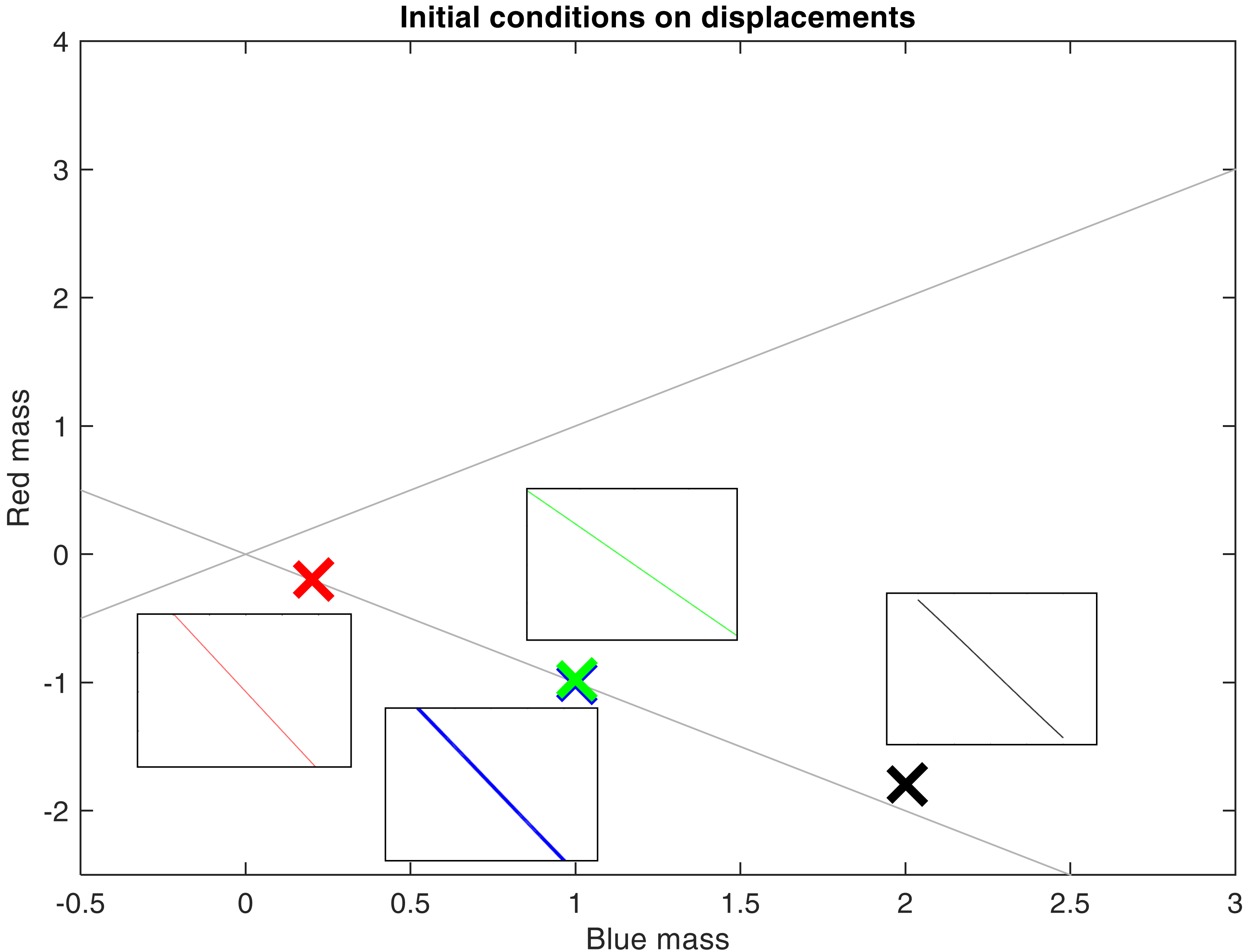
Is this a nonlinear mode ?







Initial conditions on displacements
4
3
2
1
0
-1
-2
Red mass
-0.5
0
0.5
1
1.5
2
2.5
3
Blue mass


Initial conditions on displacements
Red mass
Blue mass
4
3
2
1
0
-1
-2
-0.5
0
0.5
1
1.5
2
2.5
3

The 2DOF example

$$\ddot{q}_1 + (2q_1 - q_2) + 0.5q_1^3 = 0$$
$$\ddot{q}_2 + (2q_2 - q_1) = 0$$

How do we calculate nonlinear modes ?

Assumption of harmonic motion: $q_{1,2} \cong A, B cos\omega t$

The 1-term harmonic balance method

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\Rightarrow \cos^3\theta = \frac{1}{4}\left(\cos 3\theta + 3\cos\theta\right)$$

$$\ddot{q}_1 + 2q_1 - q_2 + \frac{1}{2}q_1^3 = 0 \longrightarrow -\omega^2 A + 2A - B + \frac{3}{8}A^3 = 0 \quad (1)$$

$$\ddot{q}_2 + 2q_2 - q_1 = 0 \qquad -\omega^2 B + 2B - A = 0 \rightarrow \boxed{B = \frac{A}{2-\omega^2}} \quad (2)$$

$$(1)\ \&\ (2) \Rightarrow -\omega^2 A + 2A - \frac{A}{2-\omega^2} + \frac{3}{8}A^3 = 0$$

$$\Rightarrow -\omega^2 + 2 - \frac{1}{2-\omega^2} + \frac{3}{8}A^2 = 0$$

$$\Rightarrow -2\omega^2 + \omega^4 + 4 - 2\omega^2 - 1 + \frac{3}{4}A^2 - \frac{3}{8}A^2\omega^2 = 0$$

$$\Rightarrow A^2\left(\frac{3}{4} - \frac{3}{8}\omega^2\right) = -\omega^4 + 4\omega^2 - 3$$

$$\Rightarrow \boxed{A = \pm\sqrt{\frac{8(\omega^2-1)(\omega^2-3)}{3(\omega^2-2)}}}$$

Fundamental difference between LNMs and NNMs

$$\ddot{q}_1 + (2q_1 - q_2) = 0$$
$$\ddot{q}_2 + (2q_2 - q_1) = 0$$

$q_{1,2} = A, B cos\omega t$

$$A = B,\ \omega_1 = 1 \text{ rad/s}$$

$$A = -B,\ \omega_2 = \sqrt{3} \text{ rad/s}$$

$$\ddot{q}_1 + (2q_1 - q_2) + 0.5q_1^3 = 0$$
$$\ddot{q}_2 + (2q_2 - q_1) = 0$$

$q_{1,2} \cong A, B cos\omega t$

$$A = \pm\sqrt{\frac{8(\omega^2 - 3)(\omega^2 - 1)}{3(\omega^2 - 2)}}$$

$$B = \frac{A}{2 - \omega^2}$$

What do you observe ?

Fundamental difference between LNMs and NNMs

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$q_{1,2} = A, B cos\omega t$

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$$B = \frac{A}{2 - \omega^2}$$

Modal shapes depend on frequency

Fundamental difference between LNMs and NNMs

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$$B = \frac{A}{2 - \omega^2}$$

The natural frequency changes
(but existence conditions !)

Fundamental difference between LNMs and NNMs

$$\ddot{q}_1 + (2q_1 - q_2) = 0$$
$$\ddot{q}_2 + (2q_2 - q_1) = 0$$

$q_{1,2} = A, B cos\omega t$

$$A = B,\ \ \omega_1 = 1\ \text{rad/s}$$

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$q_{1,2} \cong A, B cos\omega t$

$$A = \pm\sqrt{\frac{8(\omega^2 - 3)(\omega^2 - 1)}{3(\omega^2 - 2)}}$$

$$B = \frac{A}{2 - \omega^2}$$

$$\omega_1 \in \left[1, \sqrt{2}\right[\ \ \text{rad/s}$$

$$\omega_2 \in \left[\sqrt{3}, +\infty\right[\ \ \text{rad/s}$$

Existence conditions for NNM

1. Frequency-energy dependence

Useful graphical representation

$$\ddot{q}_1 + (2q_1 - q_2) + 0.5q_1^3 = 0$$
$$\ddot{q}_2 + (2q_2 - q_1) = 0$$

Initial conditions: $[q_1(0) \quad q_2(0) \quad \dot{q}_1(0) \quad \dot{q}_2(0)] = [A \quad B \quad 0 \quad 0]$

Total energy =
initial potential energy : $\mathrm{E} = \mathrm{V} = \frac{A^2}{2} + \frac{(B-A)^2}{2} + \frac{B^2}{2} + \frac{0.5A^4}{4}$

A frequency-energy plot is calculated by

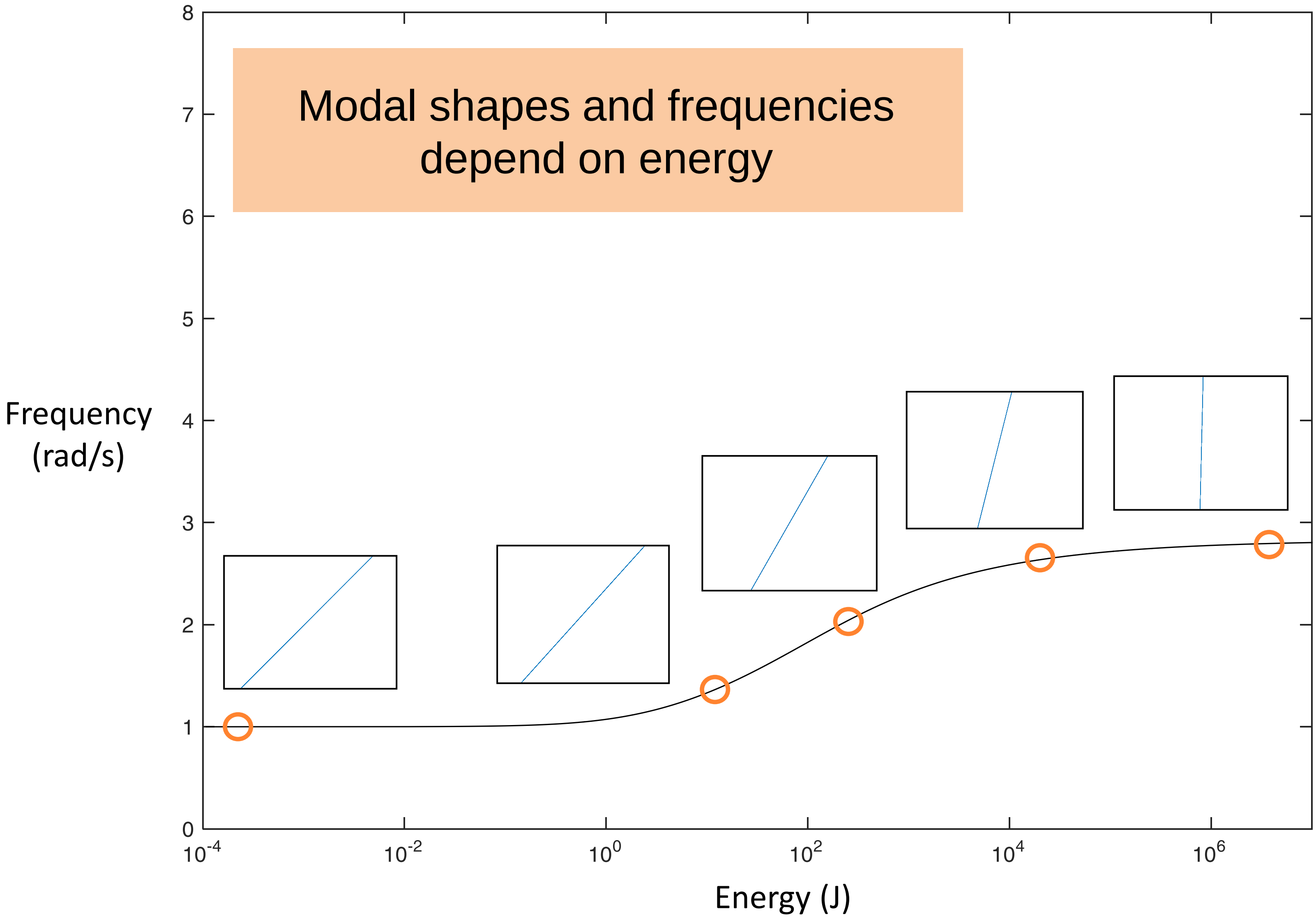
- Selecting a frequency in the interval provided by the existence conditions,
- Calculating A and B according to the analytical formulas
- Calculating the corresponding total energy
- Representing the frequency as a function of the total energy

In Matlab

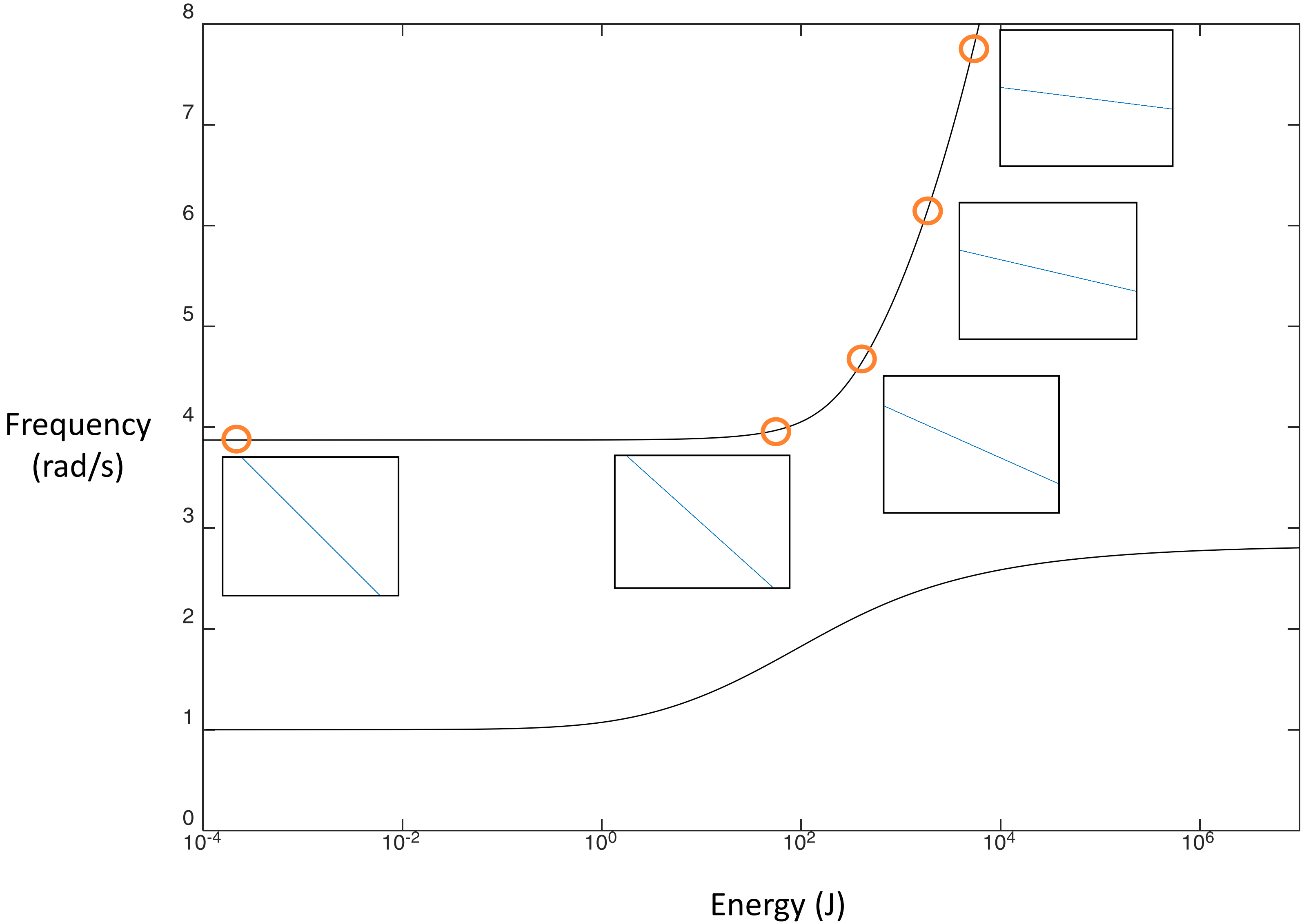
HB1_2DOF_FEP.m

```matlab
clear all
close all

cpt=1;
for omeg=1.00001:0.001:sqrt(2)
    A=sqrt(8*(omeg^2-3)*(omeg^2-1)/3/((omeg^2-2)));
    B=A/(2-omeg^2);
    NRJ(cpt)=(A-B)^2/2+A^2/2+B^2/2+0.5*A^4/4;
    freq(cpt)=omeg;
    AIP(cpt)=A;
    BIP(cpt)=B;
    cpt=cpt+1;
end
semilogx(NRJ,freq,'k')

cpt=1;
for omeg=sqrt(3)+0.0000001:0.001:4
    A=sqrt(8*(omeg^2-3)*(omeg^2-1)/3/((omeg^2-2)));
    B=A/(2-omeg^2);
    NRJ2(cpt)=(A-B)^2/2+A^2/2+B^2/2+0.5*A^4/4;
    freq2(cpt)=omeg;
    AOP(cpt)=A;
    BOP(cpt)=B;
    cpt=cpt+1;
end
hold on
semilogx(NRJ2,freq2,'k')
```

The in-phase NNM in the FEP



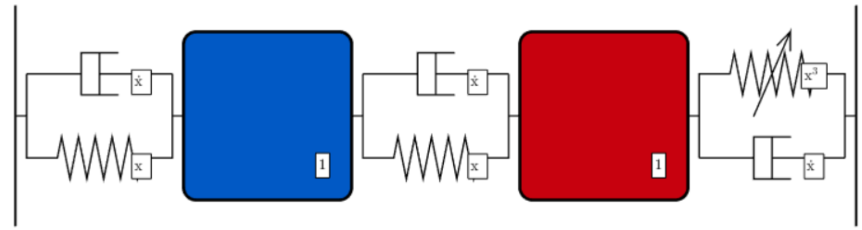
The out-of-phase NNM in the FEP

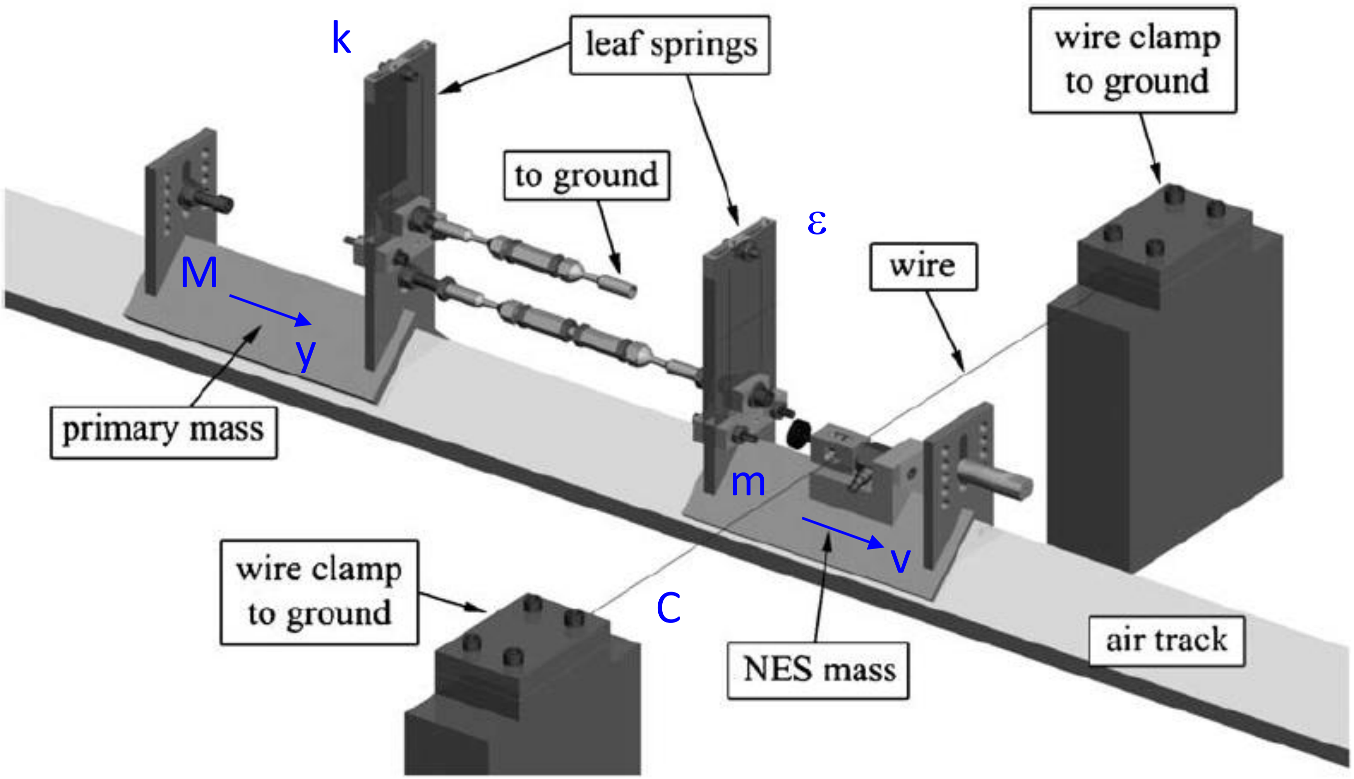


Experimental evidence of frequency-energy dependence

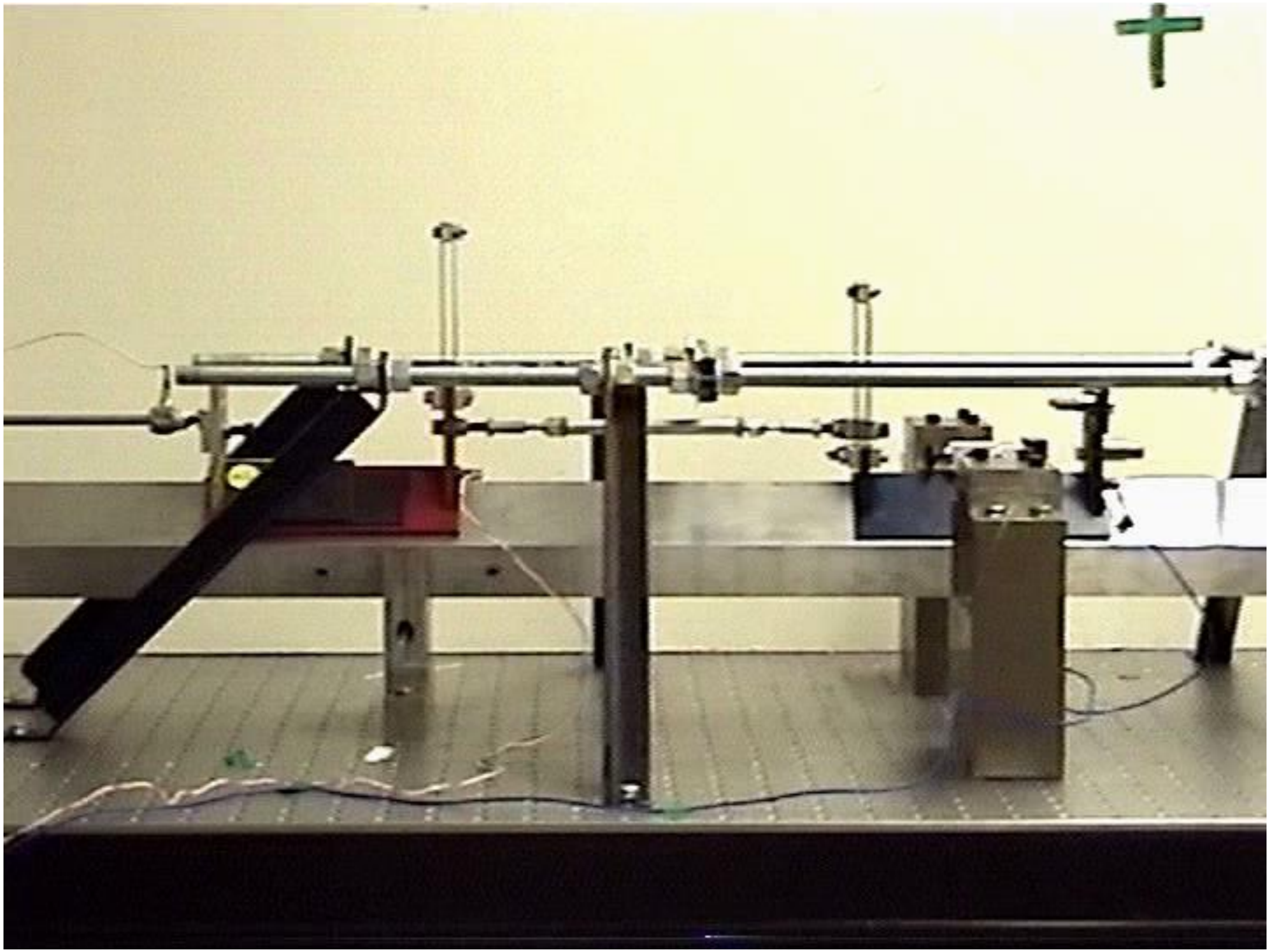
$$M\ddot{y} + \epsilon\lambda_1\dot{y} + \epsilon\lambda(\dot{y}-\dot{v}) + \epsilon(y-v) + ky = 0$$

$$m\ddot{v} + \epsilon\lambda_2\dot{v} + \epsilon\lambda(\dot{v}-\dot{y}) + \epsilon(v-y) + Cv^3 = 0$$

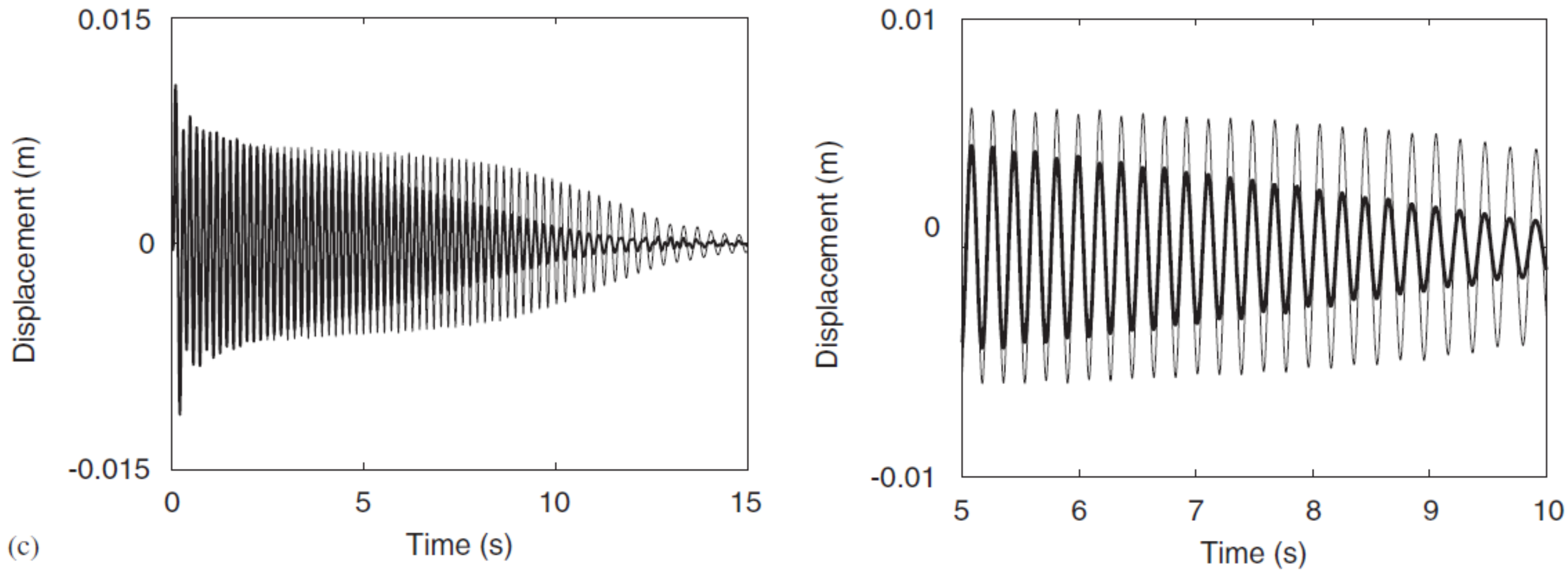




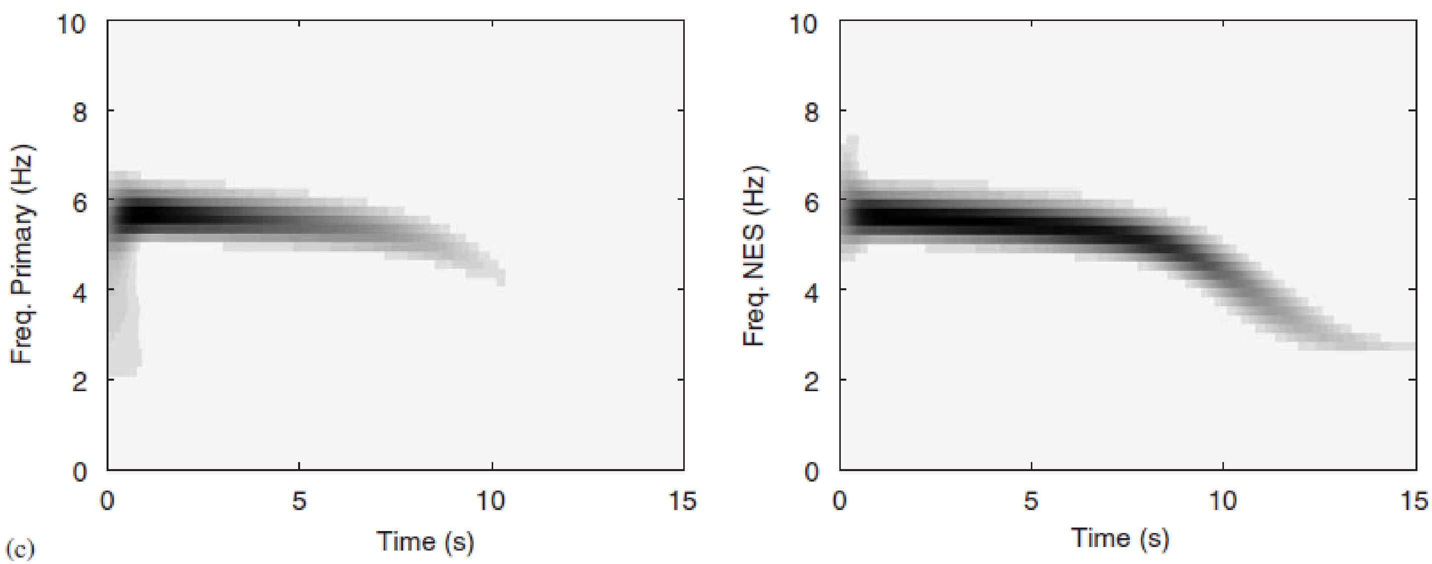
What you see is a nonlinear mode



Time series and time frequency analysis

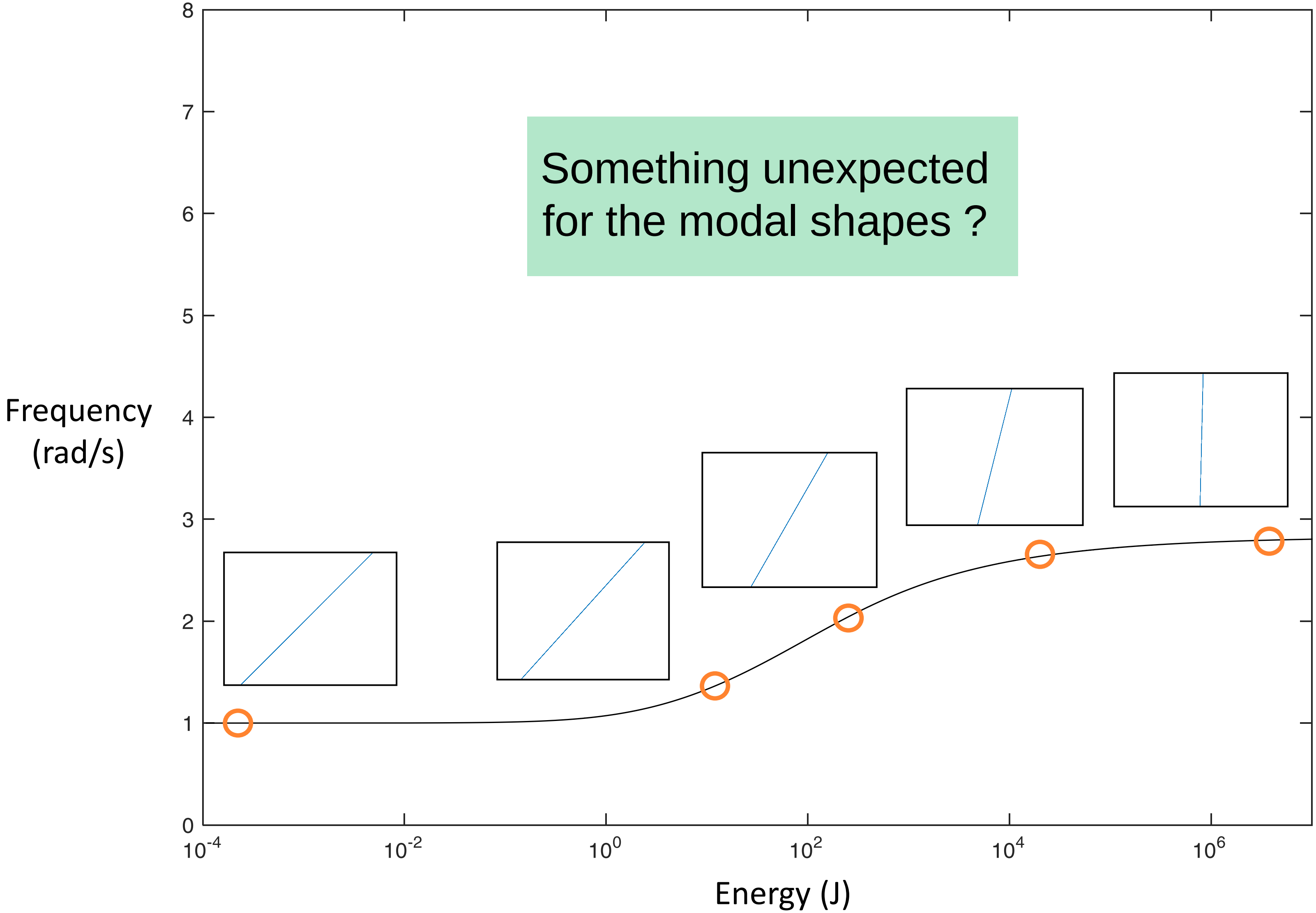


(c)



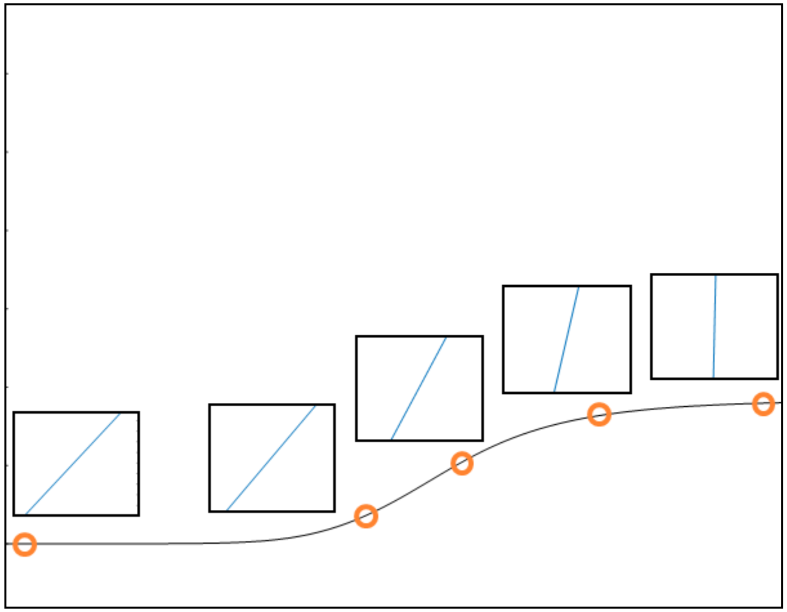
(c)

The in-phase NNM (1-term HB)

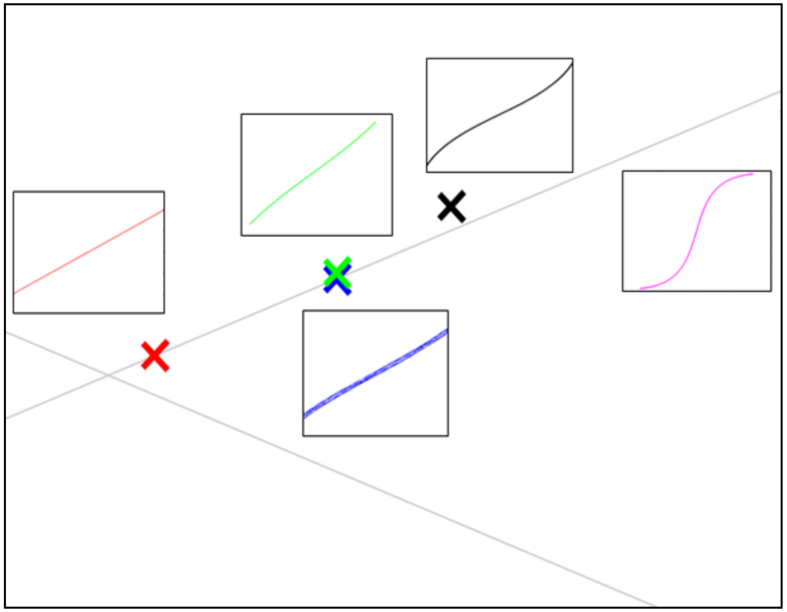


How come ?

HB with 1 harmonics



Numerical simulation



Limitation of a 1-term harmonic balance method

$$\ddot{q}_1 + (2q_1 - q_2) + 0.5q_1^3 = 0$$
$$\ddot{q}_2 + (2q_2 - q_1) = 0$$

$q_{1,2} \cong A, B cos\omega t$

$$A = \pm\sqrt{\frac{8(\omega^2 - 3)(\omega^2 - 1)}{3(\omega^2 - 2)}}$$

$$B = \frac{A}{2 - \omega^2}$$

$$q_1 = \frac{A}{B} q_2 = \frac{8 - \omega^2}{7} q_2$$

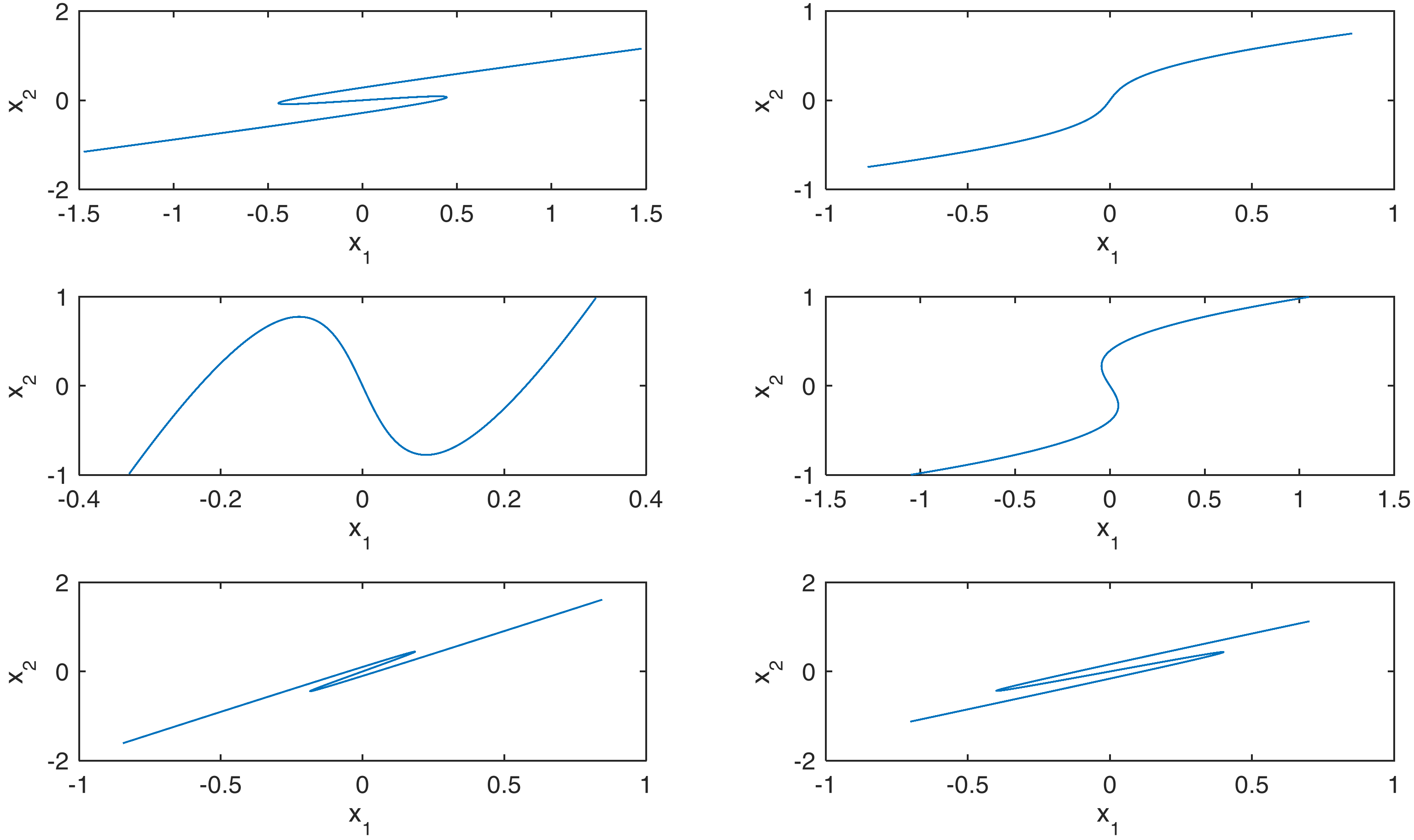
Frequency-dependent linear relation imposed between x_1 *and* x_2

2. Harmonics

The mode curvature is induced by harmonics

$$x_1 = A cos\omega t + B cos 3\omega t$$

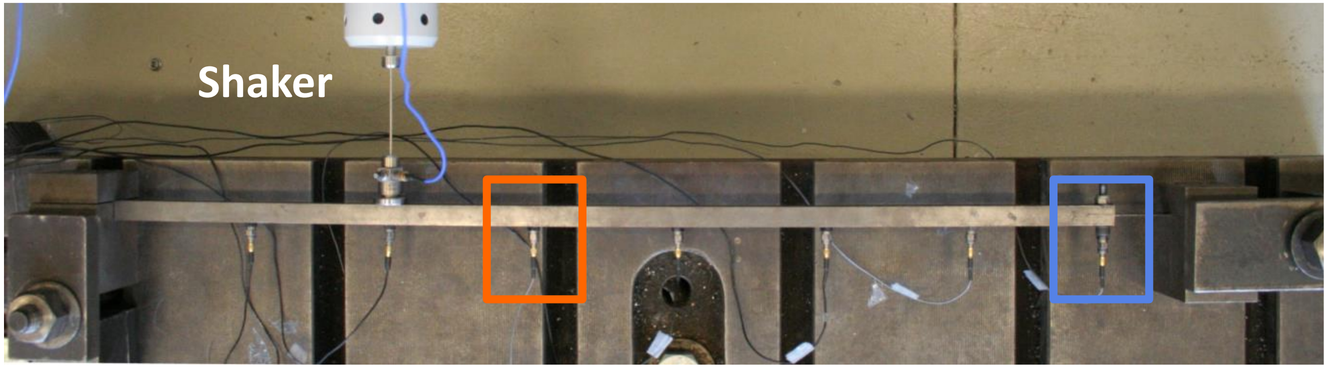
$$x_2 = C cos\omega t + D cos 3\omega t$$

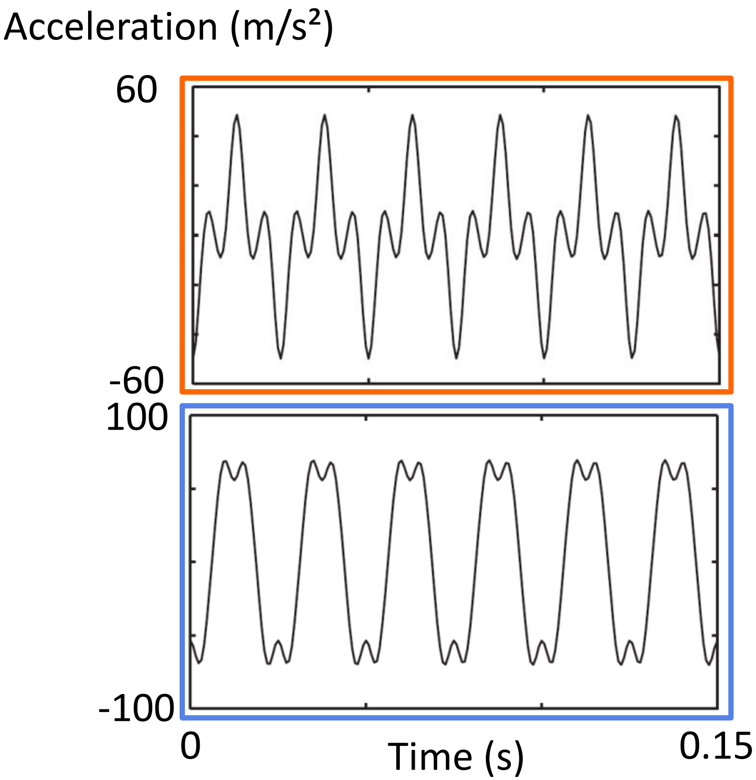


In Matlab

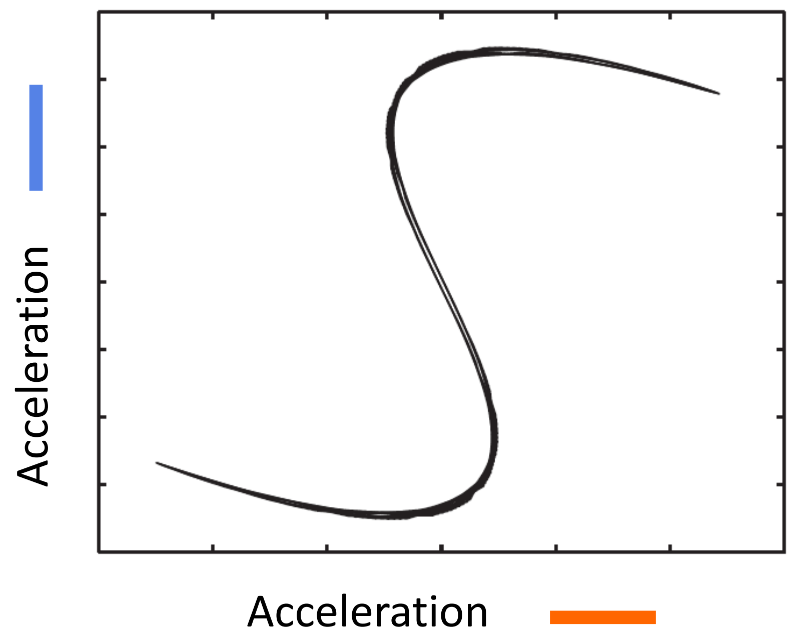
```
f=input('1=sin/sin 2=cos/cos: ');
w1=input('Enter harmonics 1:');
w2=input('Enter harmonics 2:');

temps=[0:0.01:50];

if f==1
    for k=1:6
        x1=rand(1)*sin(w1*temps)+rand(1)*sin(w2*temps);
        x2=rand(1)*sin(w1*temps)+rand(1)*sin(w2*temps);
        subplot(3,2,k)
        plot(x1,x2)
        xlabel('x_1');
        ylabel('x_2');
        set(gcf,'uni','nor','pos',[0.2 0.2 0.6 0.6])
    end
else
    for k=1:6
        x1=rand(1)*cos(w1*temps)+rand(1)*cos(w2*temps);
        x2=rand(1)*cos(w1*temps)+rand(1)*cos(w2*temps);
        subplot(3,2,k)
        plot(x1,x2)
        xlabel('x_1');
        ylabel('x_2');
        set(gcf,'uni','nor','pos',[0.2 0.2 0.6 0.6])
    end
end
```

What you see is a *real* nonlinear mode

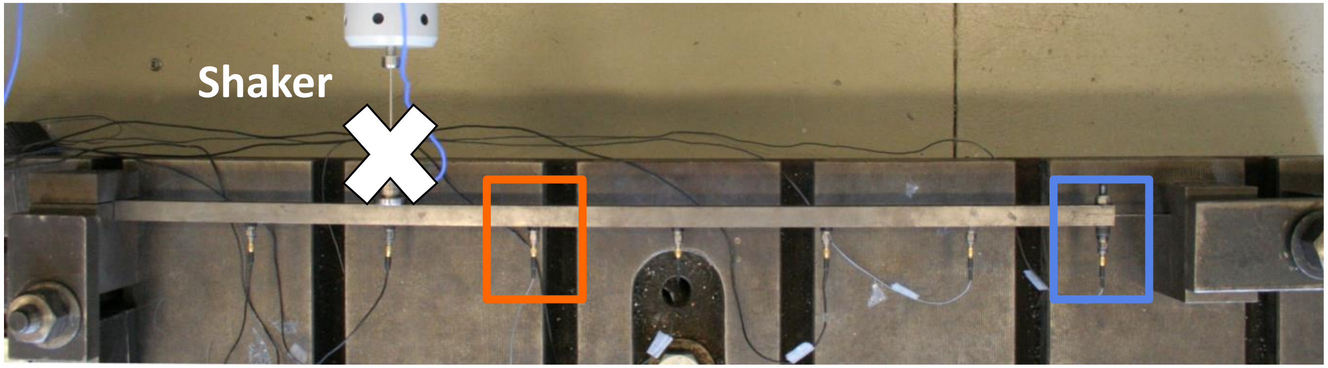


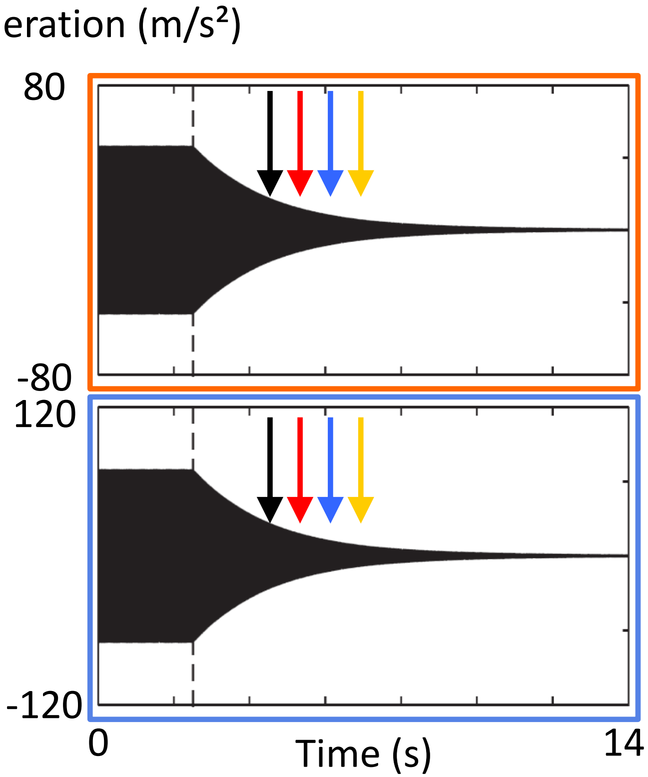


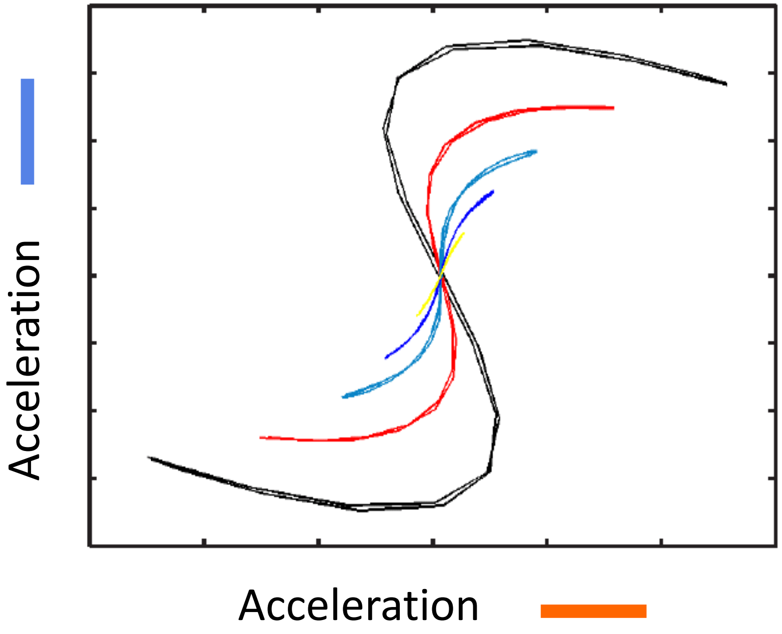
Nonlinear modes exhibit harmonics



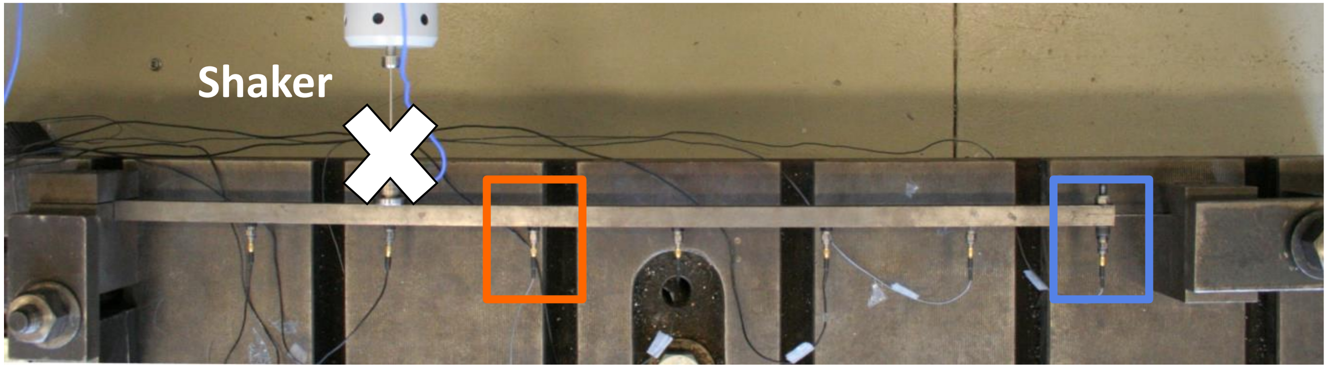
The mode shapes evolve with time

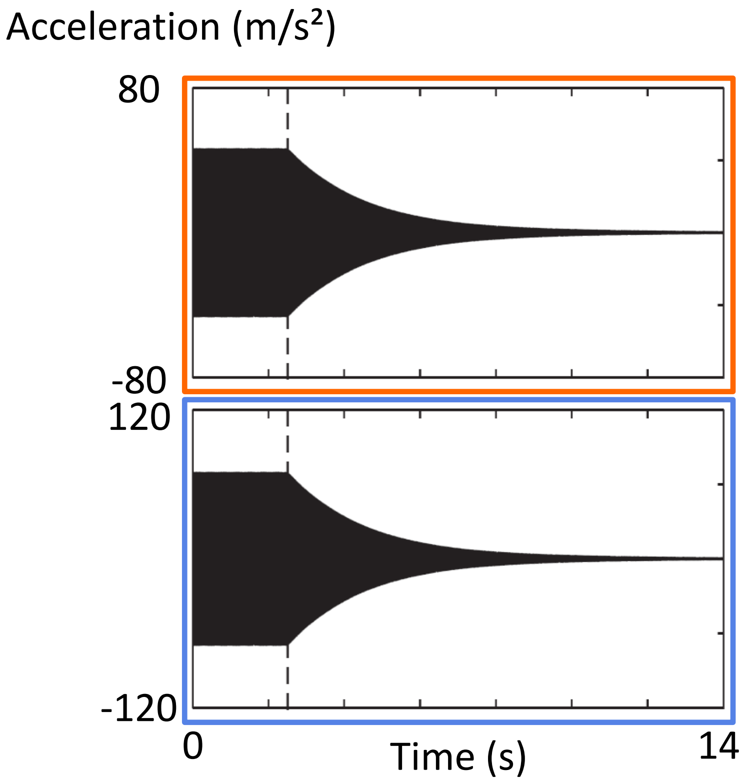


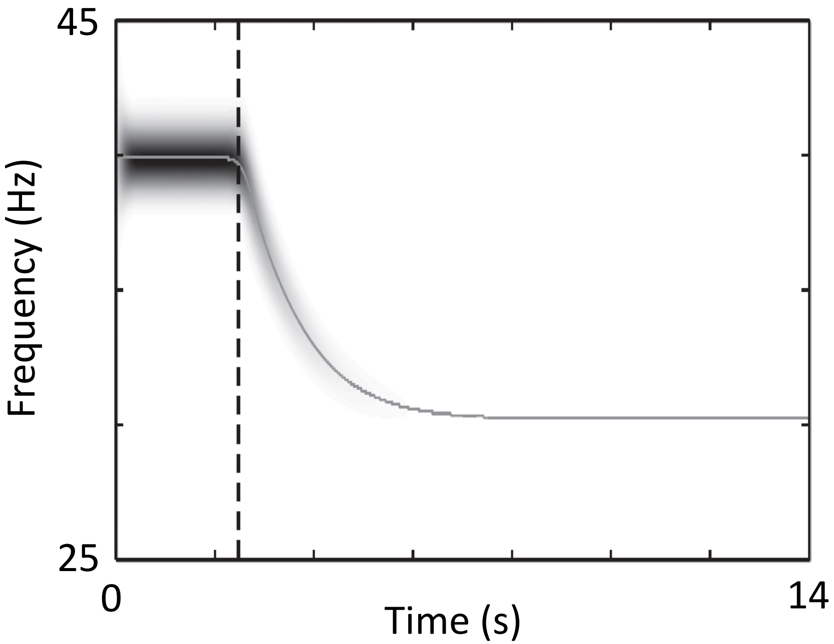




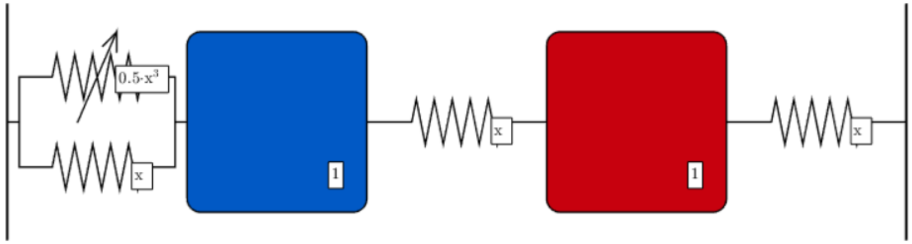
The natural frequency decreases with time



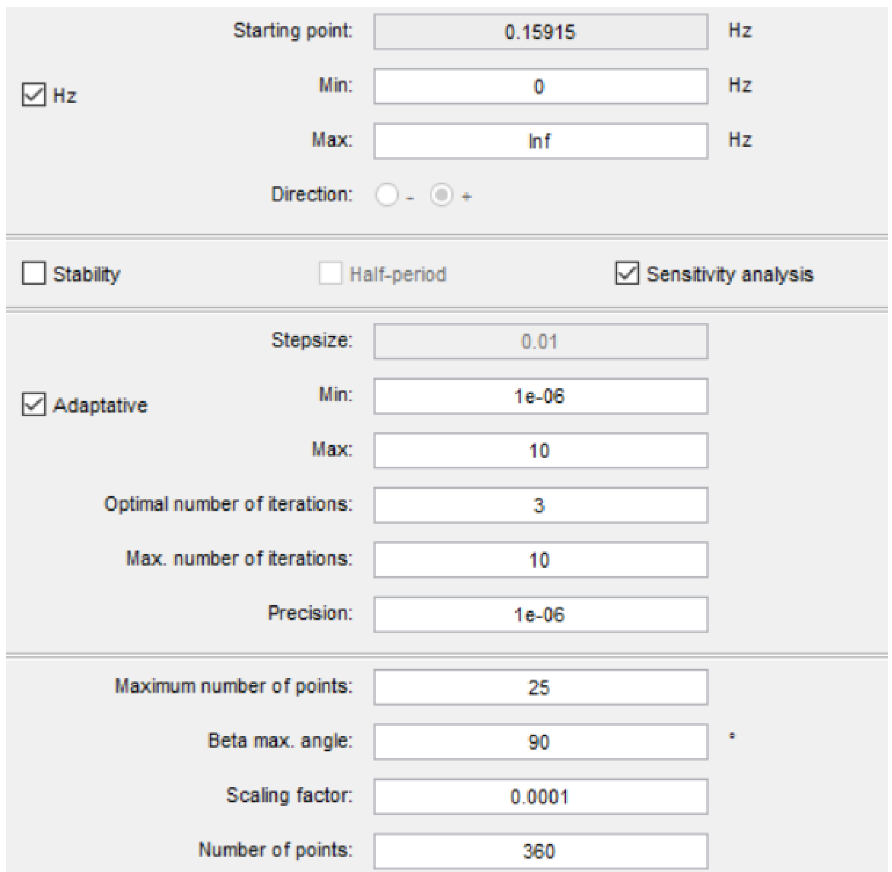




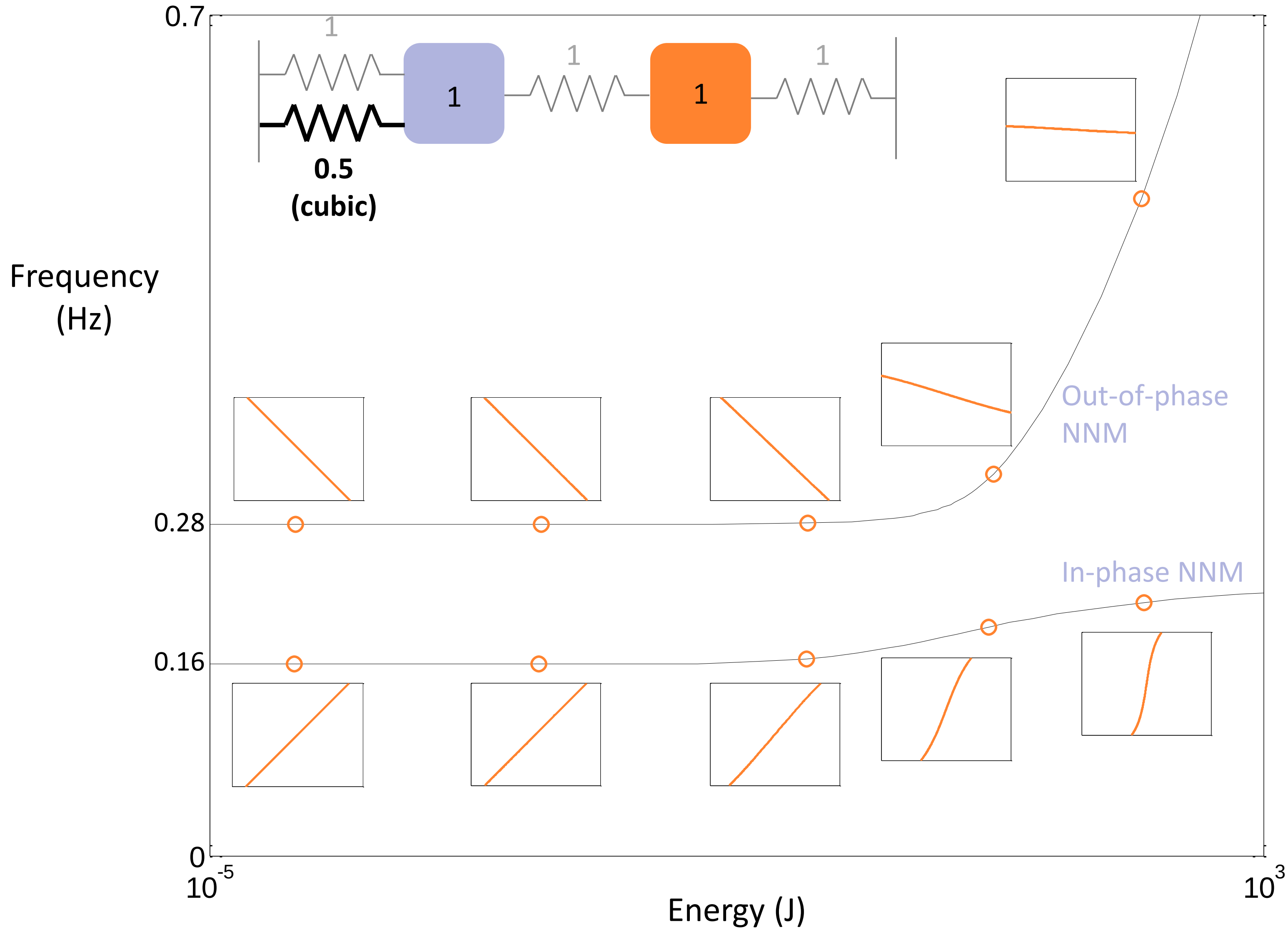
Numerical calculation



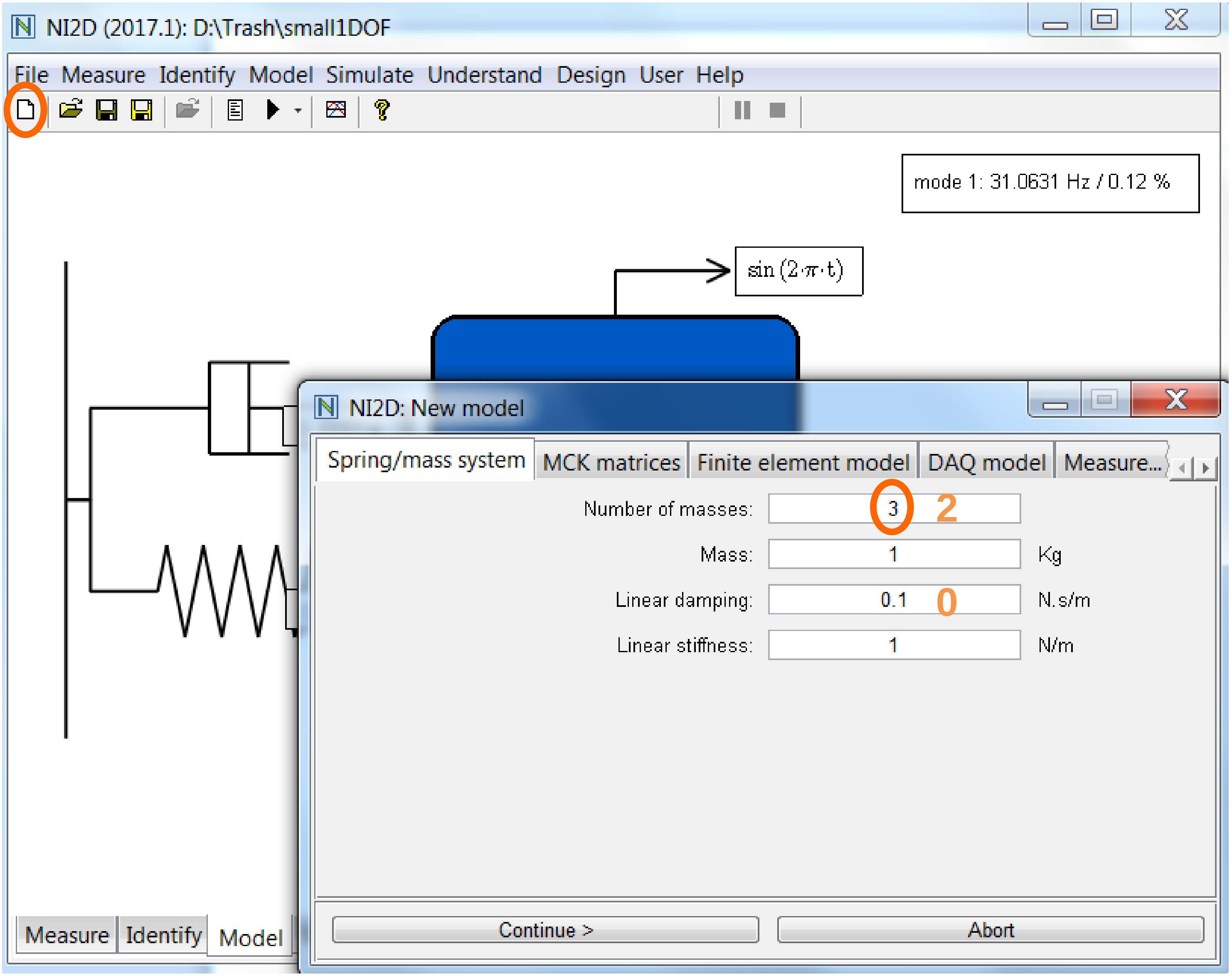
NI2D – 2DOF\NNM_ShawSystem



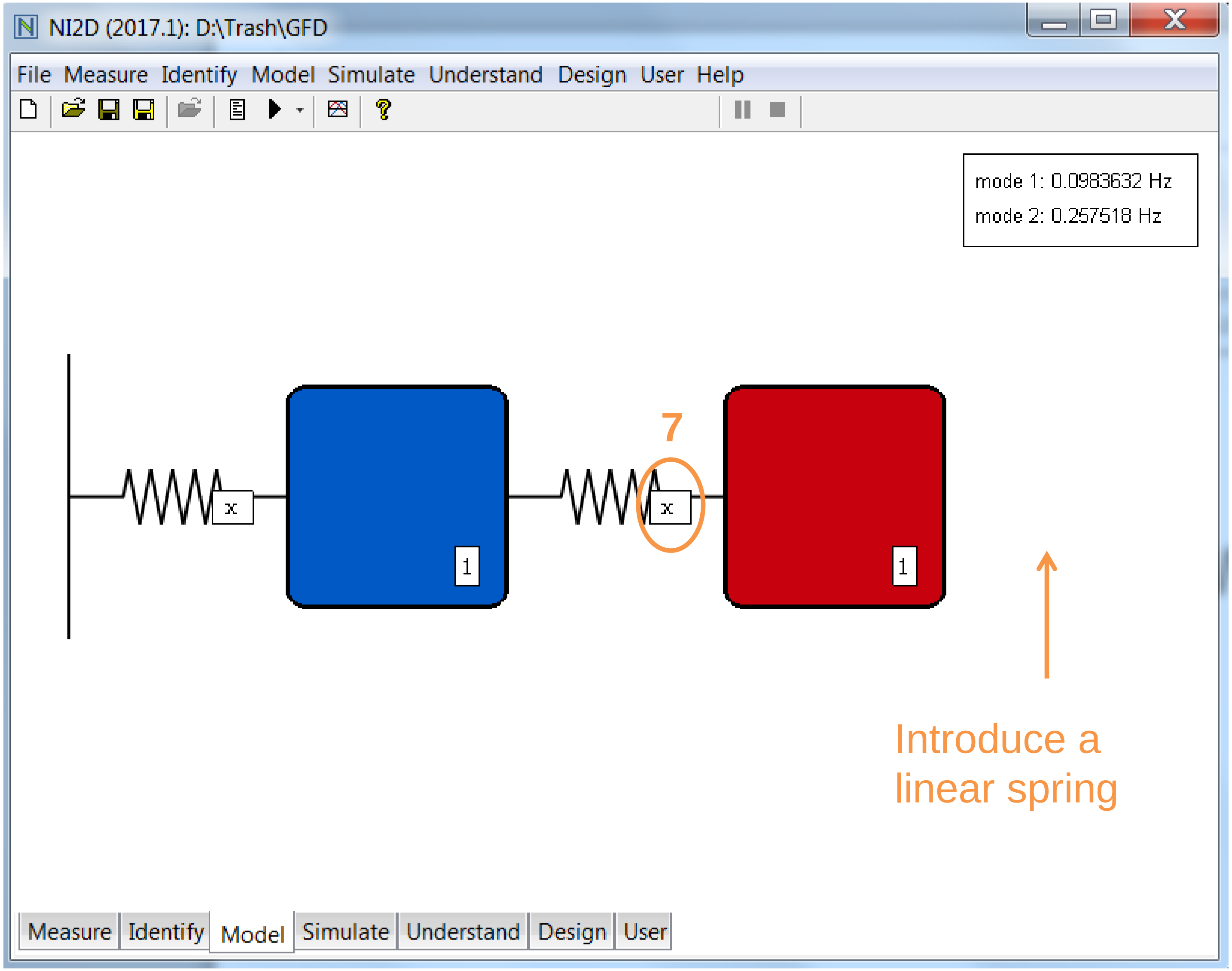
« Curved » nonlinear modes are now obtained



Do it yourself in NI2D: create a 2-DOF model

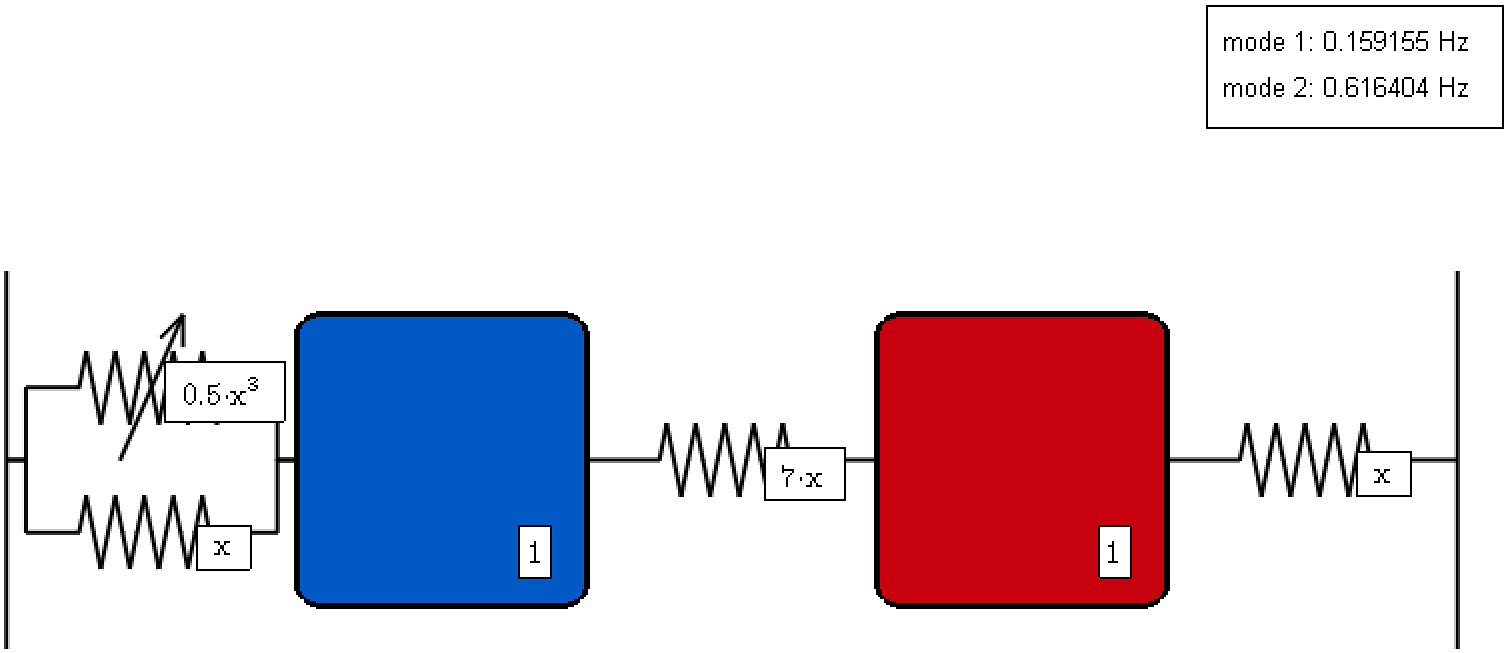


Modify the 2-DOF model

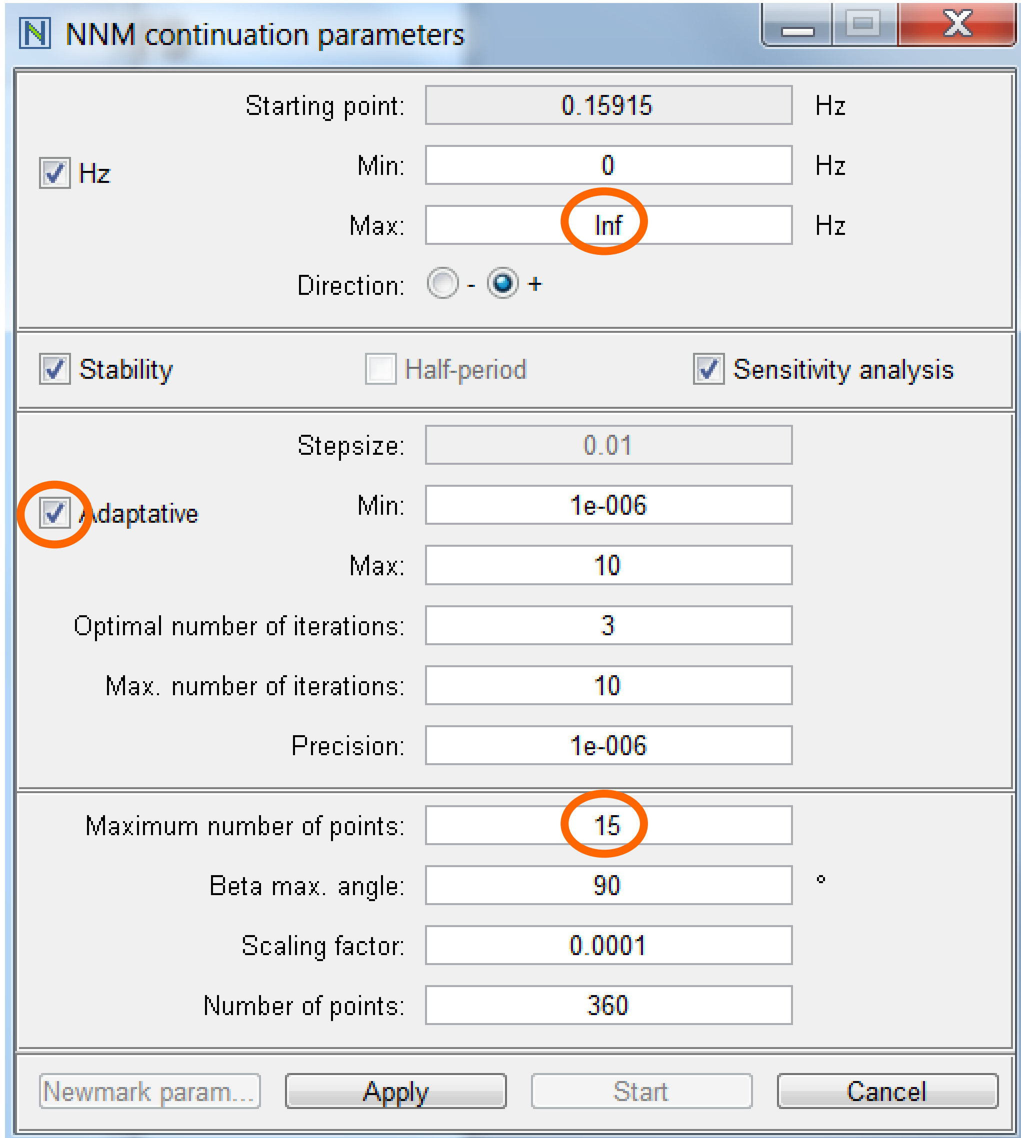


Introduce a linear spring

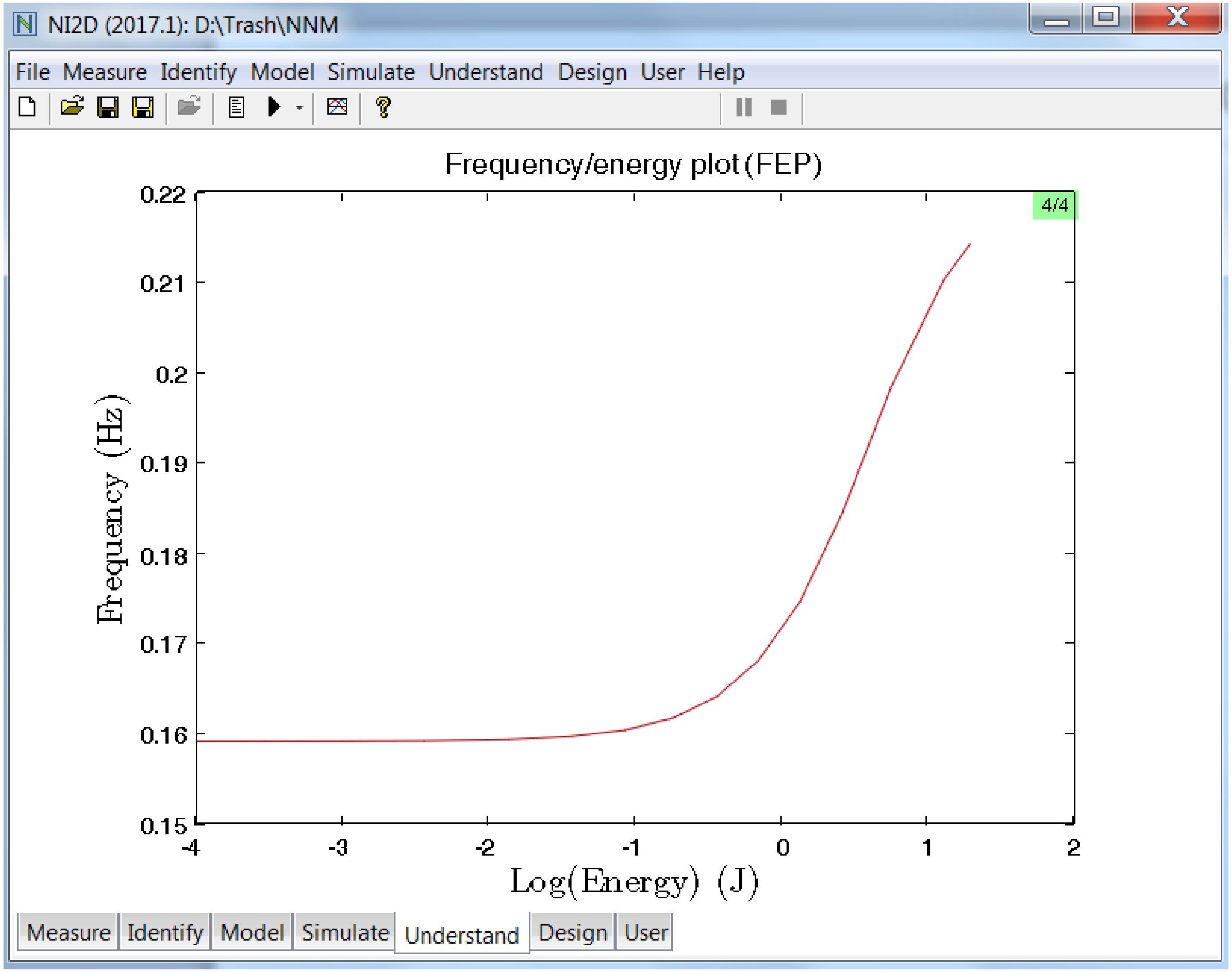
The final model



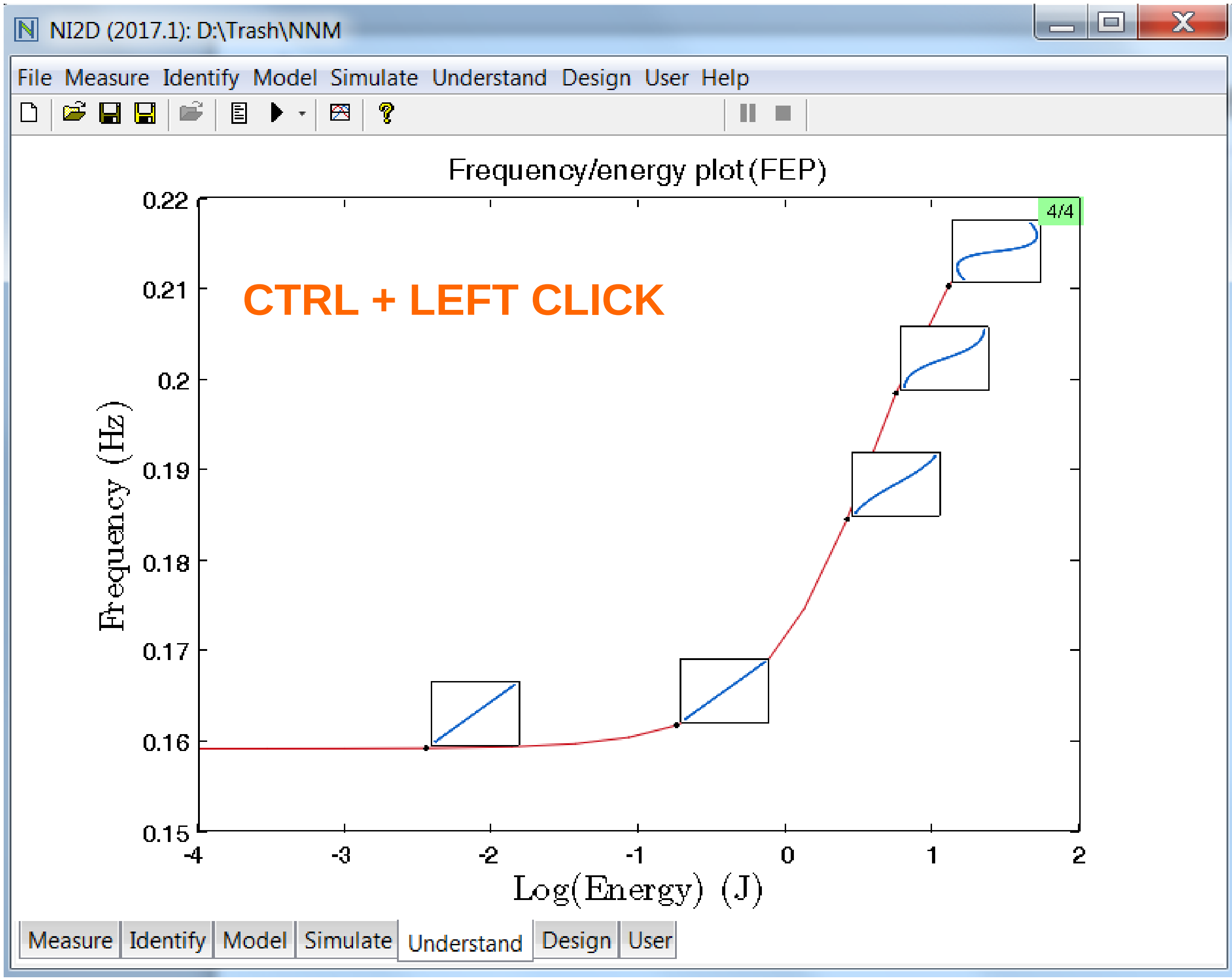
In-phase mode: set appropriate parameters



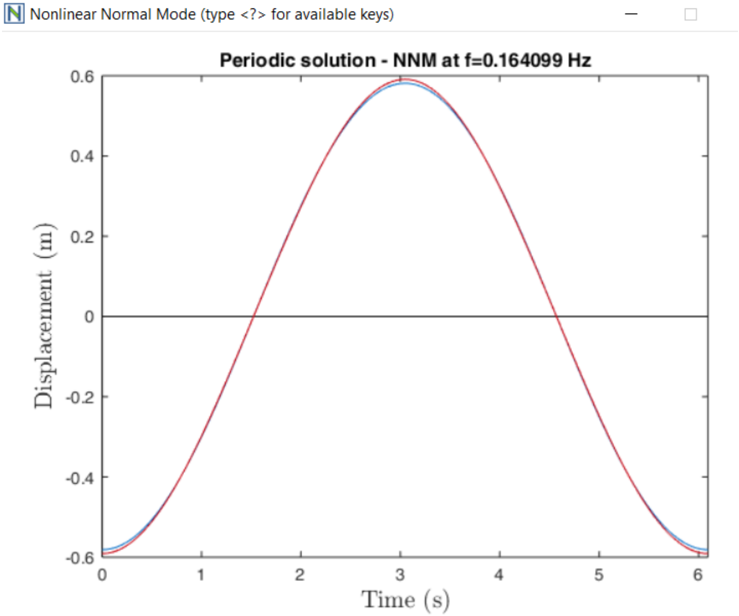
In-phase mode: resonance frequency

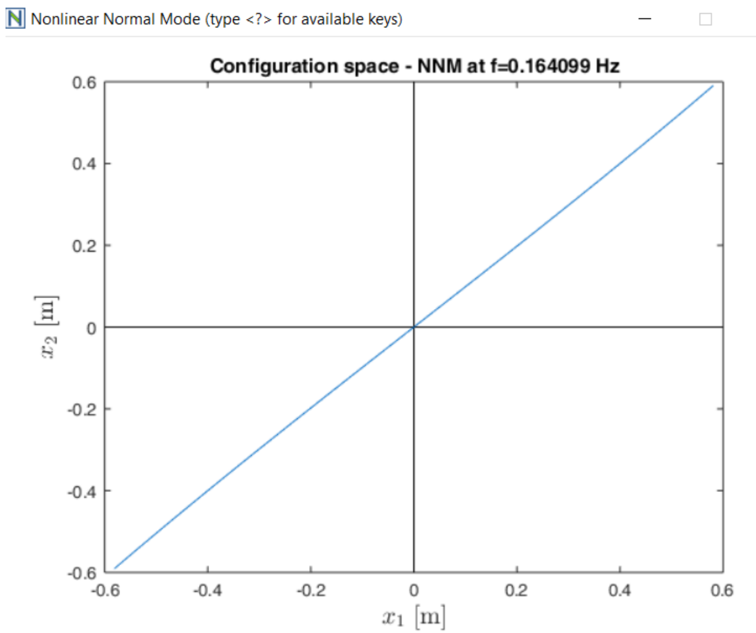


In-phase mode: mode shapes



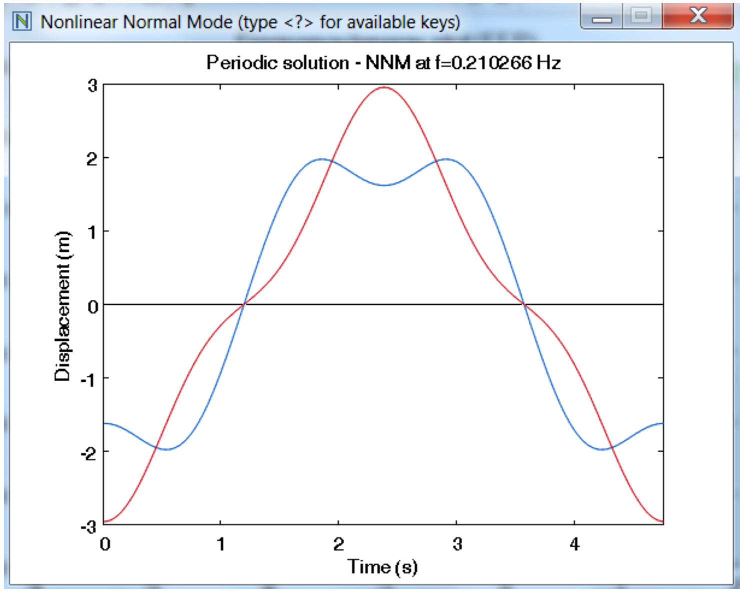
In-phase mode @ low energies

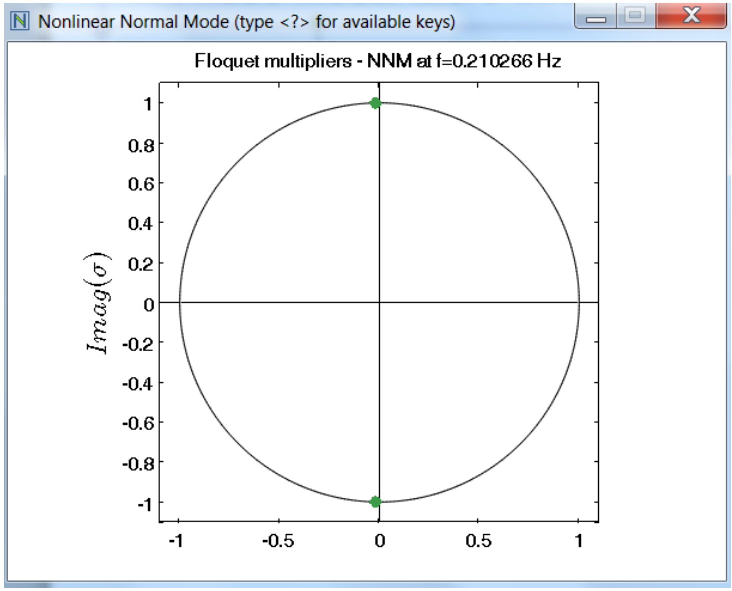




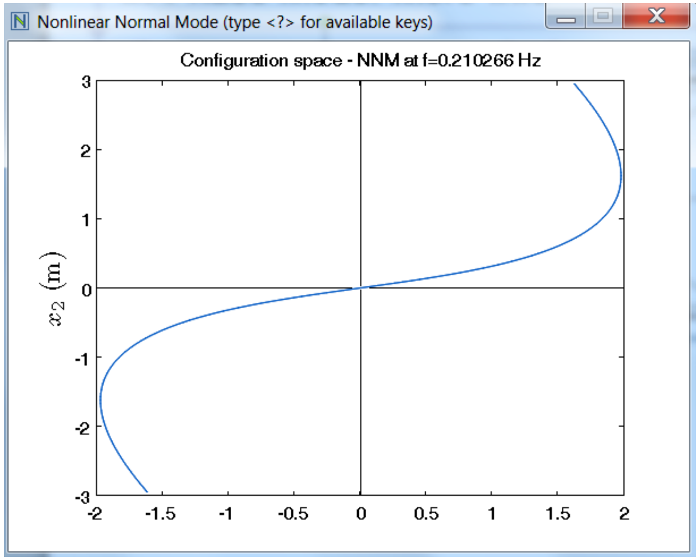
Double click + A

In-phase mode @ high energies



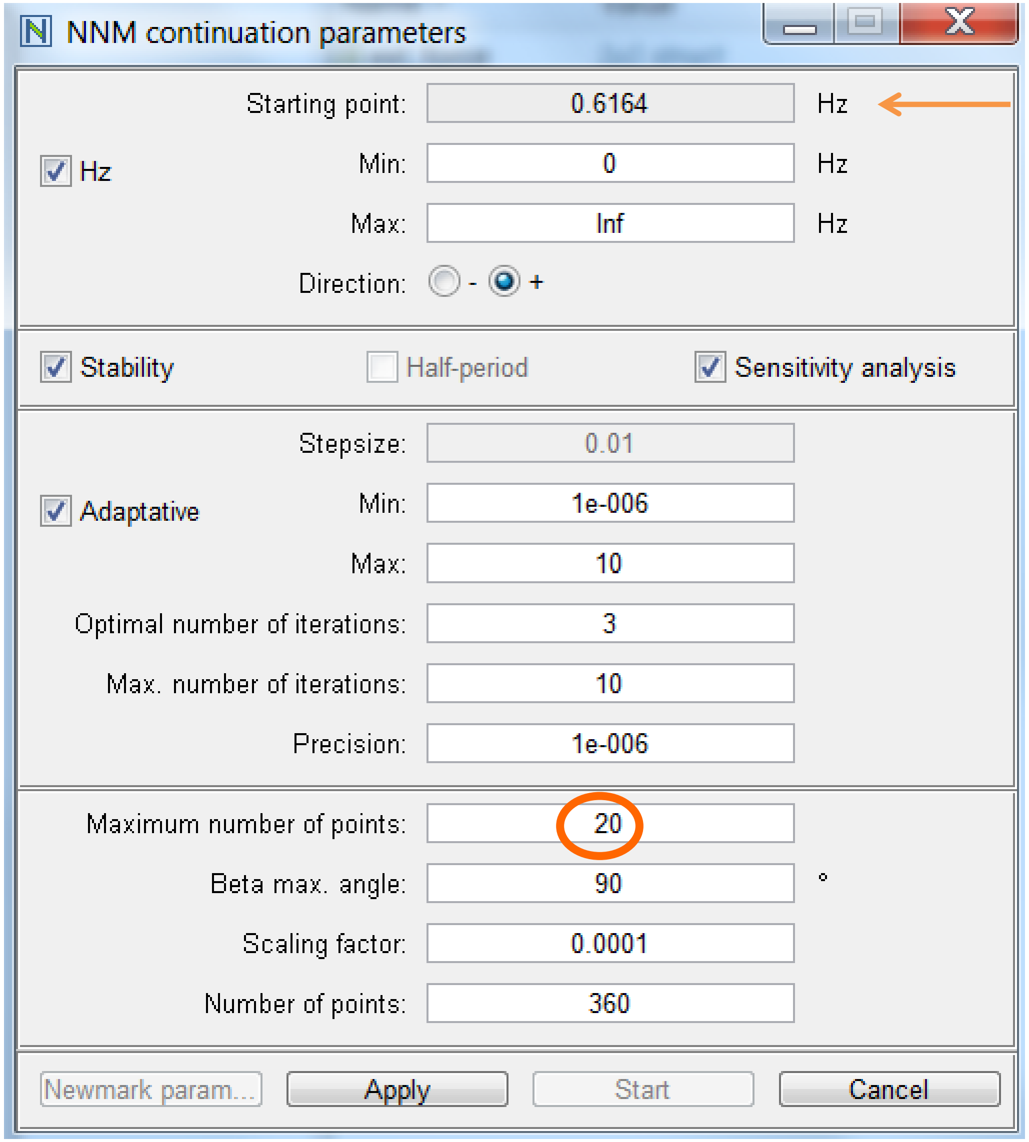


Double click + A



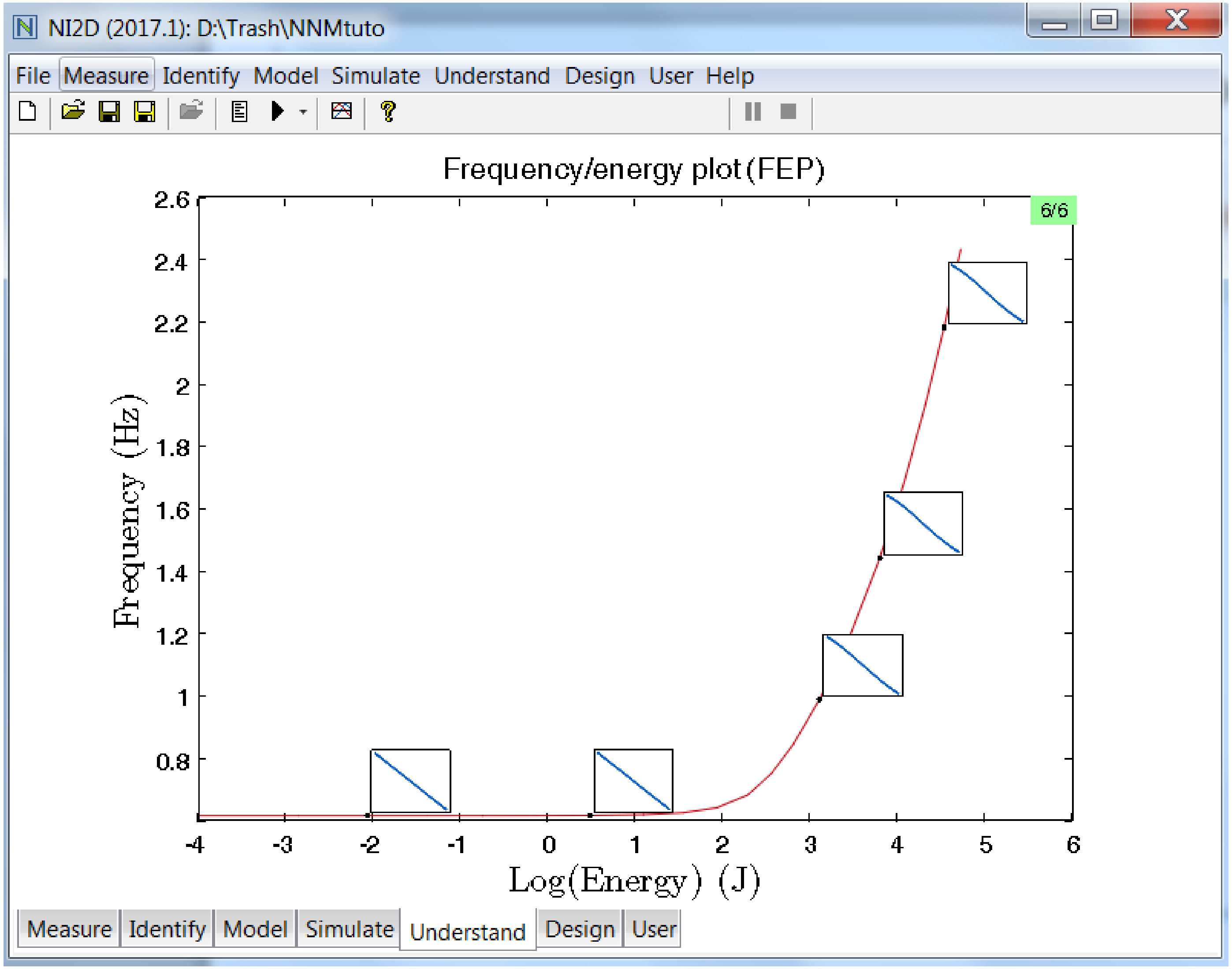
THE MOTION IS NON SYNCHRONOUS ?!?

Out-of-phase mode: set the parameters

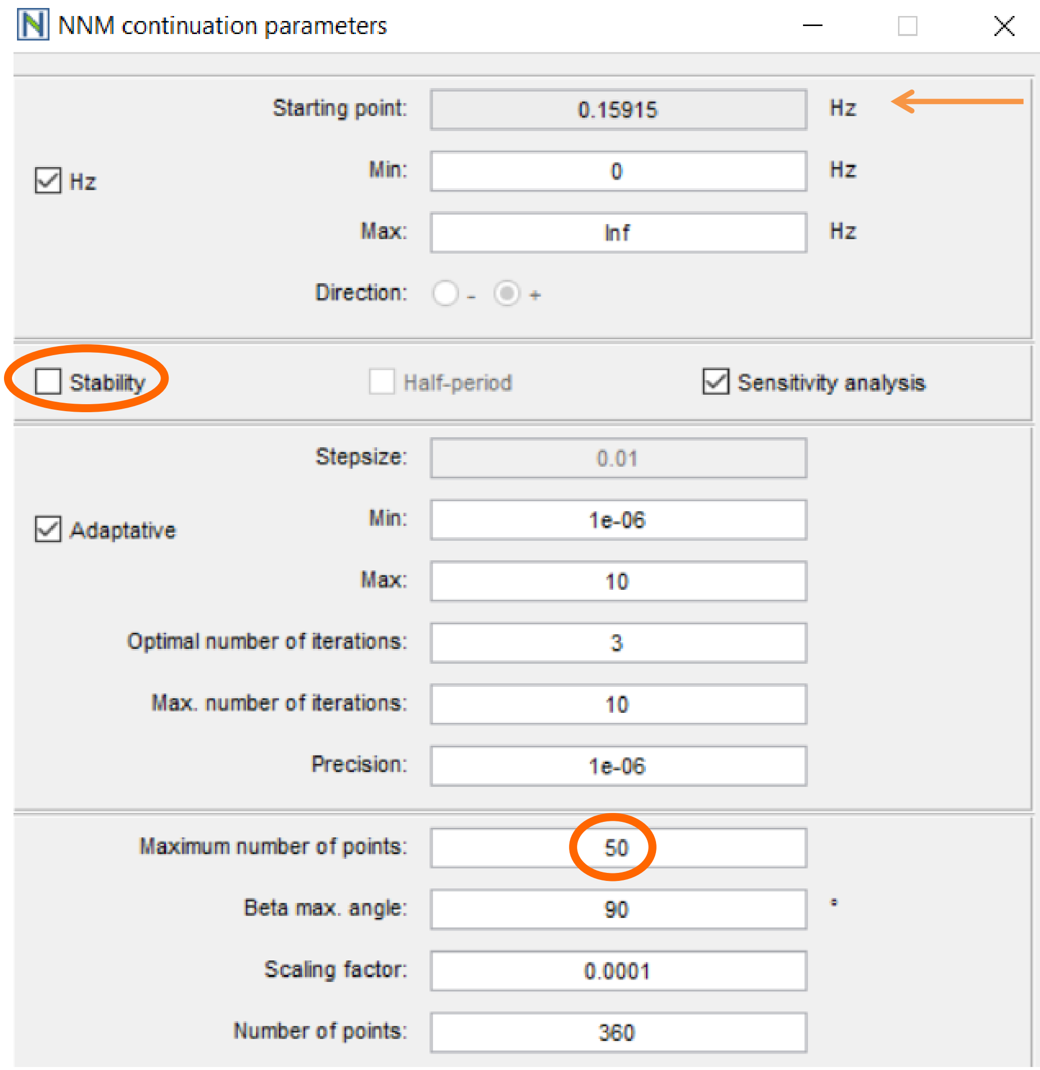


Right click

Out-of-phase mode: frequency and mode shapes

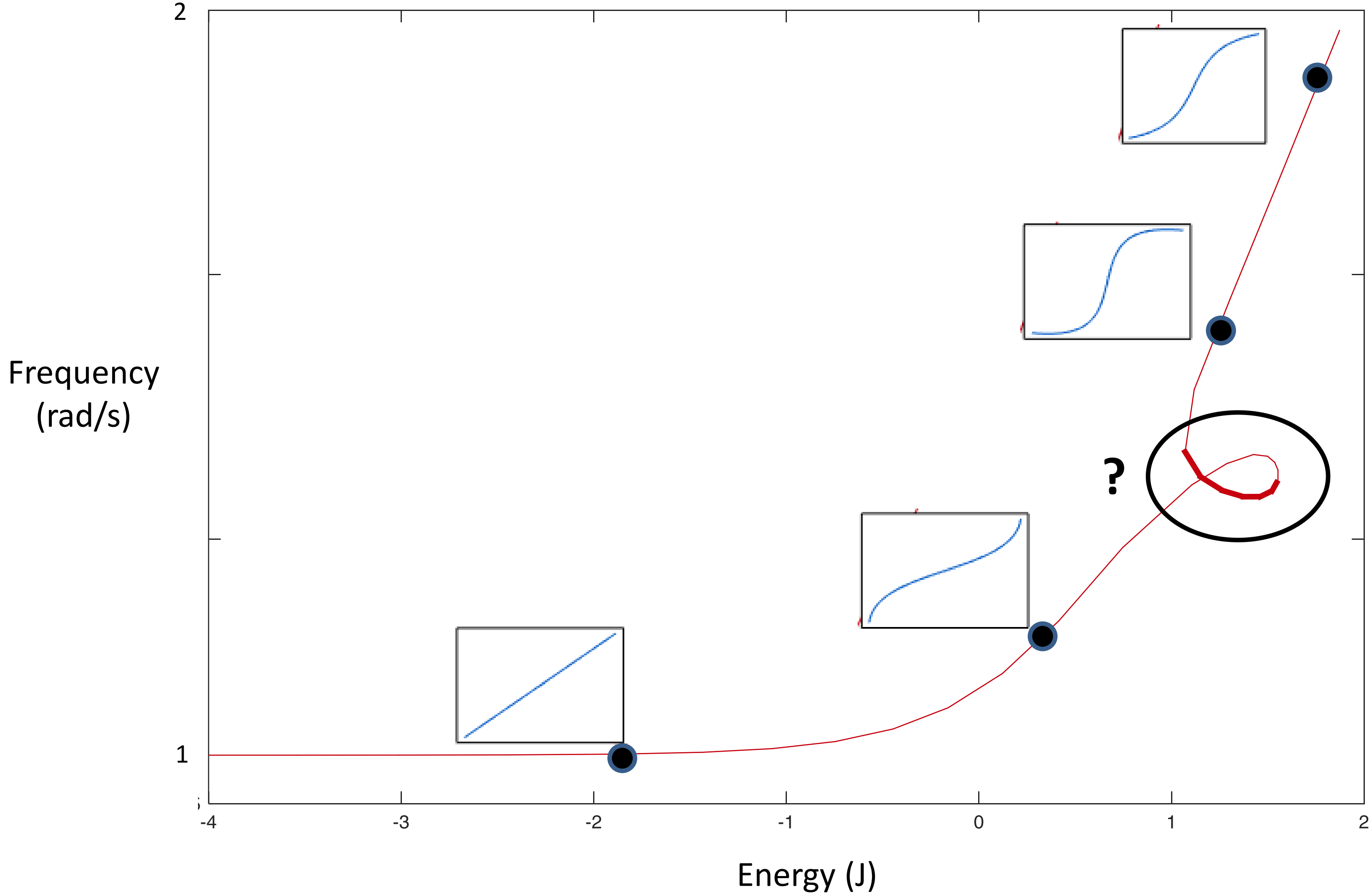


Let's go back to the in-phase mode

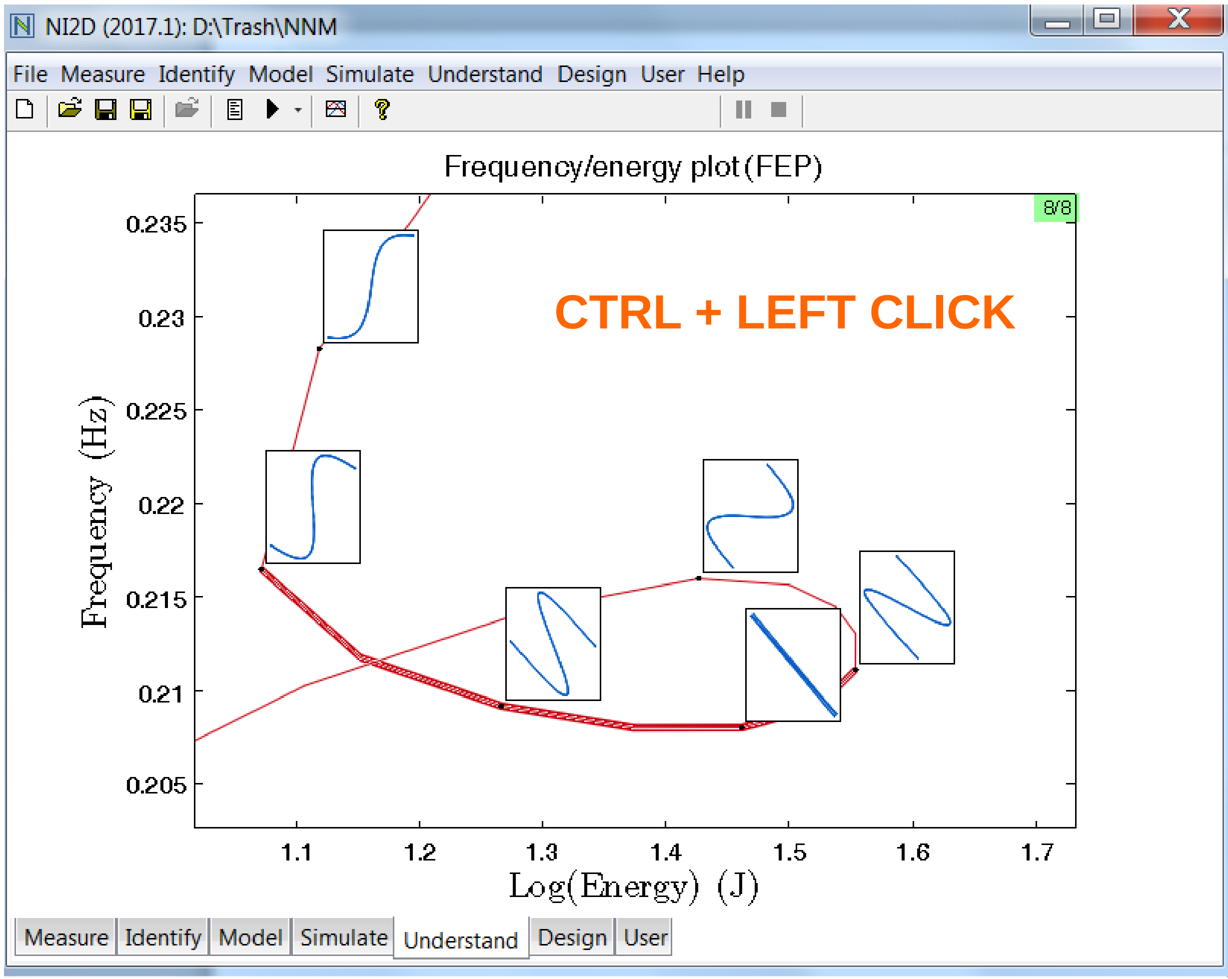


Right click for mode 2

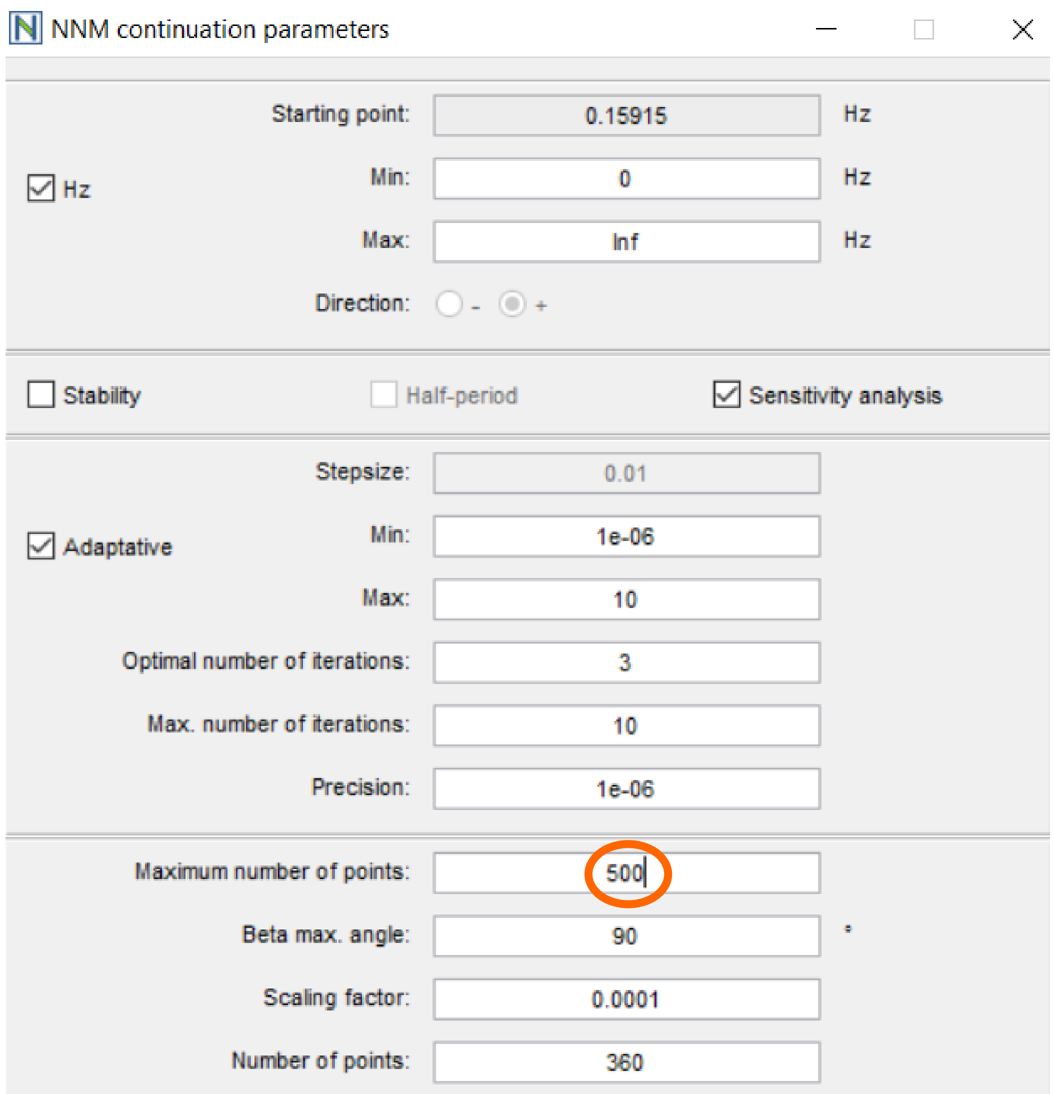
One new feature !



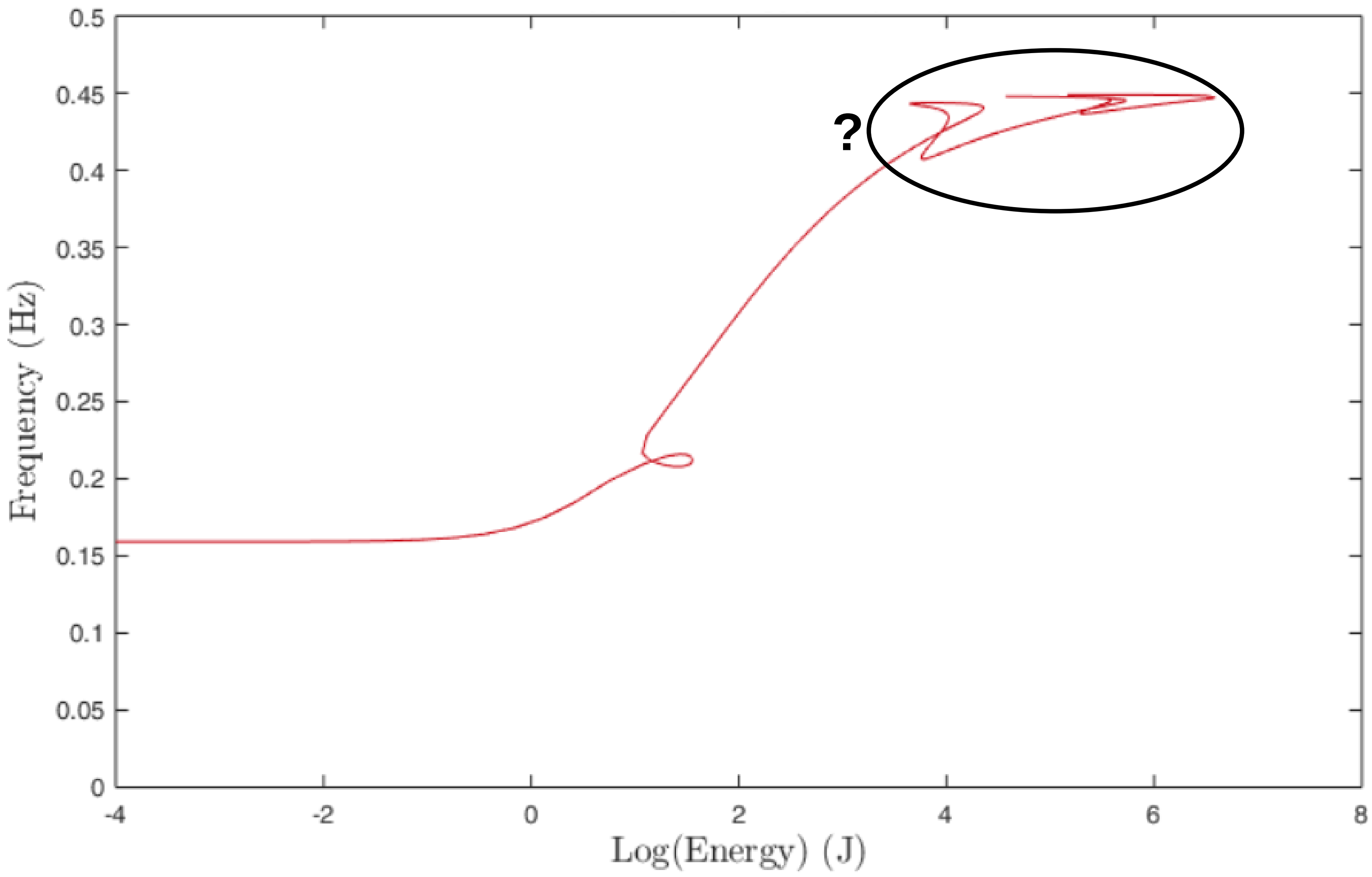
Zoom around the loop



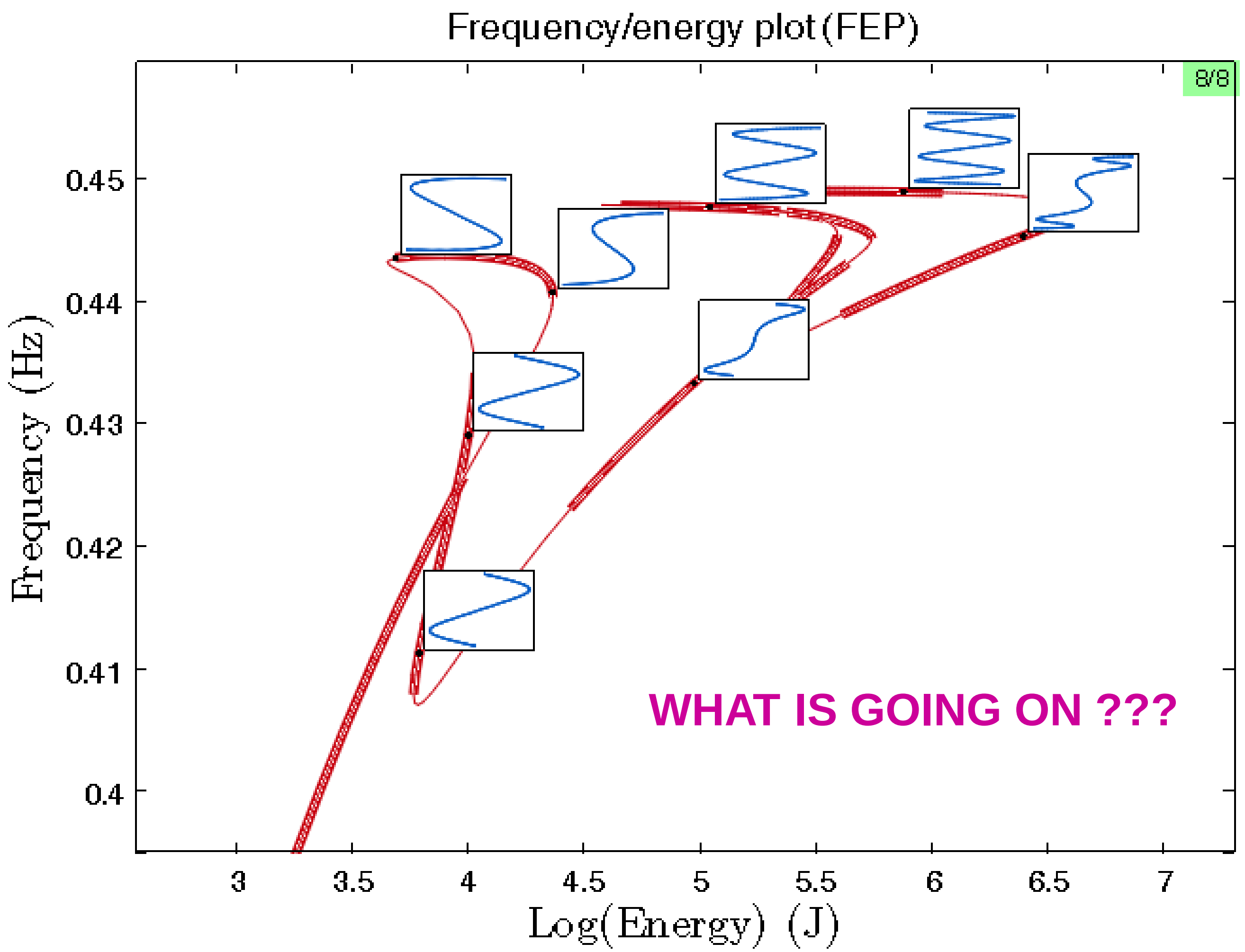
Let’s go even further



Additional loops

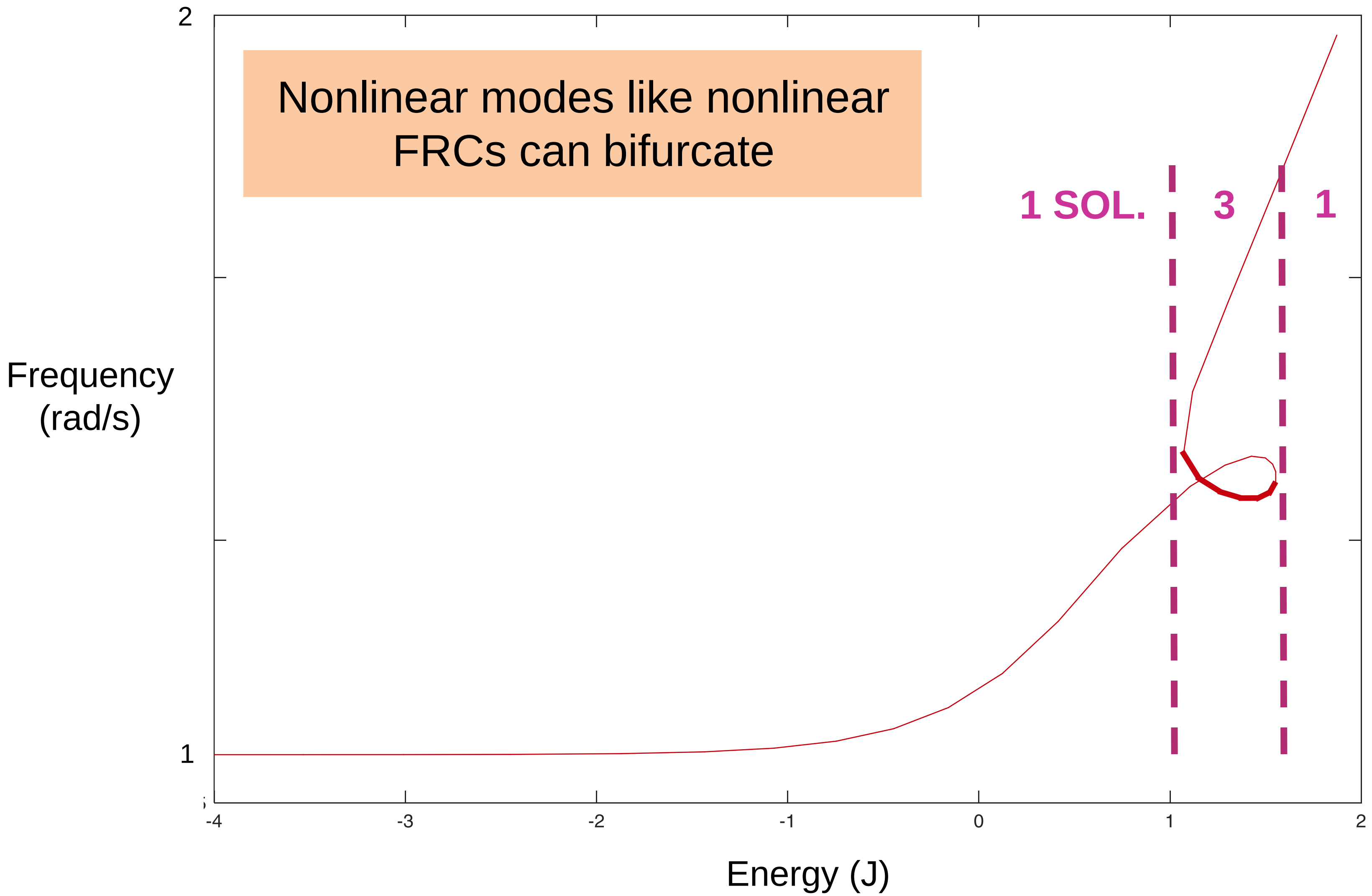


Zoom around the loops

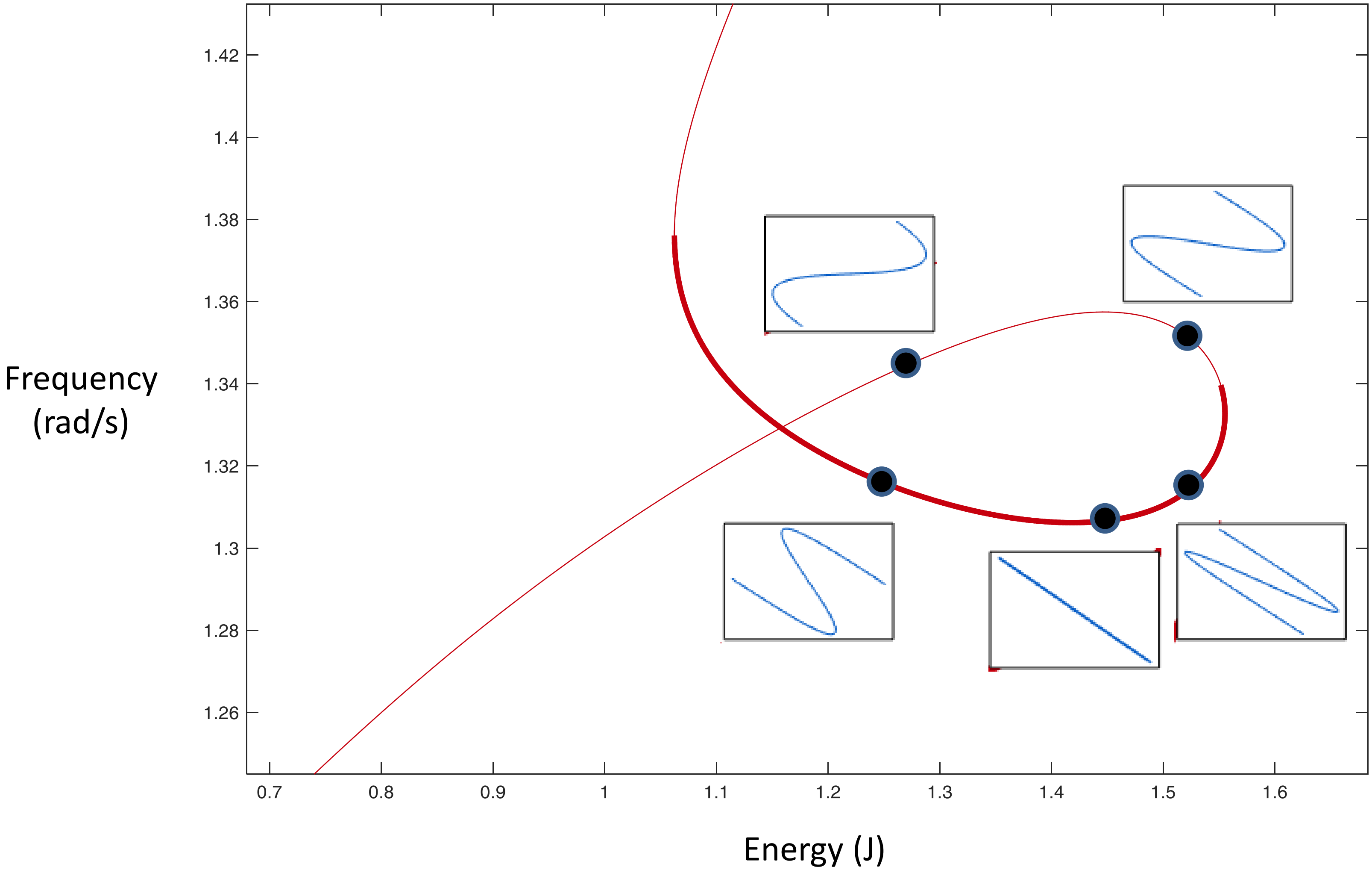


3. Bifurcations

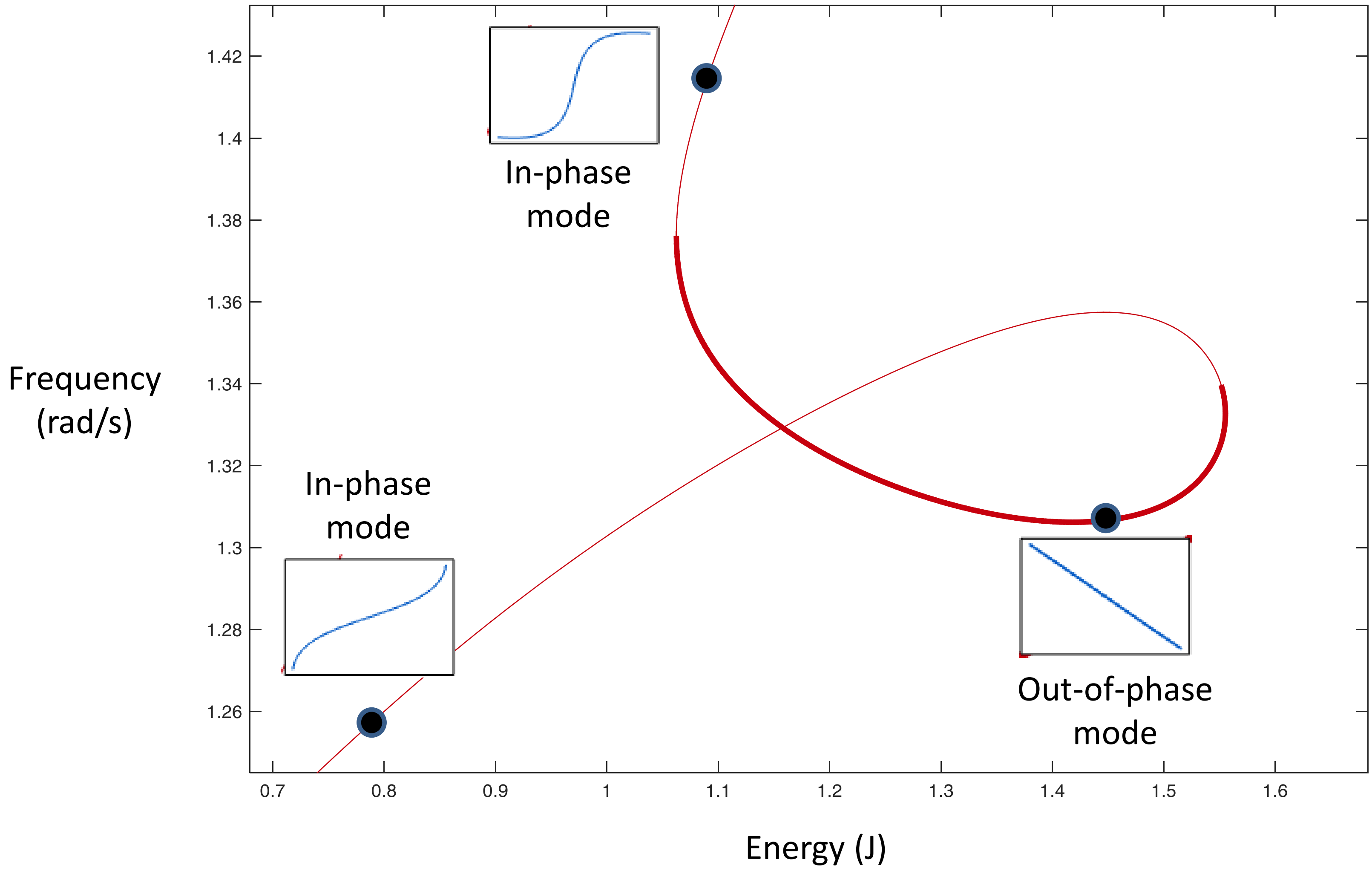
Three in-phase modes at a specific energy



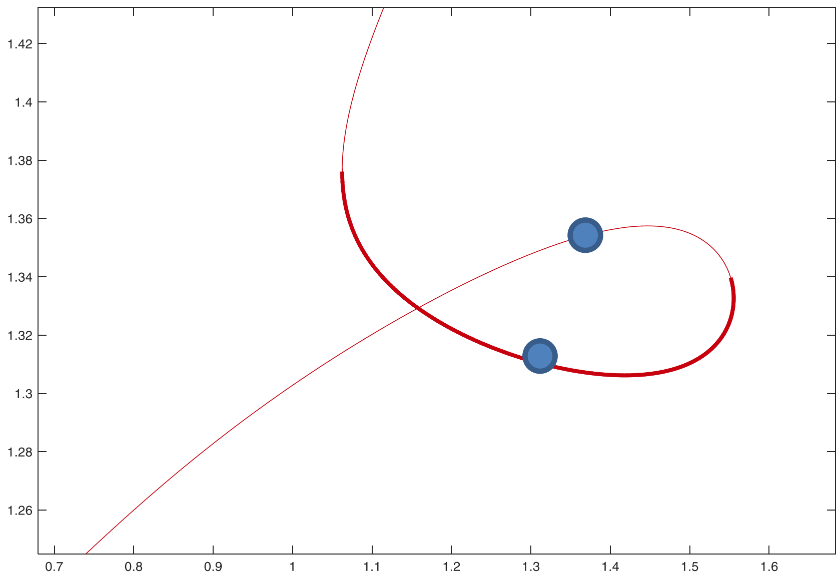
Modes on the branch present a third harmonic

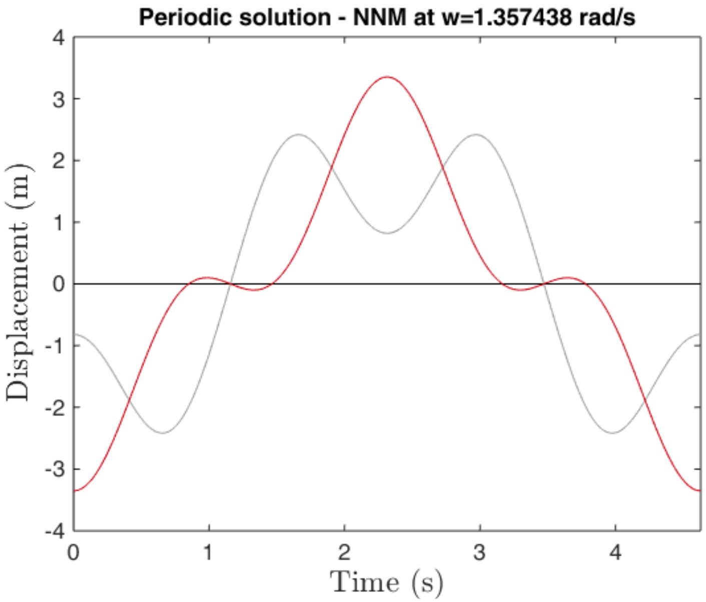


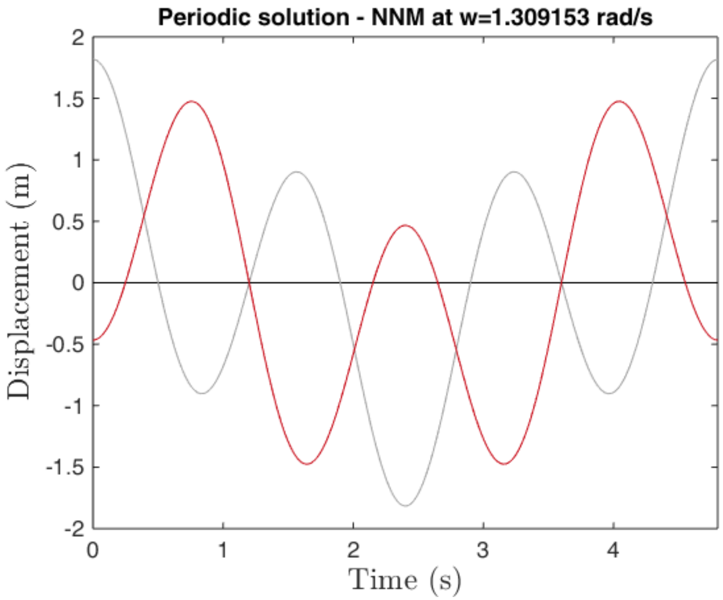
In-phase and out-of-phase modes “connected” !



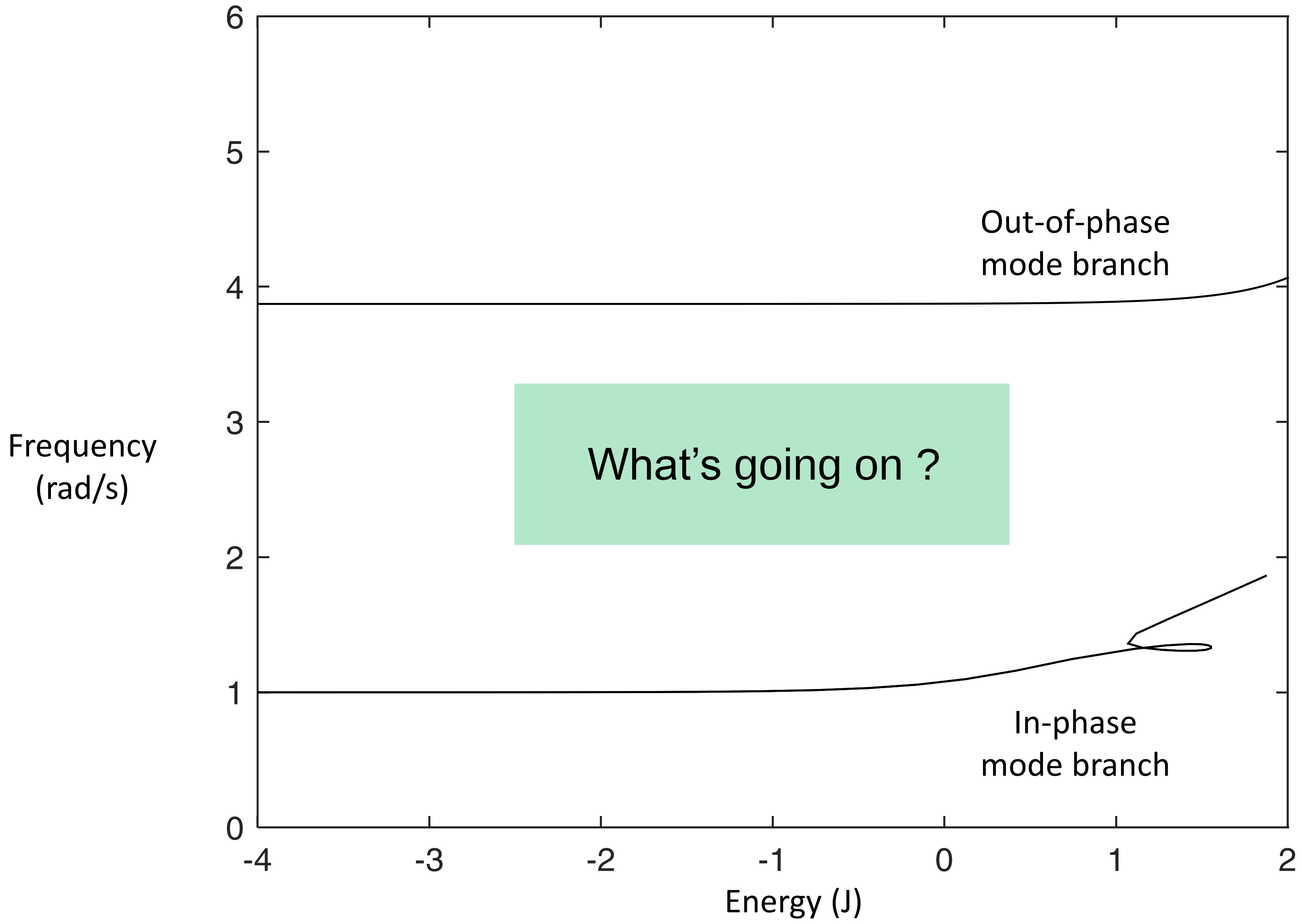
No longer a synchronous motion



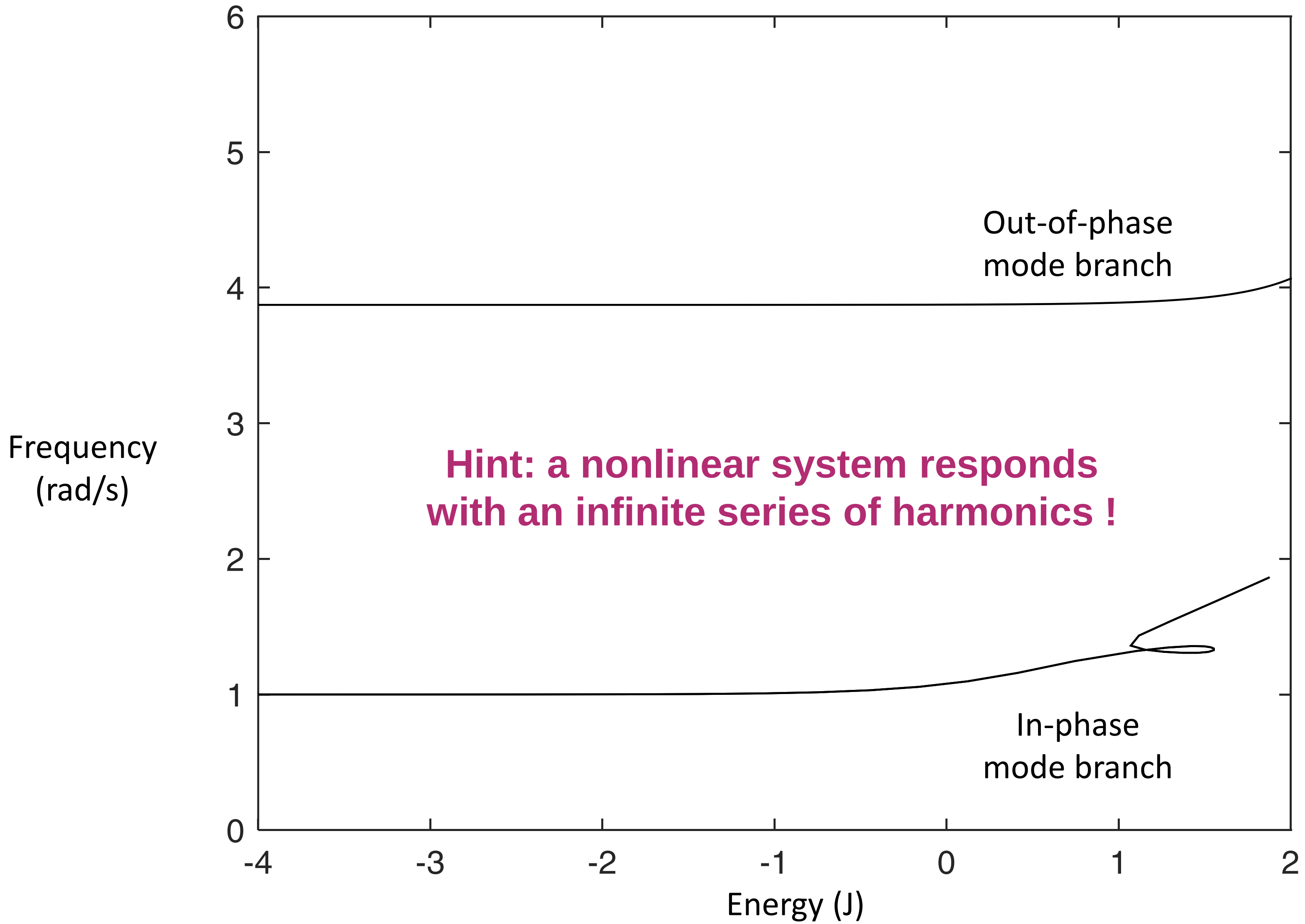




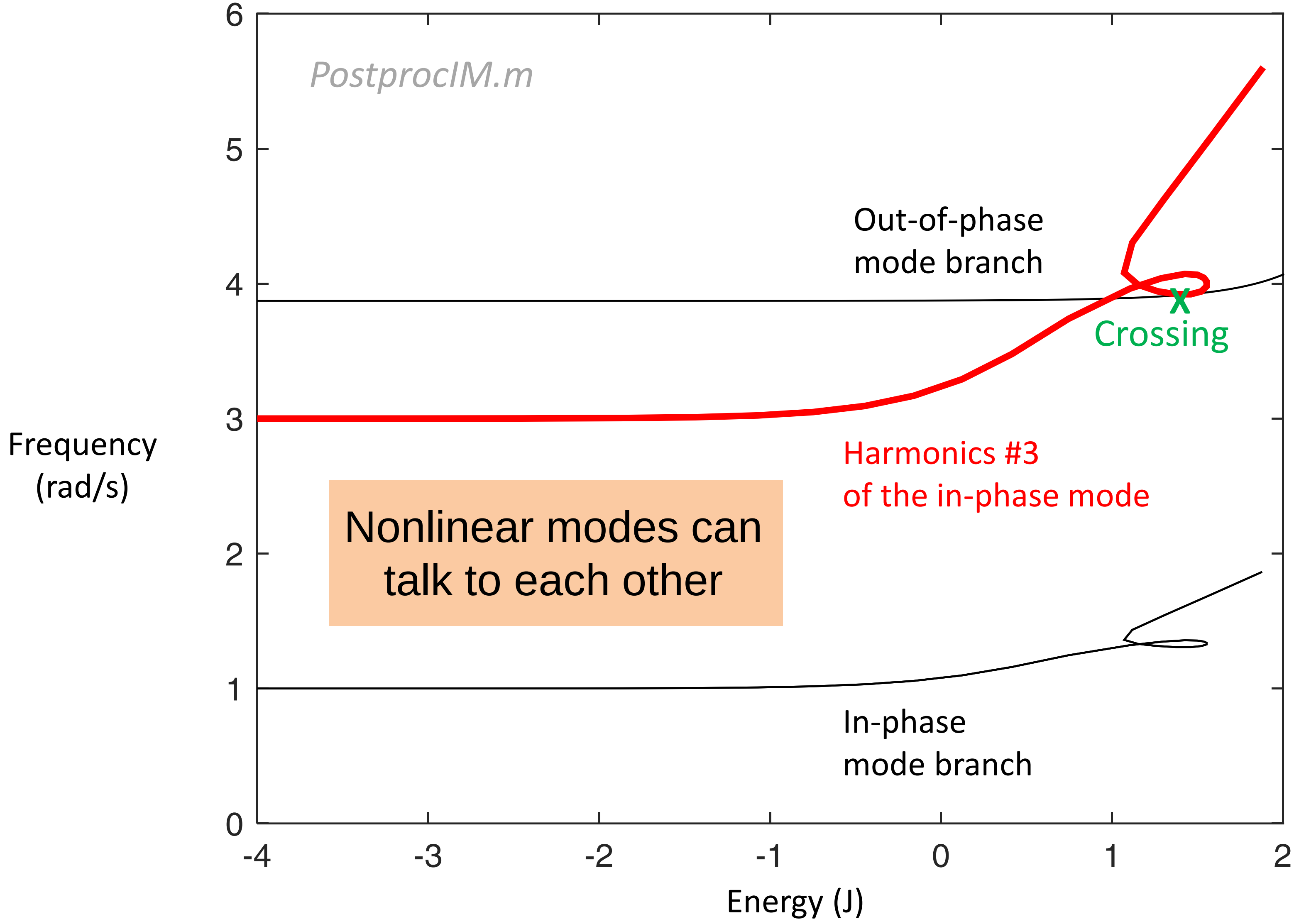
In-phase and out-of-phase modes NOT "connected" !



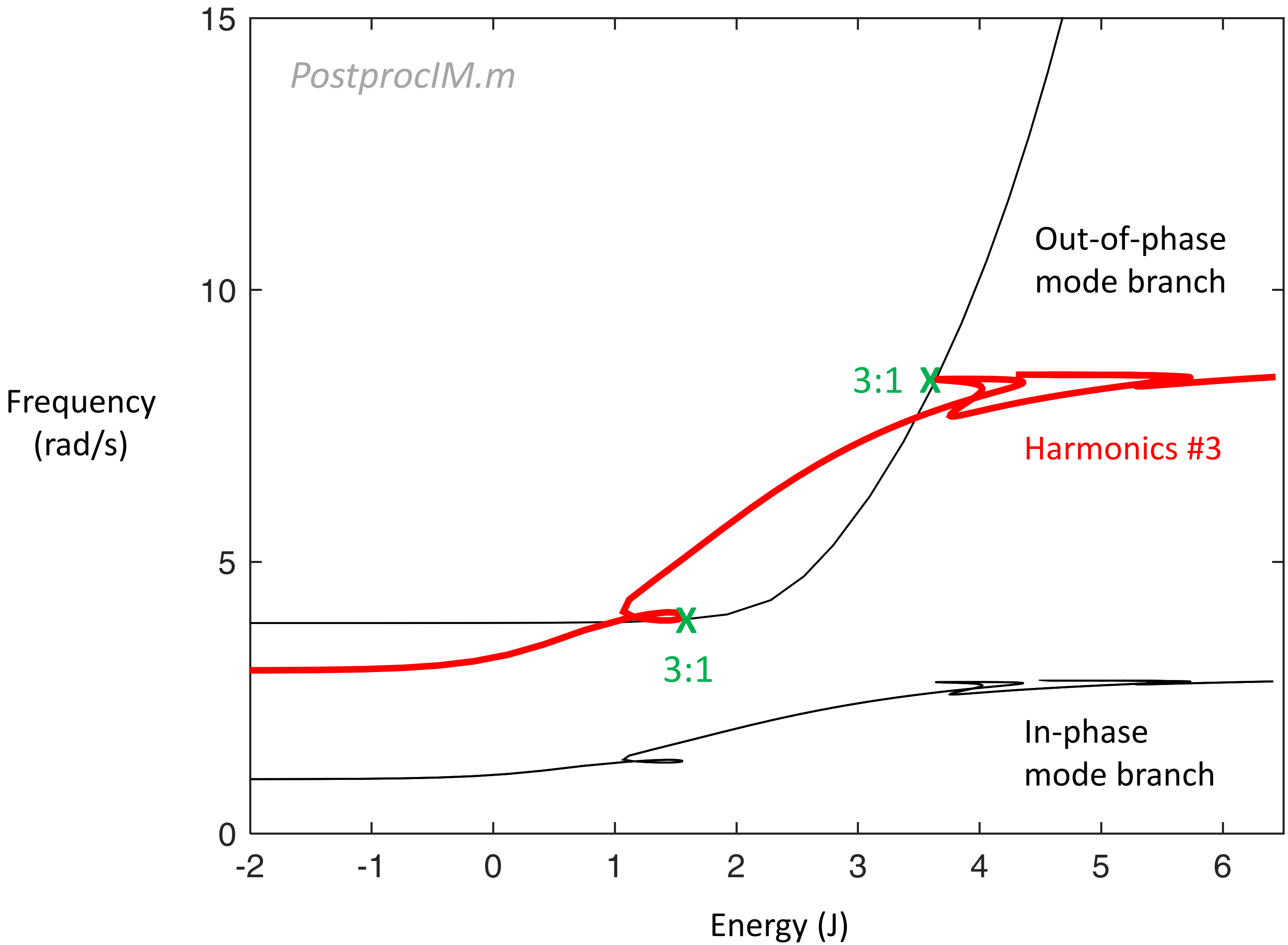
In-phase and out-of-phase modes NOT “connected” !



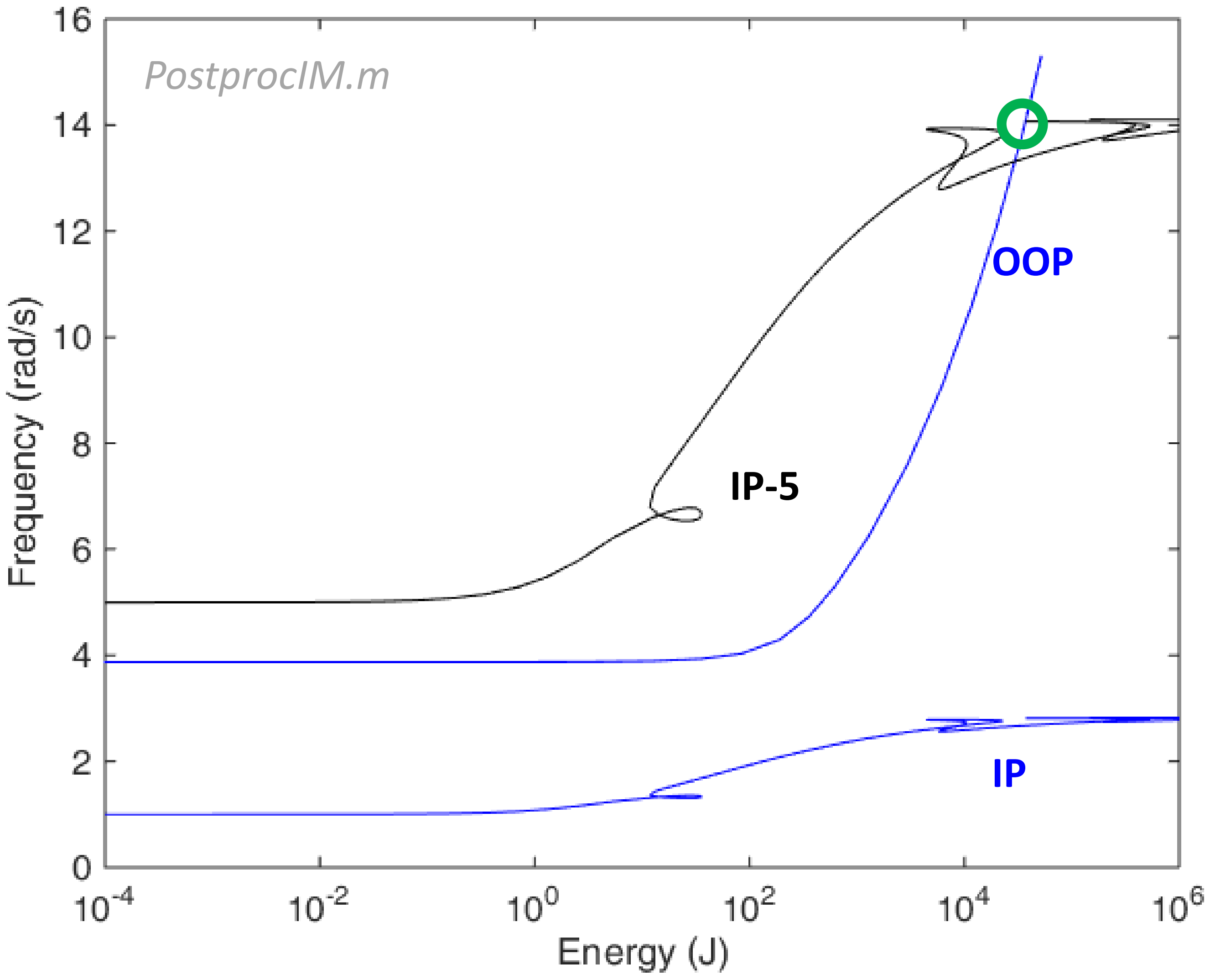
The missing piece of info of the FEP: harmonics



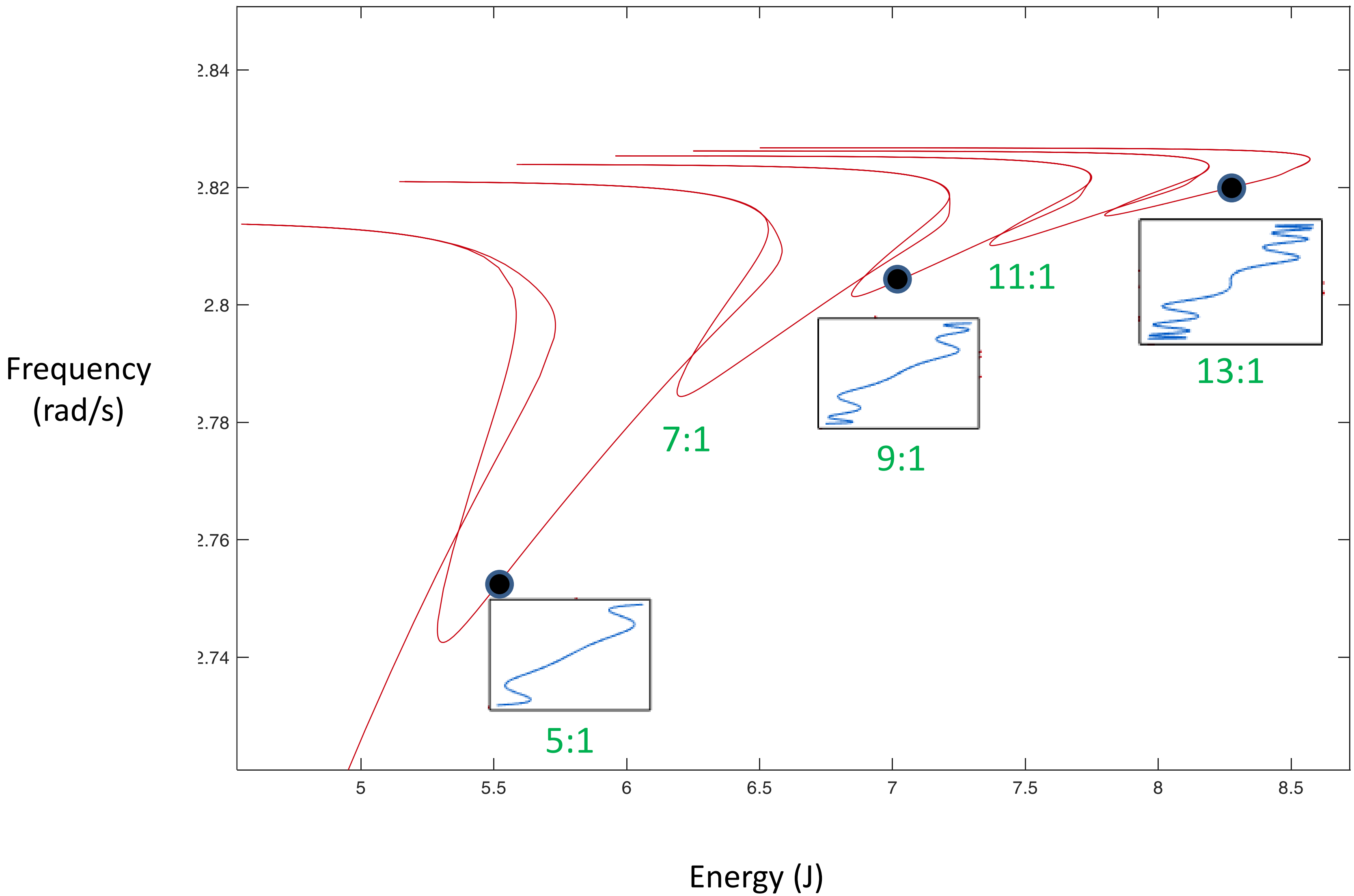
Evidence of another 3:1 modal interaction



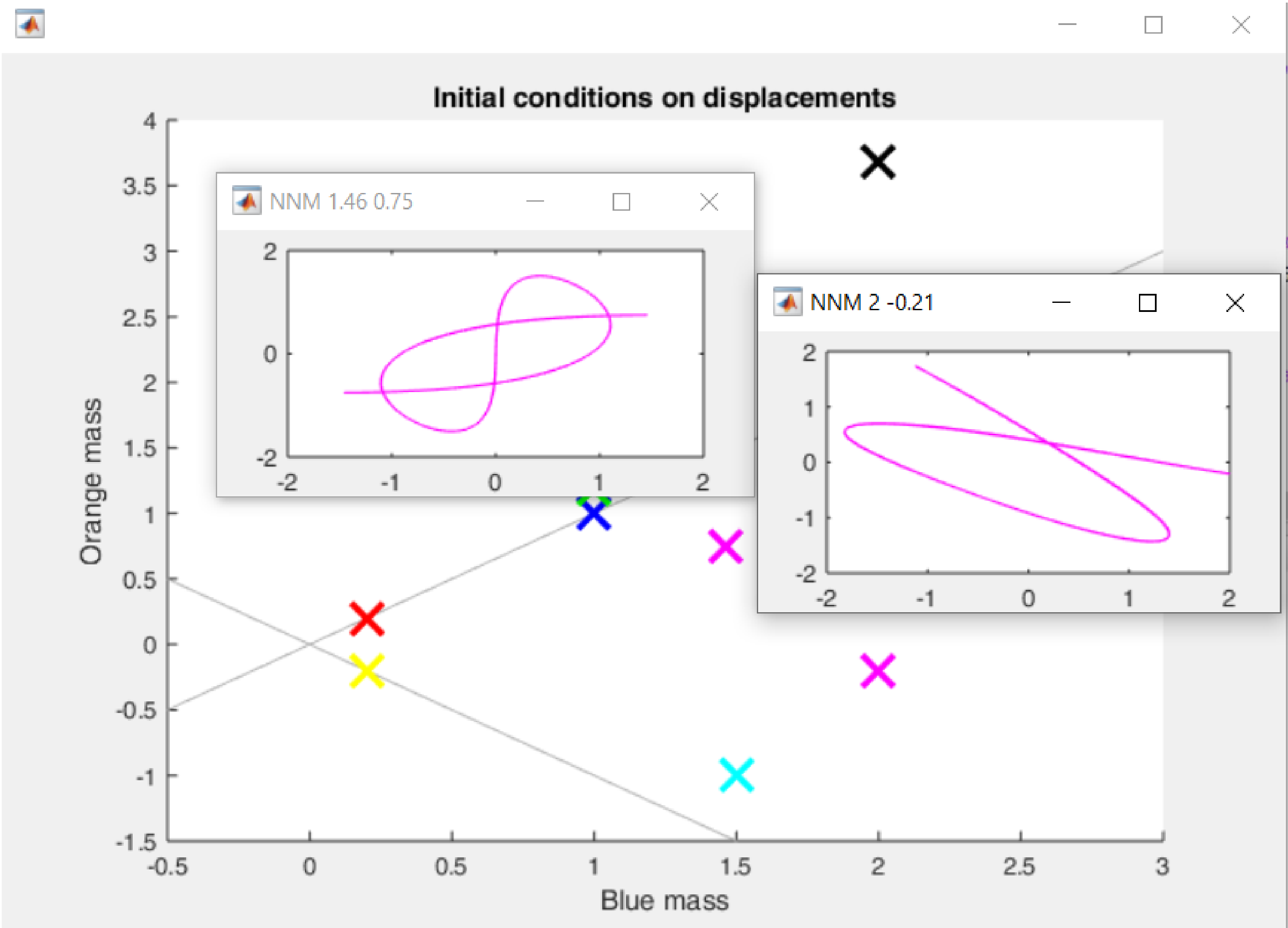
There should be a 5:1 modal interaction as well…



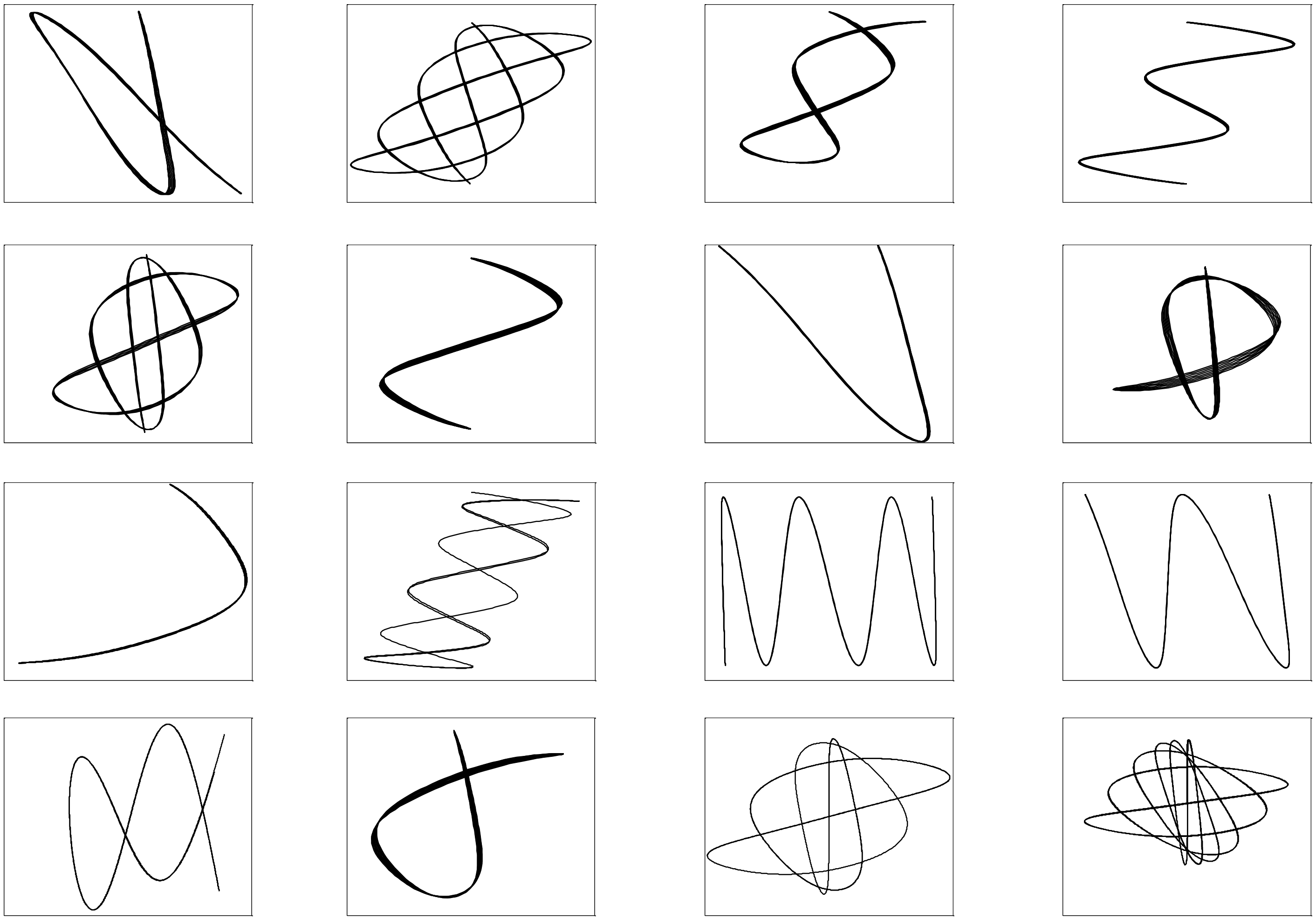
This figure can be explained with modal interactions



This figure can be explained with modal interactions

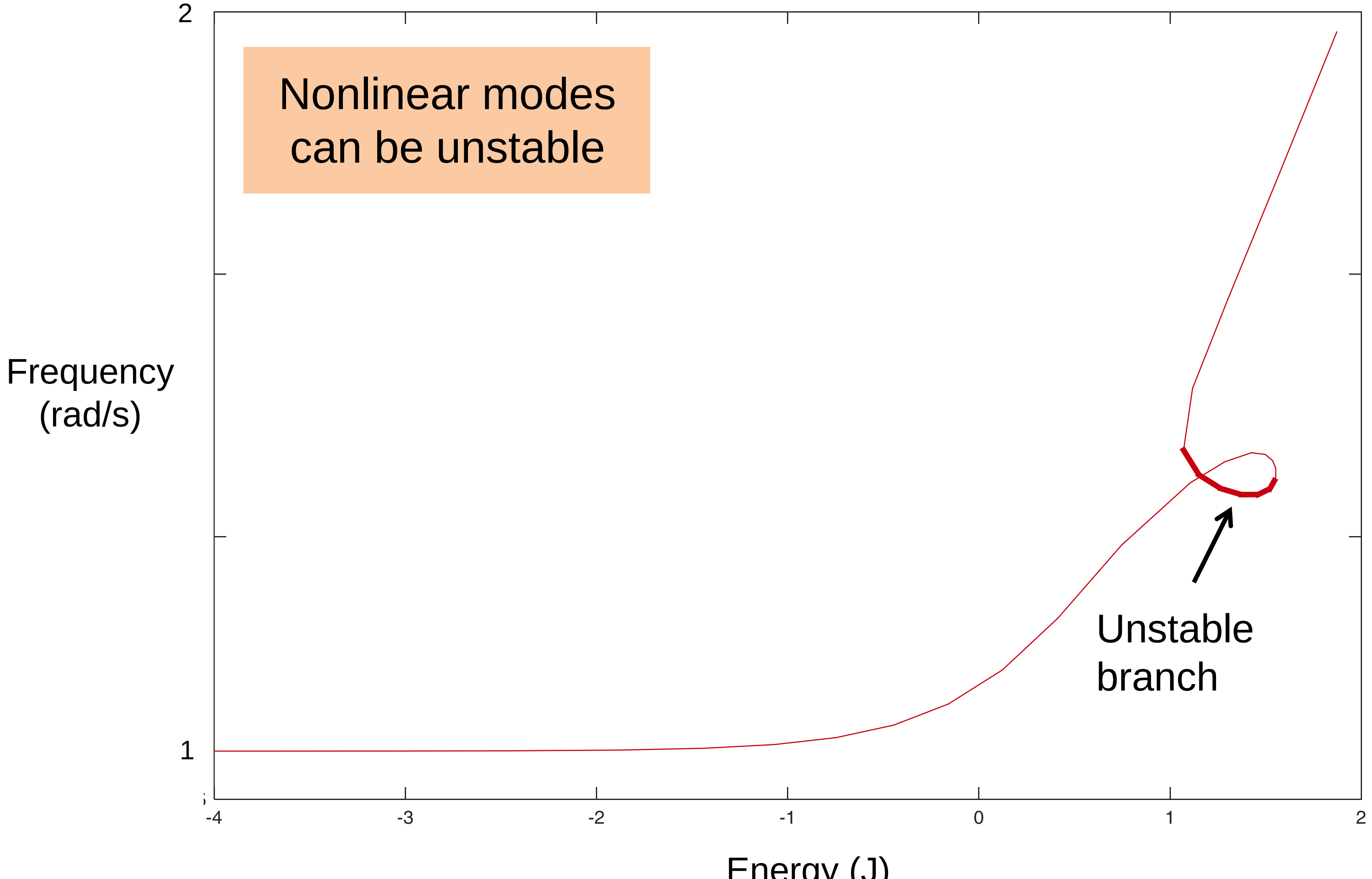


Neither abstract art nor a new alphabet



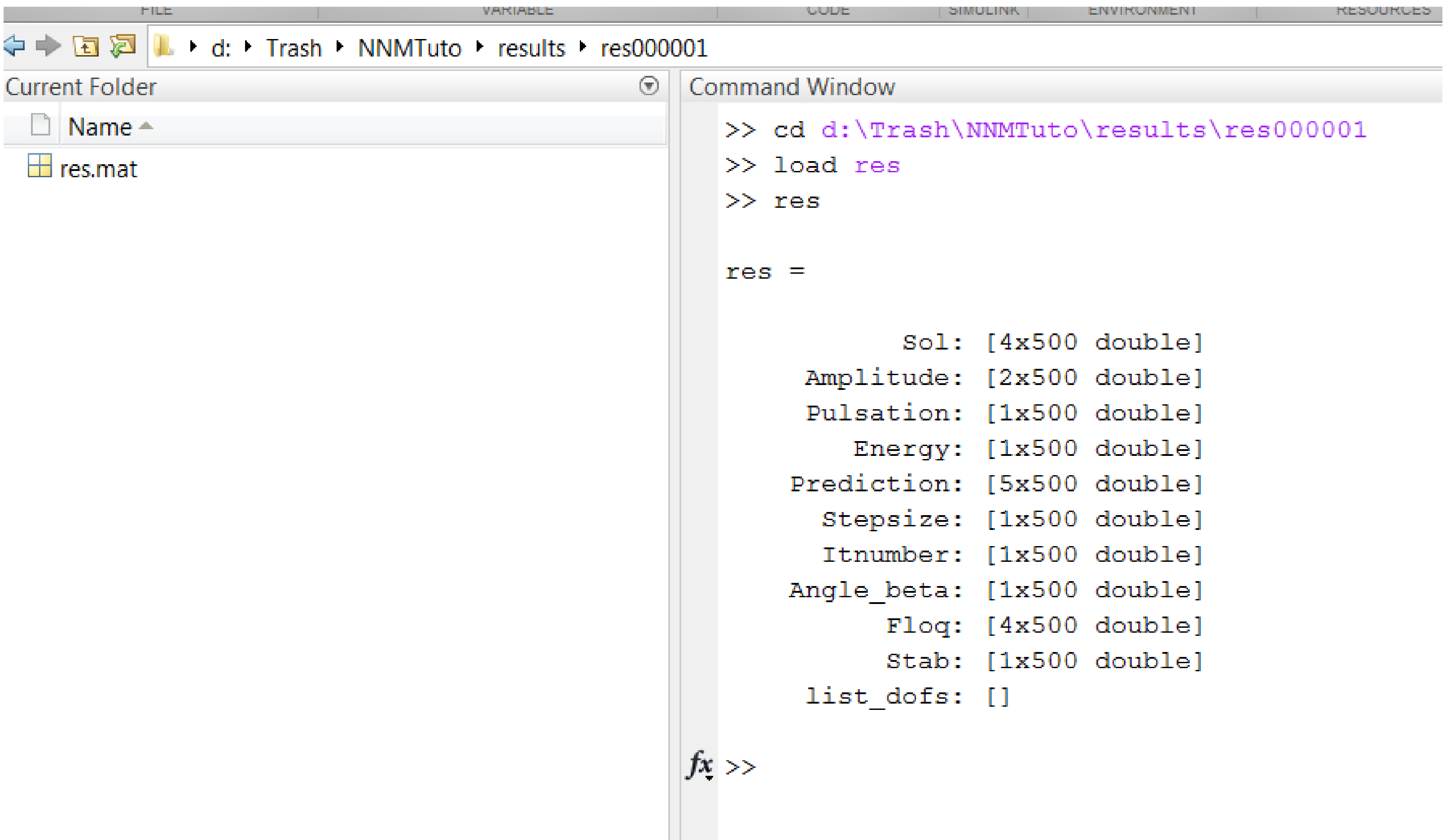
4. Stability

The in-phase mode loses stability



Let’s try to investigate what’s going on

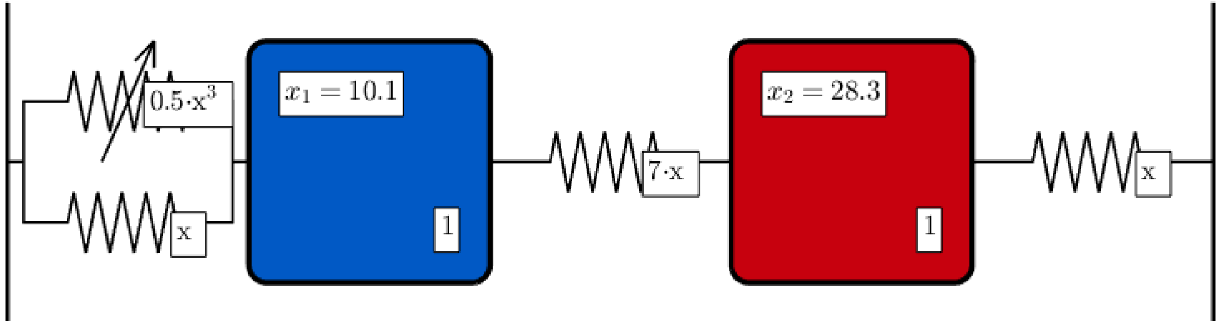
Open Matlab and load the file that contains the results



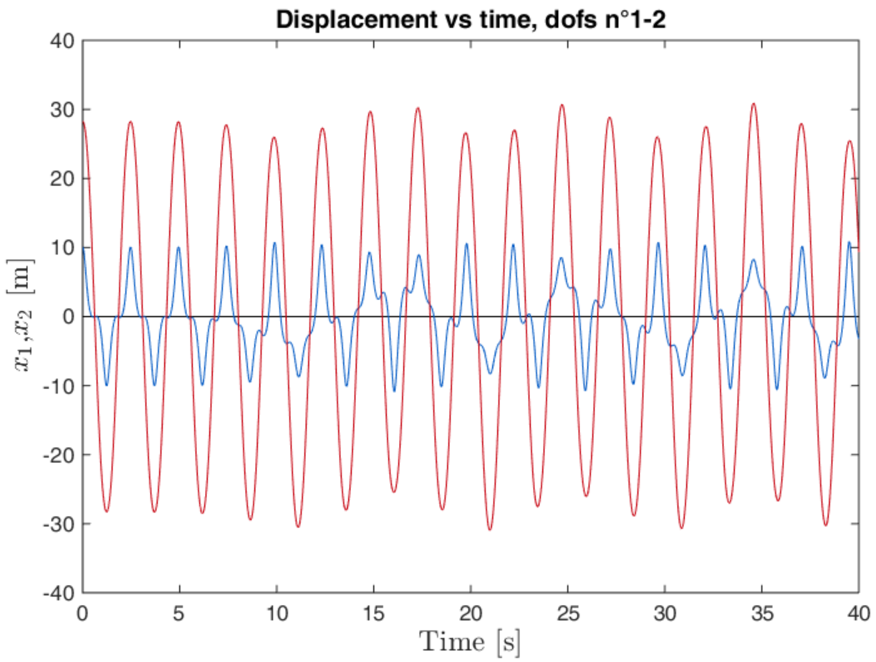
Initial conditions for an unstable mode

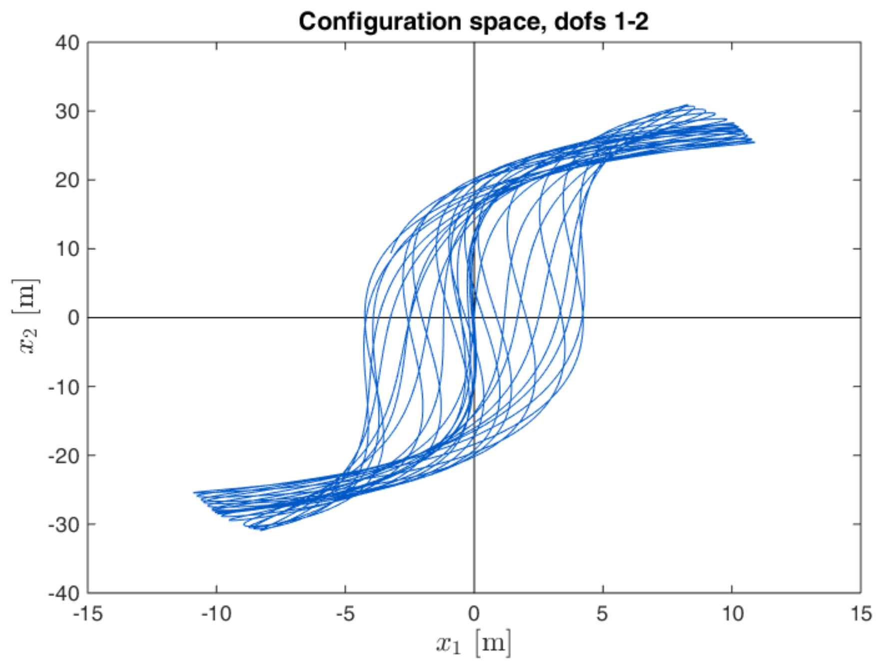
```
>> log10(res.Energy)

ans =

  Columns 1 through 10

   -4.0000   -3.5355   -3.0114   -2.4511   -1.8710   -1.4370   -1.0721   -0.7467   -0.4459   -0.1596

  Columns 11 through 20

    0.1225    0.4141    0.7460    1.1054    1.2869    1.4251    1.4983    1.5359    1.5526    1.5525

  Columns 21 through 30

    1.5215    1.4604    1.3731    1.2651    1.1511    1.0705    1.1176    1.3035    1.5701    1.8719

  Columns 31 through 40

    2.1140    2.3121    2.5284    2.7065    2.8575    2.9883    3.1036    3.2386    3.3569    3.4621

  Columns 41 through 50

    3.5862    3.6958    3.8244    3.9723    4.1396    4.2933    4.3316    4.3422    4.3508    4.3539
```

```
>> res.Sol(:,40)

ans =

   10.0700
   28.2892
         0
         0
```



Unstable nonlinear mode



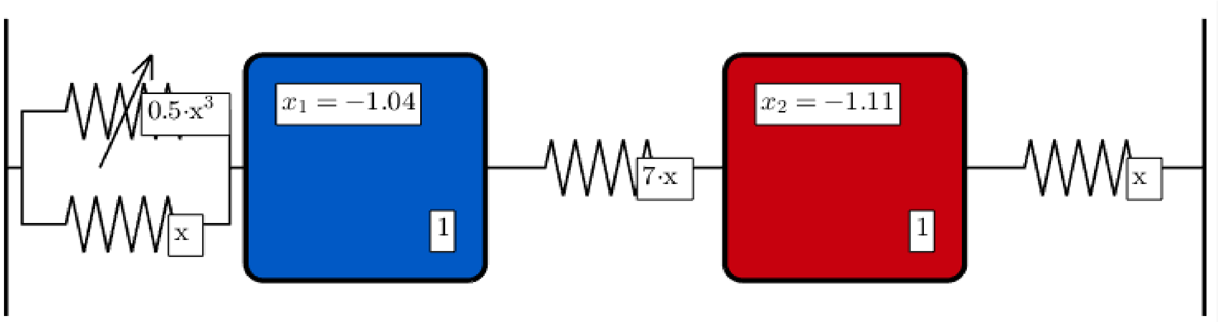


E:\Enseignement\Cours\NonlinearVibrations\Lectures\
Matlab\L04_NNM\Simulation_Stability

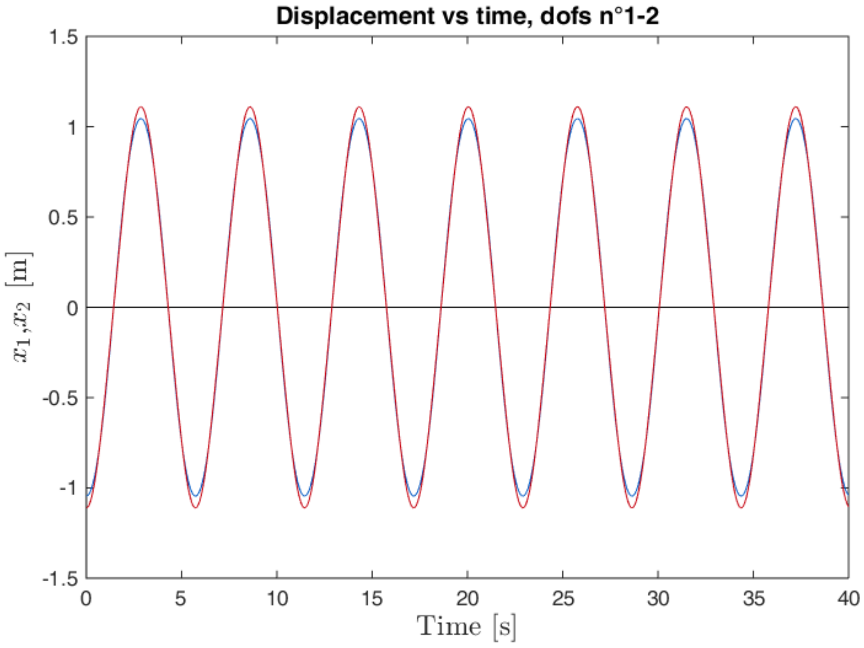
Initial conditions for a stable mode

```
>> log10(res.Energy)

ans =

  Columns 1 through 10

   -4.0000    -3.5355    -3.0114    -2.4511    -1.8710    -1.4370    -1.0721    -0.7467    -0.4459    -0.1596

  Columns 11 through 20

    0.1225     0.4141     0.7460     1.1054     1.2869     1.4251     1.4983     1.5359     1.5526     1.5525

  Columns 21 through 30

    1.5215     1.4604     1.3731     1.2651     1.1511     1.0705     1.1176     1.3035     1.5701     1.8719

  Columns 31 through 40

    2.1140     2.3121     2.5284     2.7065     2.8575     2.9883     3.1036     3.2386     3.3569     3.4621

  Columns 41 through 50

    3.5862     3.6958     3.8244     3.9723     4.1396     4.2933     4.3316     4.3422     4.3508     4.3539
```

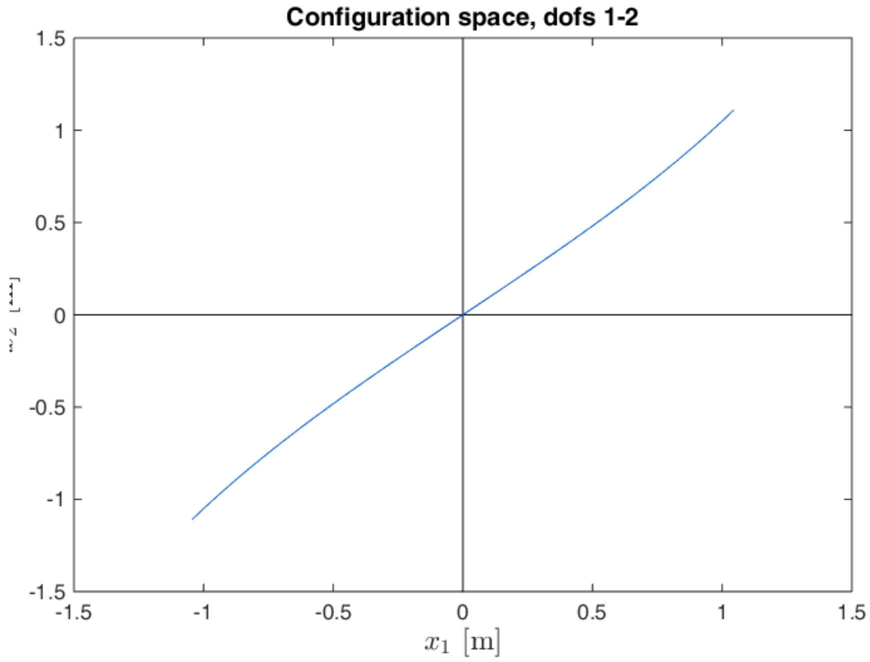
```
>> res.Sol(:,11)

ans =

   -1.0449
   -1.1101
         0
         0
```



Stable mode



Displacement vs time, dofs n°1-2
1.5
1
0.5
0
-0.5
-1
-1.5
0
5
10
15
20
25
30
35
40
Time [s]
x_1,x_2 [m]



Configuration space, dofs 1-2
1.5
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1
1.5
x_1 [m]

5. Numerical computation ?

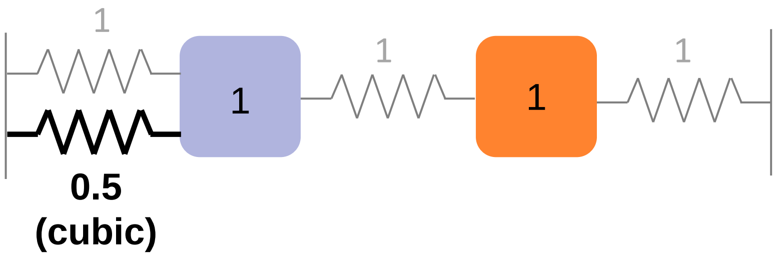
Basic idea

An nonlinear mode is a periodic motion.

▼

1. Look for periodic solutions !

Naive approach



$z_0 \rightarrow z(t, z_0)$
using numerical time integration

Case 1: $x_1(0) = 1$ and $x_2(0) = 1$ ⟶ ~~Periodic solution~~

Case 2: $x_1(0) = 1$ and $x_2(0) = 1.08$ ⟶ ~~Periodic solution~~

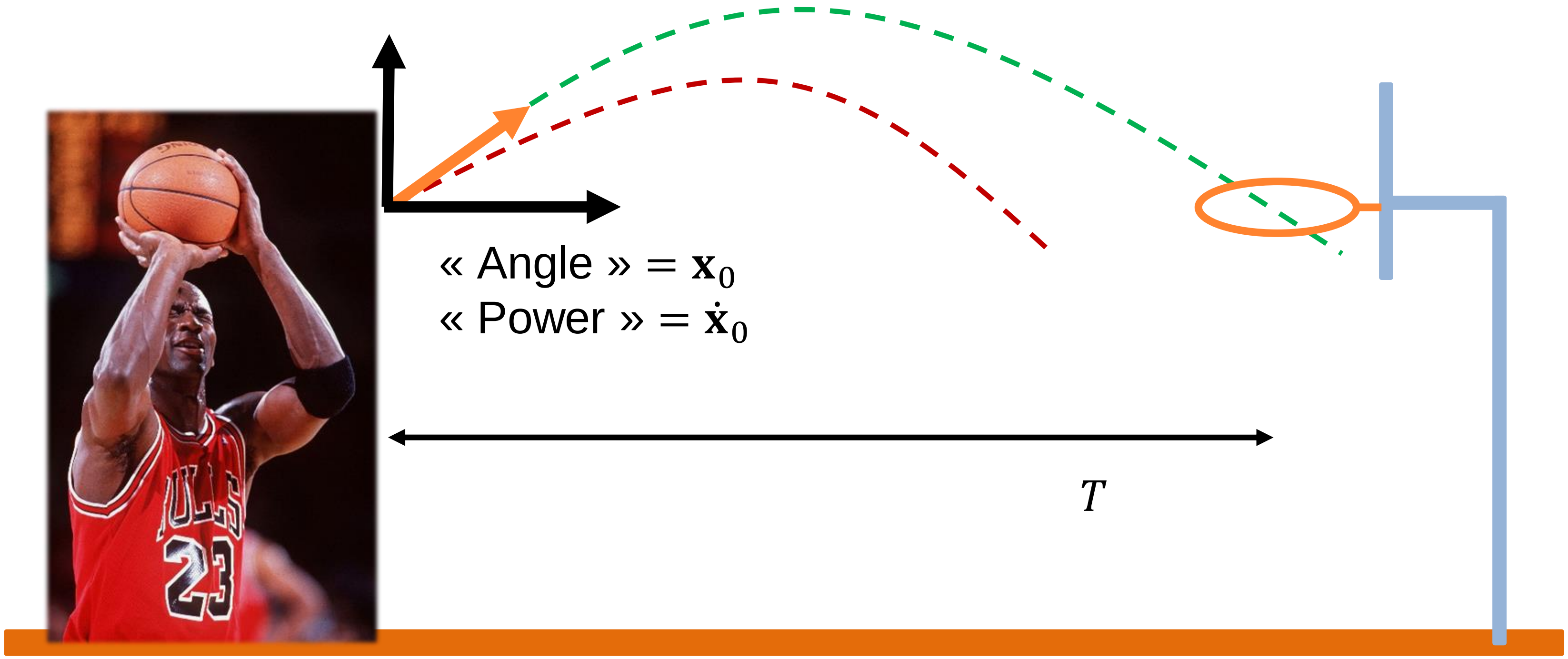
Case 3: $x_1(0) = 1$ and $x_2(0) = 1.16$ ⟶ ~~Periodic solution~~

Case 4: $x_1(0) = 1$ and $x_2(0) = 1.24$ ⟶ ~~Periodic solution~~

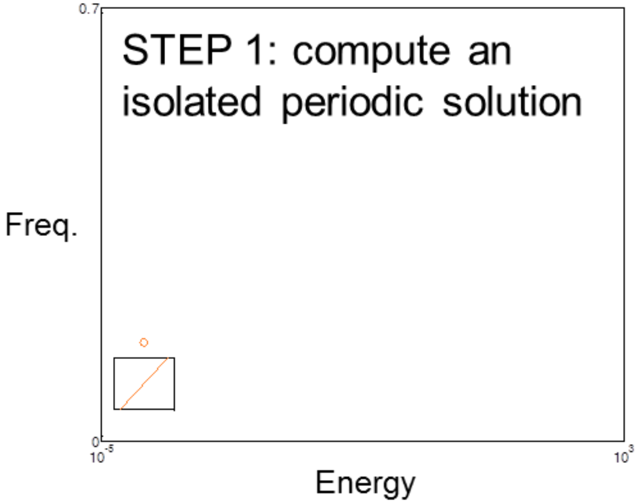
Case 5: $x_1(0) = 1$ and $x_2(0) = 1.19$ ⟶ Periodic solution !

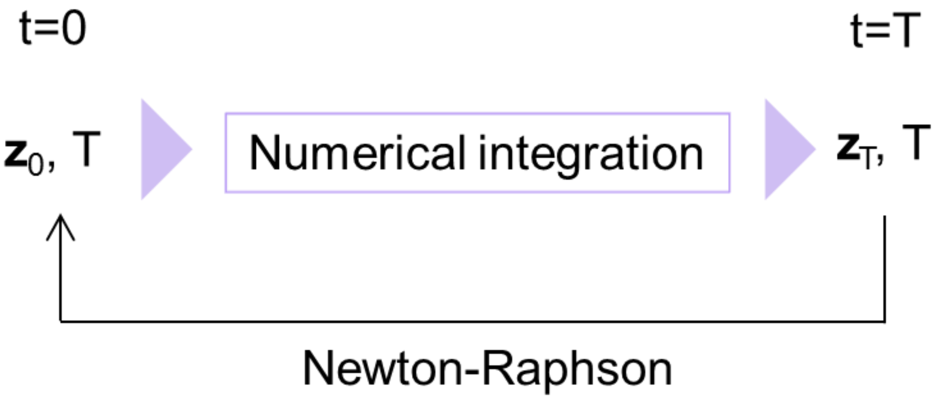
Shooting technique

Optimisation of the initial state of a system $[\mathbf{x}_0 \quad \dot{\mathbf{x}}_0]^T$ to obtain a periodic solution after time integration over a period T.



A more robust approach





Newton-Raphson

Example: find the zero of $f(x) = \frac{1}{2}(x-1)^2$

$$f(x_1) = f(x_0) + \left.\frac{df(x)}{dx}\right|_{x=x_0}(x_1 - x_0) = 0$$

$$\frac{1}{2}(x_0-1)^2 + (x_0-1)(x_1-x_0) = 0$$

$$x_1 = x_0 + \frac{-\frac{1}{2}(x_0-1)^2}{(x_0-1)} = x_0 - \frac{1}{2}(x_0-1) = \frac{x_0+1}{2}$$

$$x_{j+1} = \frac{x_j+1}{2}$$

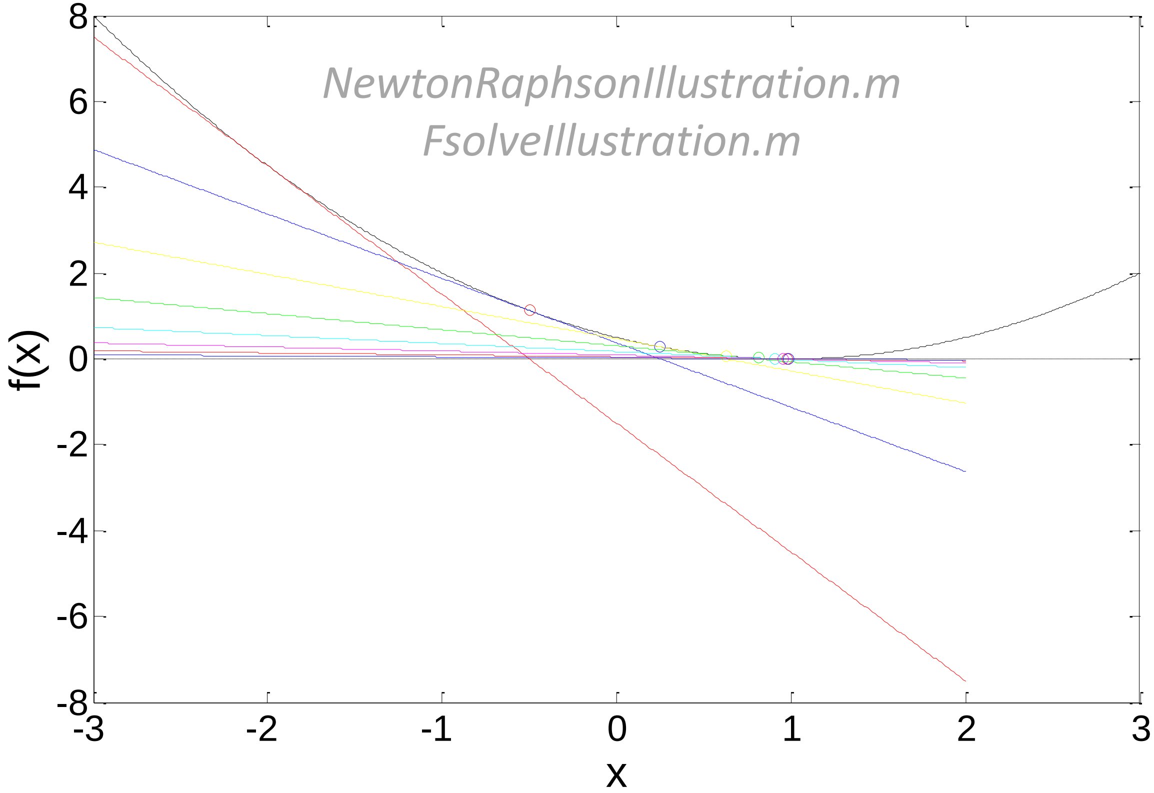
Matlab: homemade or fsolve

```
function NewtonRaphsonIllustration

clear all;close all;clc

plot([-3:0.01:3],0.5*([-3:0.01:3]-1).^2,'k',[-3:3],[0 0 0 0 0 0 0],'k--');hold on;pause

StartingPoint=-2;

while abs(0.5*(StartingPoint-1)^2)>0.0001
    plot([-3:0.01:3],0.5*(StartingPoint-1)^2+(StartingPoint-1)*([-3:0.01:3]-StartingPoint))

    StartingPoint=0.5*(StartingPoint+1),pause
end
SolutionFound=[StartingPoint 0.5*(StartingPoint-1)^2]
```

```
function FsolveIllustration

clear all;close all;clc

StartingPoint=-2;
SolutionFound=fsolve('Quadratic',StartingPoint)
```

```
function y=Quadratic(x)

y=0.5*(x-1)^2
```

Result


NewtonRaphsonIllustration.m
FsolveIllustration.m
f(x)
x
8
6
4
2
0
-2
-4
-6
-8
-3
-2
-1
0
1
2
3

State space formulation

$$M\ddot{x}(t) + Kx(t) + f_{nl}\{x(t), \dot{x}(t)\} = 0$$

$$\dot{z} = g(z, t)$$

State-space form

$$z^T = [x^T \;\; \dot{x}^T]$$

where

$$g(z) = \begin{bmatrix} \dot{x} \\ -M^{-1}[Kx + f_{nl}(x, \dot{x})] \end{bmatrix}$$

Shooting algorithm

$$\mathbf{H}(\mathbf{z}_{p0}, T) \equiv \mathbf{z}_p(T, \mathbf{z}_{p0}) - \mathbf{z}_{p0} = \mathbf{0}$$

Periodicity condition
(2-point BVP)

Numerical solution through iterations:

$$\mathbf{H}\left(\mathbf{z}_{p0}^{(0)}, T^{(0)}\right) + \left.\frac{\partial \mathbf{H}}{\partial \mathbf{z}_{p0}}\right|_{\left(\mathbf{z}_{p0}^{(0)}, T^{(0)}\right)} \Delta \mathbf{z}_{p0}^{(0)} + \left.\frac{\partial \mathbf{H}}{\partial T}\right|_{\left(\mathbf{z}_{p0}^{(0)}, T^{(0)}\right)} \Delta T^{(0)} + \xcancel{\text{H.O.T.}} = 0$$

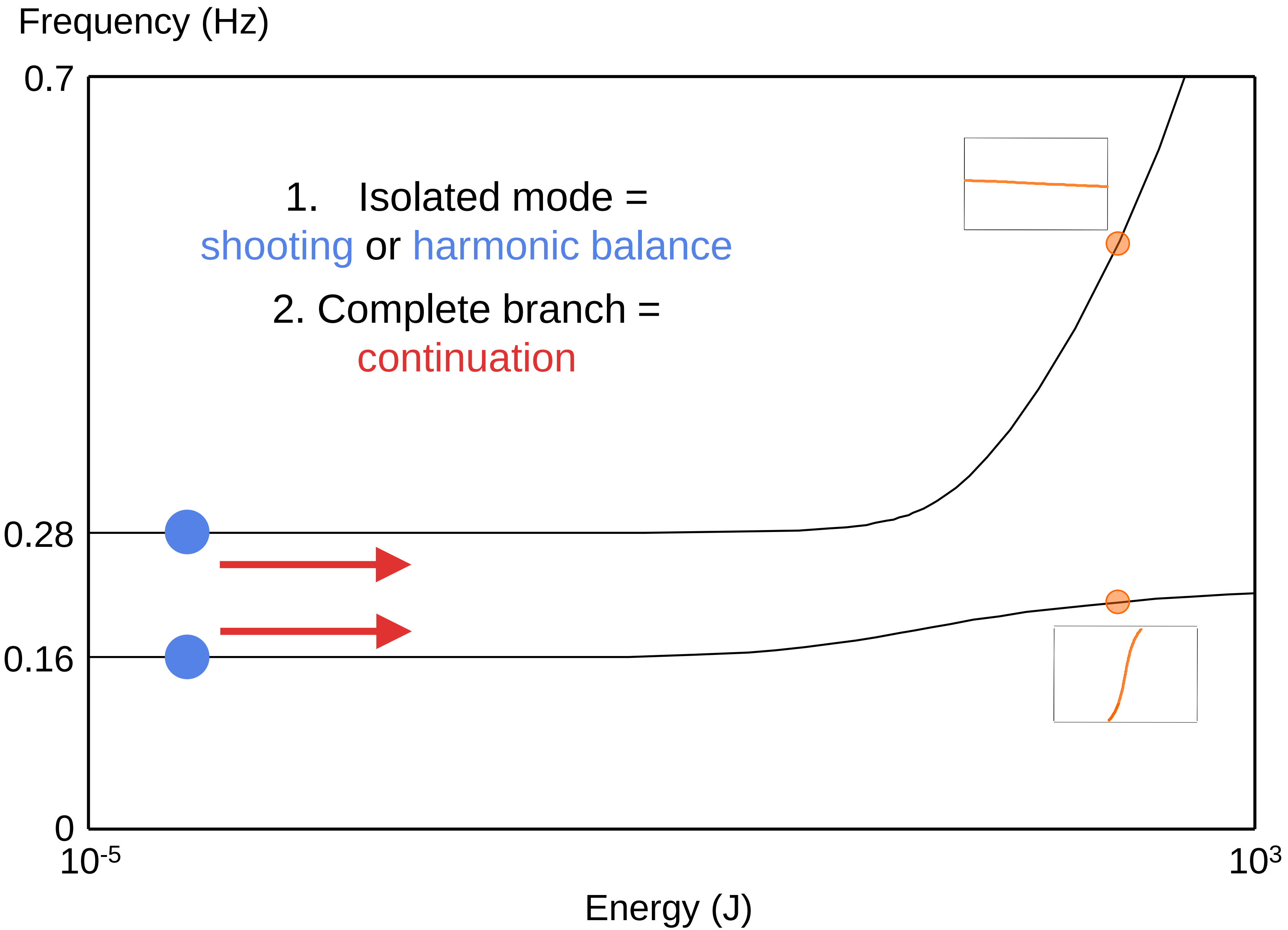
$$\frac{\partial \mathbf{H}}{\partial \mathbf{z}_0}(\mathbf{z}_0, T) = \left.\frac{\partial \mathbf{z}(t, \mathbf{z}_0)}{\partial \mathbf{z}_0}\right|_{t=T} - \mathbf{I}$$

2n x 2n — Monodromy matrix

$$\begin{aligned} \frac{\partial \mathbf{H}}{\partial T}(\mathbf{z}_0, T) &= \left.\frac{\partial \mathbf{z}(t, \mathbf{z}_0)}{\partial t}\right|_{t=T} \\ &= \mathbf{g}\left(\mathbf{z}\left(T, \mathbf{z}_0\right)\right) \end{aligned}$$

2n x 1

Combining shooting with continuation



More details

Nonlinear normal modes, Part II: Toward a practical computation using numerical continuation techniques

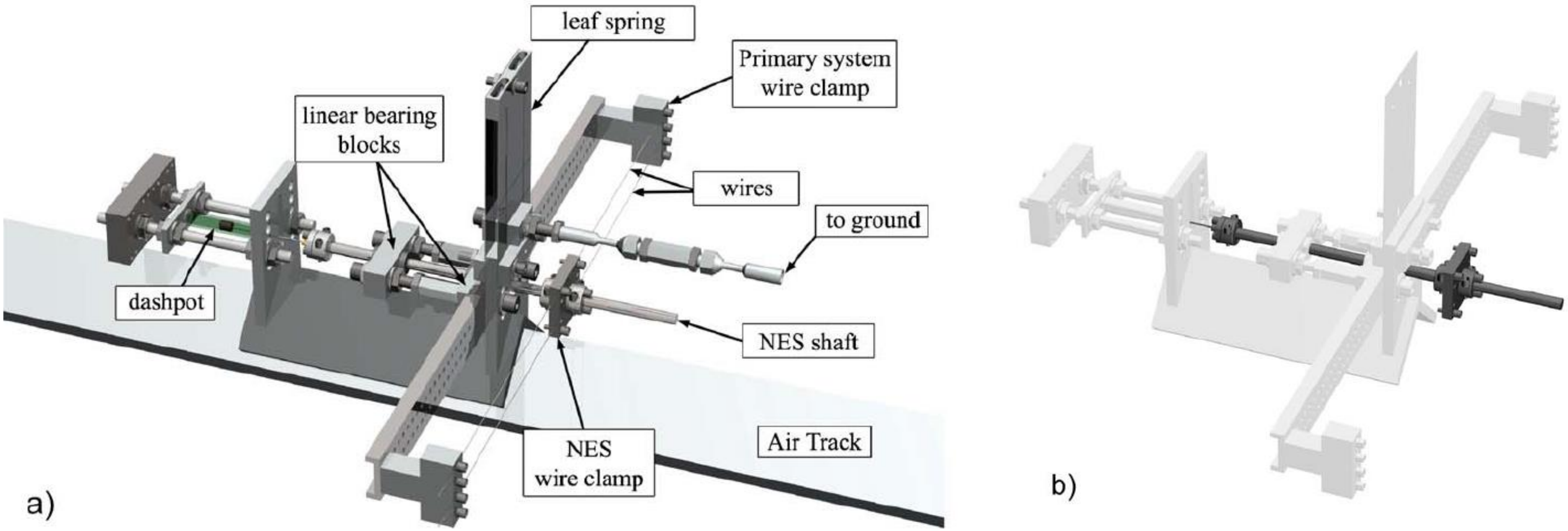
M. Peeters, R. Viguié, G. Sérandour, G. Kerschen *, J.-C. Golinval

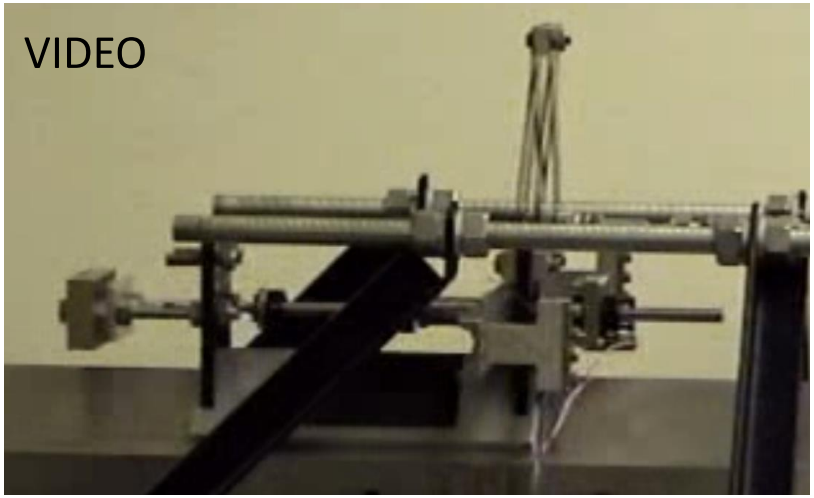
Structural Dynamics Research Group, Department of Aerospace and Mechanical Engineering, University of Liège, 1 Chemin des Chevreuils (B52/3), B-4000 Liège, Belgium

Mechanical Systems and Signal Processing 23 (2009) 195–216

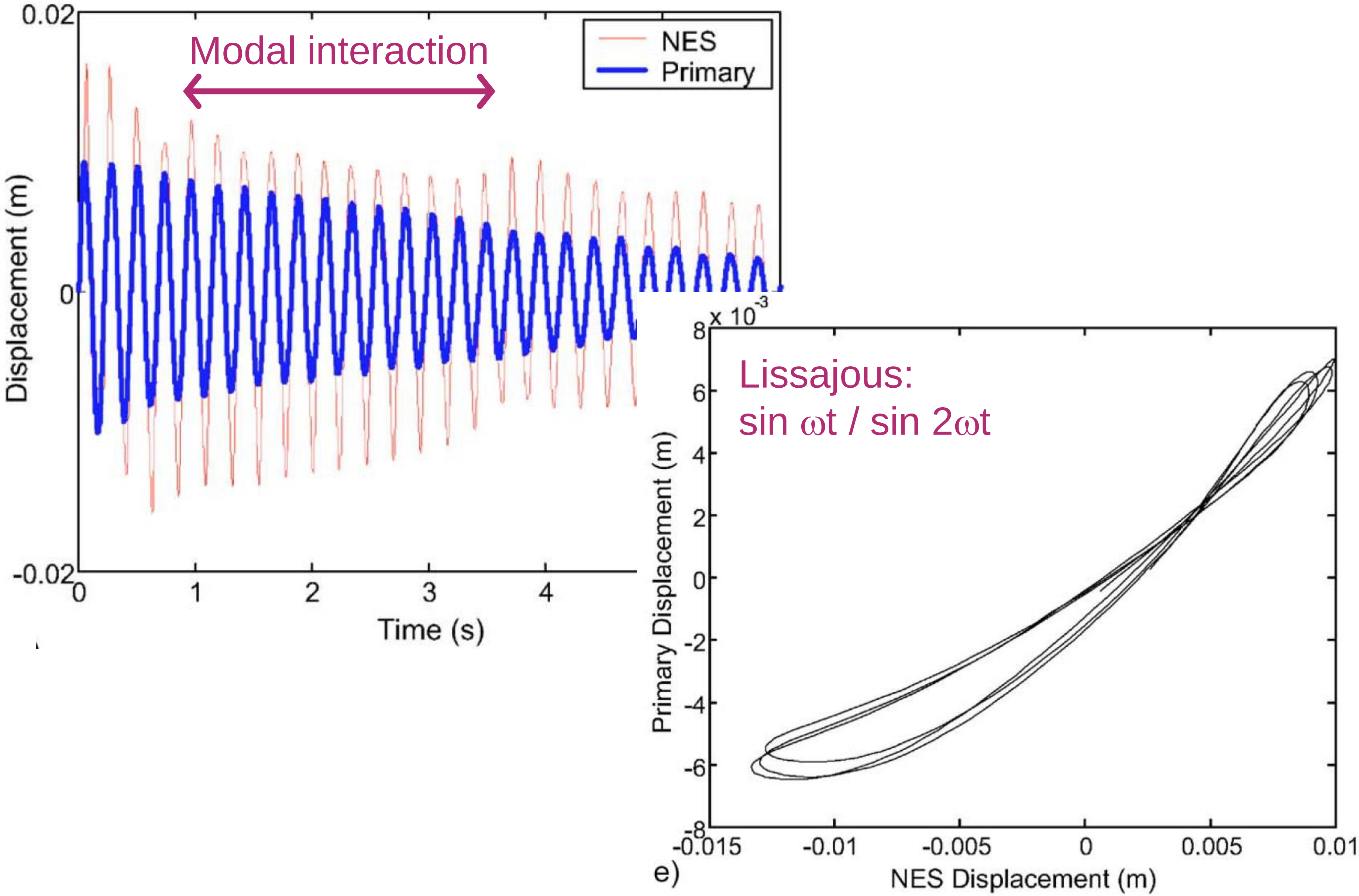
6. Existence ?

Do modal interactions exist in reality ?



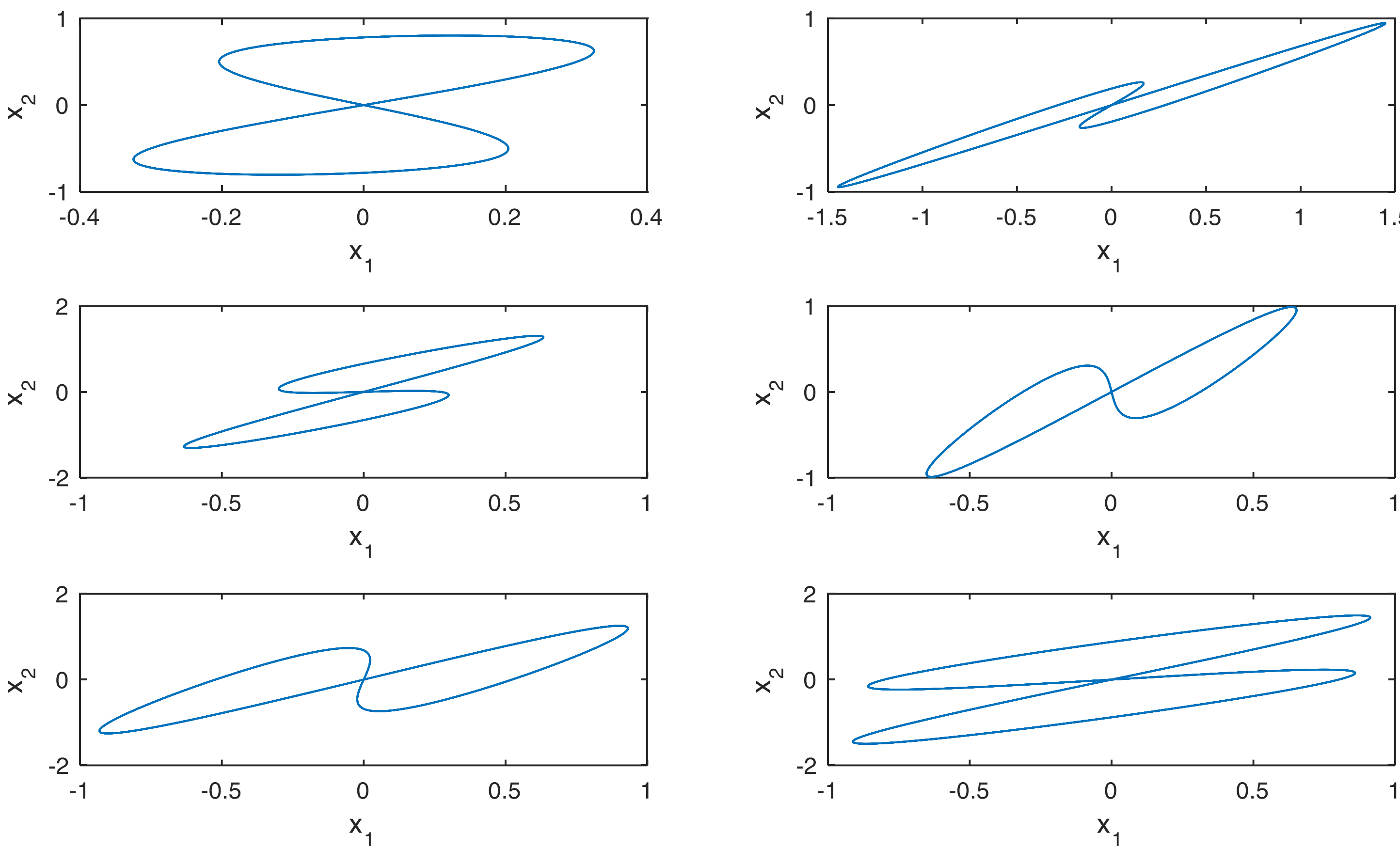


It seems


0.02
Modal interaction
NES
Primary
0
-0.02
0
1
2
3
4
Time (s)
Displacement (m)
x 10^-3
8
6
4
2
0
-2
-4
-6
-8
Lissajous:
sin ωt / sin 2ωt
Primary Displacement (m)
-0.015
-0.01
-0.005
0
0.005
0.01
NES Displacement (m)
e)

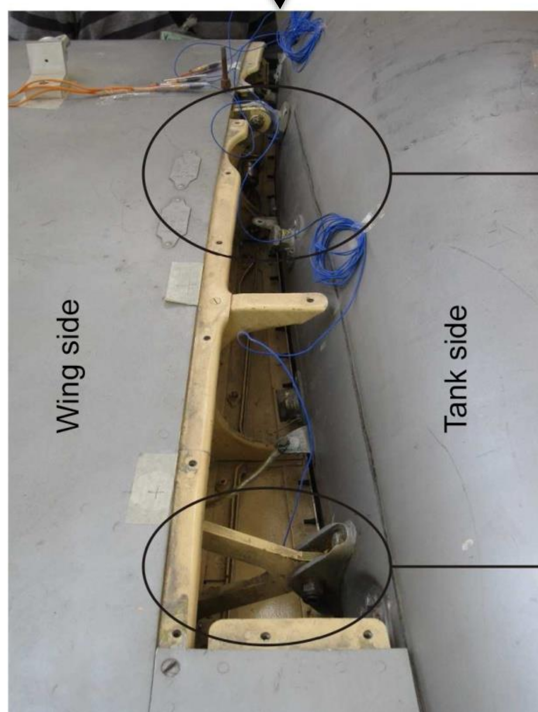
Lissajous curves…

```
>> LissajousNNM
1=sin/sin 2=cos/cos: 1
Enter harmonics 1:1
Enter harmonics 2:2
```

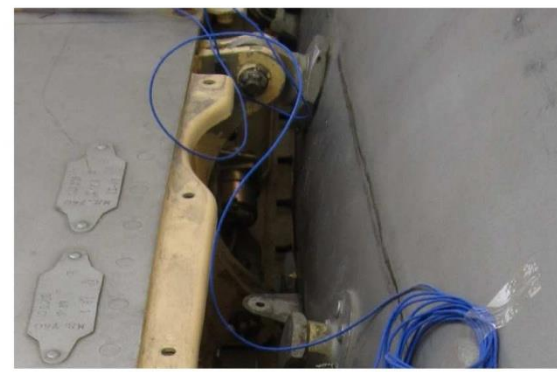


Do they exist in complex systems ?

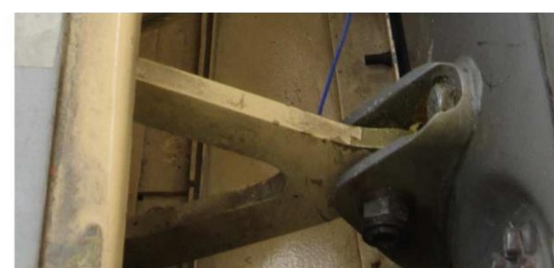
Bolted connections between external fuel tank and wing tip



Front connection



Rear connection



The testing campaign

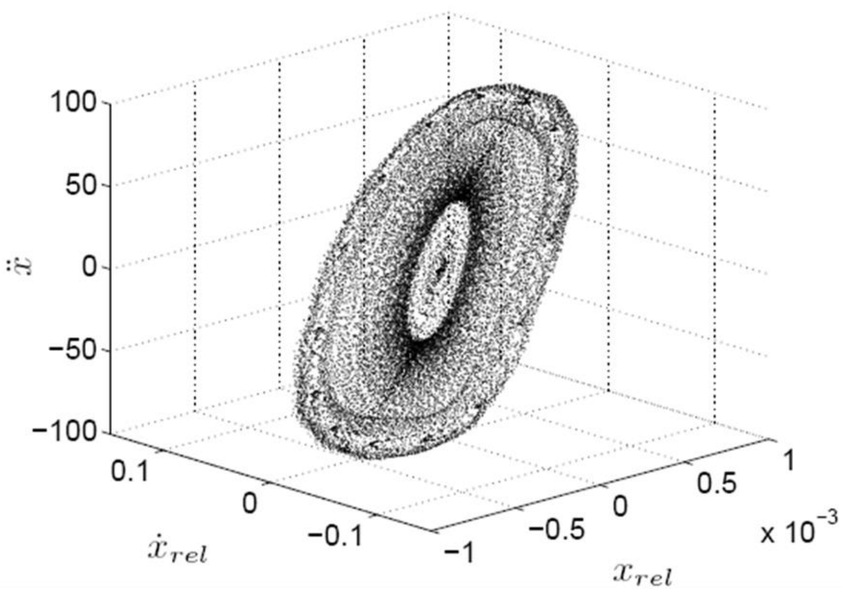
IMG_1679.avi

IMG_1680.MOV

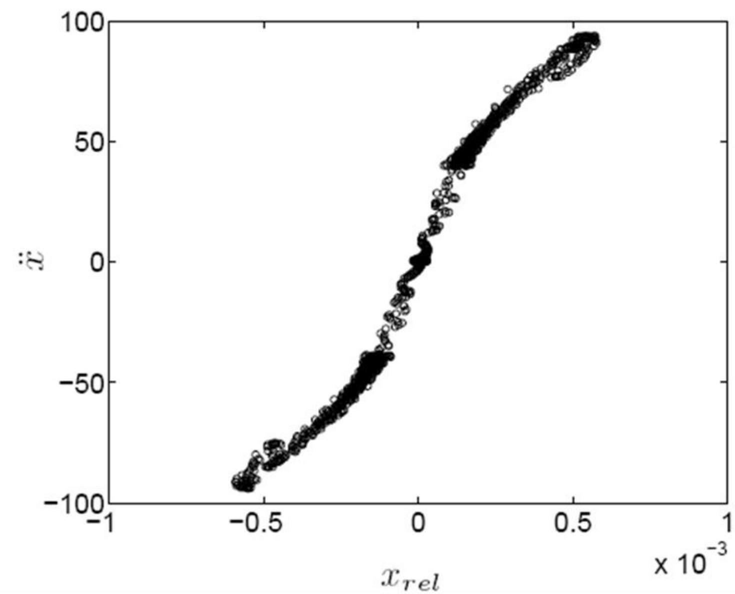
Softening nonlinearity in the bolted connections

Swept sine testing

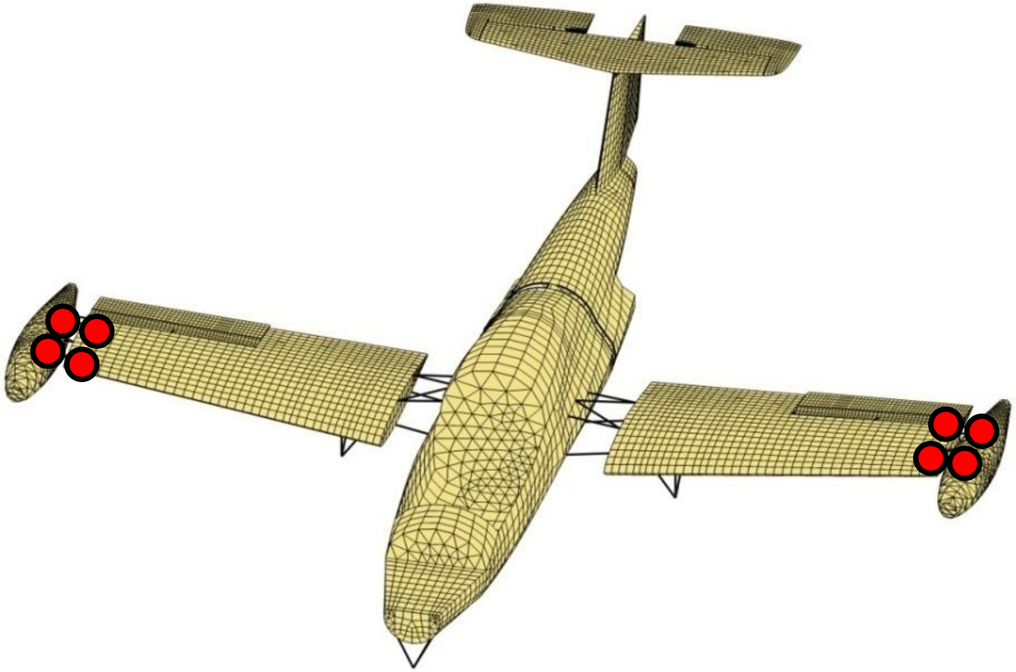
Accelaration surface method



Stiffness curve



Finite element model reduction



Finite element model
(2D shells and beams, 85000 DOFs)

Condensation of the linear components of the model

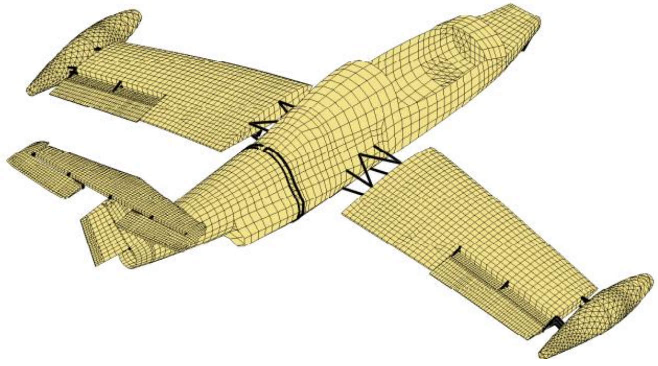
Craig-Bampton technique

- 8 remaining nodes
- 500 internal modes

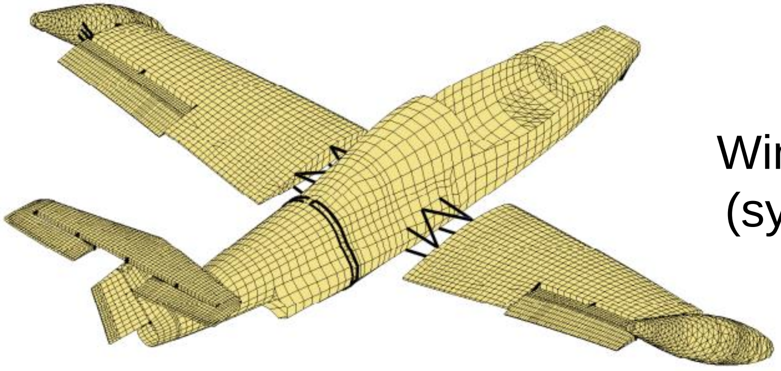
Reduced model accurate in [0-100] Hz, 548 DOFs

A close look at two modes

Mode	Freq. (Hz)	Mode	Freq. (Hz)
1	0.0936	13	21.2193
2	0.7260	14	22.7619
3	0.9606	15	23.6525
4	1.2118	16	25.8667
5	1.2153	17	28.2679
6	1.7951	18	29.3309
7	2.1072	19	31.0847
8	2.5157	20	34.9151
9	3.5736	21	39.5169
10	8.1913	22	40.8516
11	9.8644	23	47.3547
12	16.1790	24	52.1404



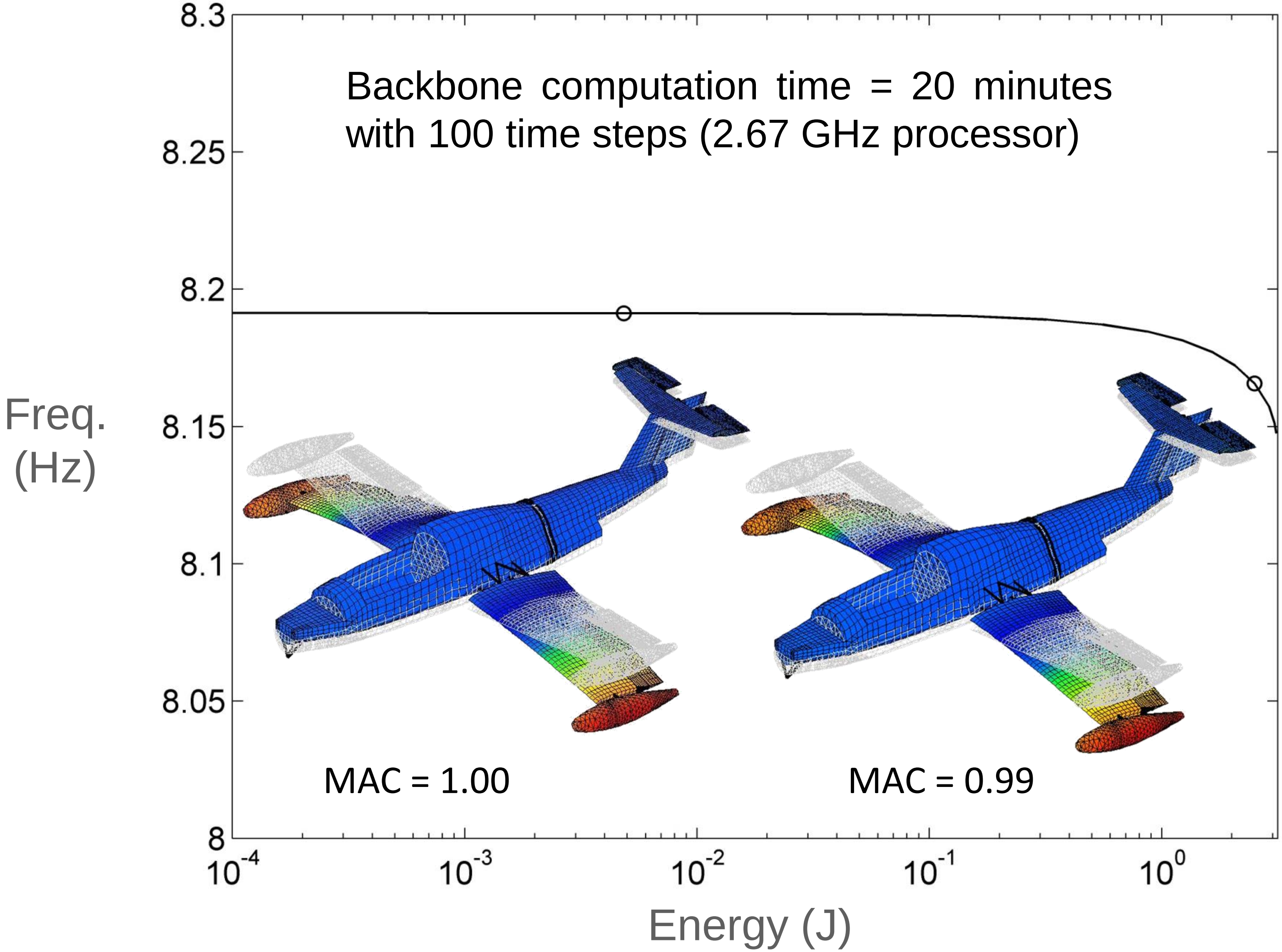
Wing bending



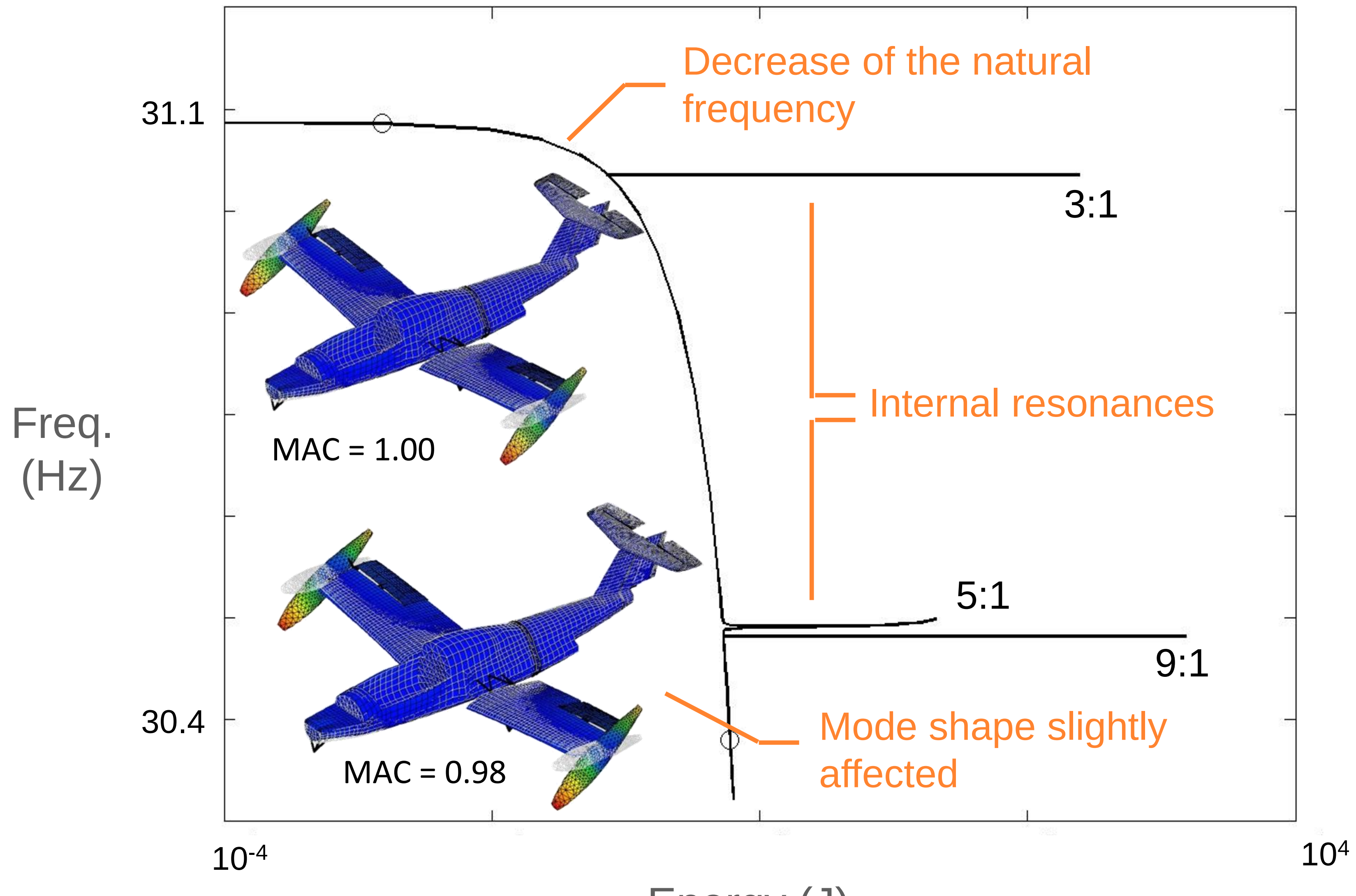
Wing torsion (symmetric)

The first wing bending mode is not affected

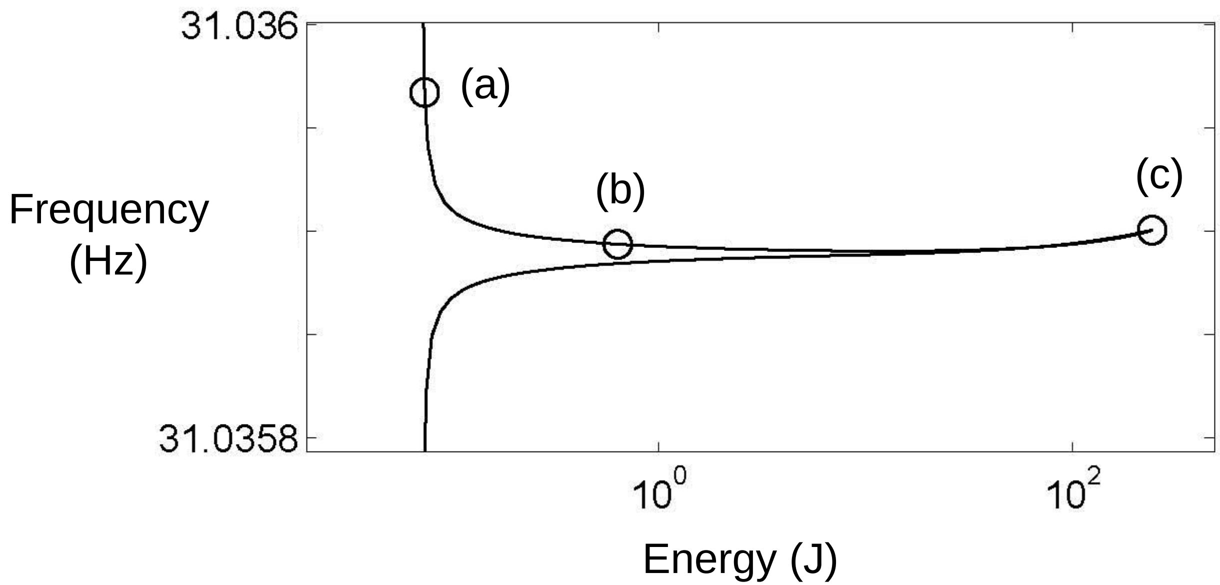
E:\NI2D\Marketing\DemosToolbox\Aircraft_ONERA\NNMcont

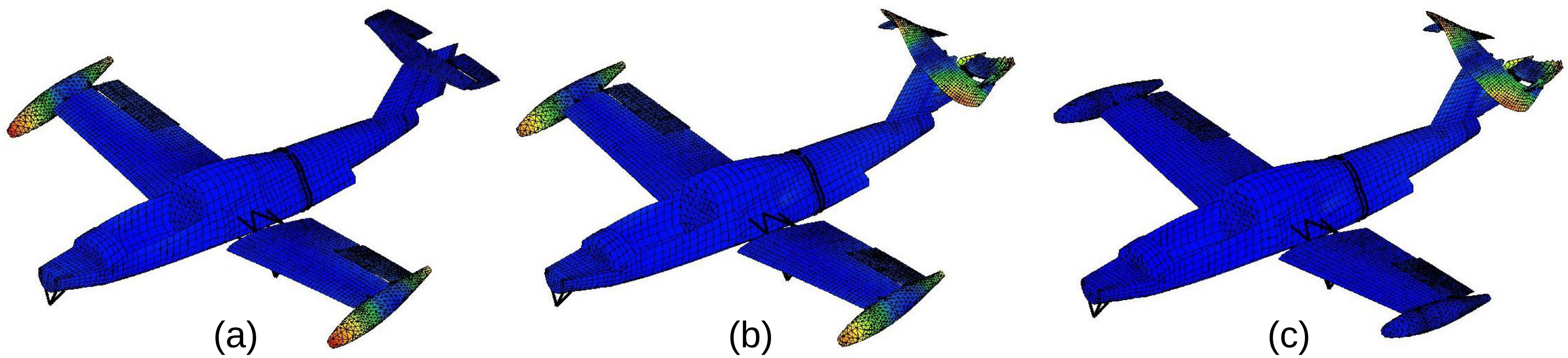


The first wing torsional mode is nonlinear

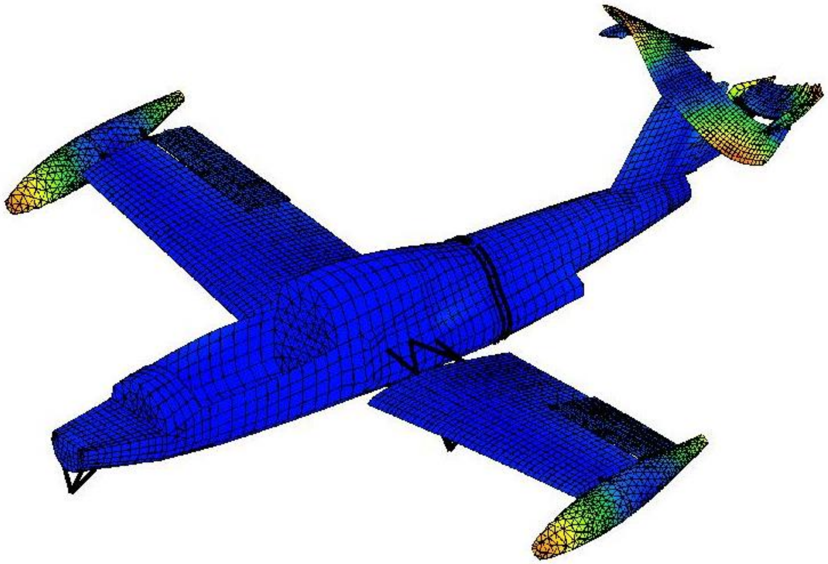


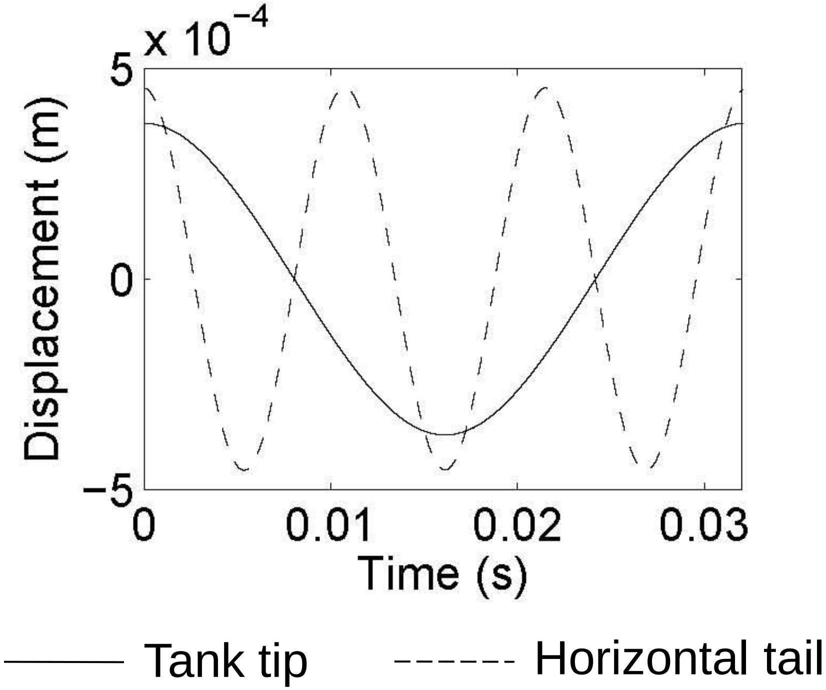
Close-up of the 3:1 modal interaction



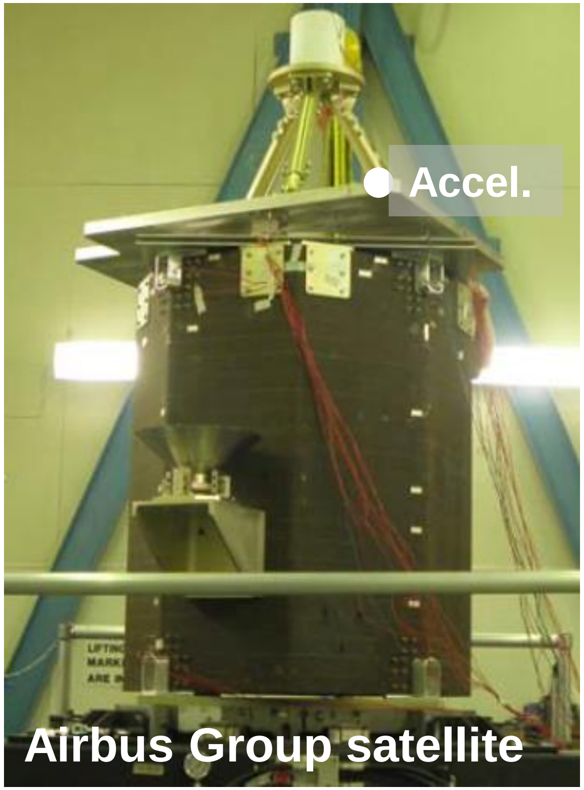


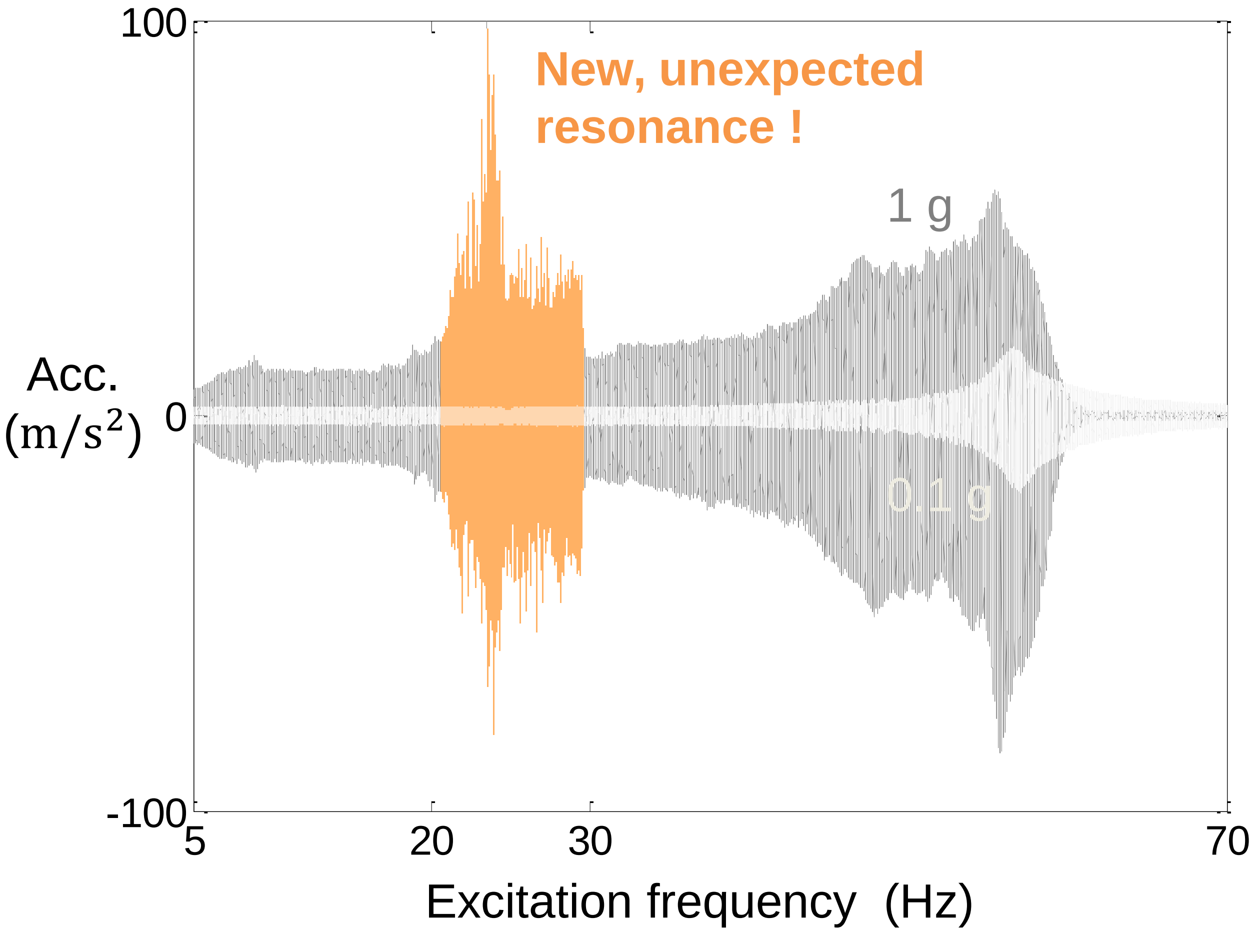
No resemblance with any linear mode



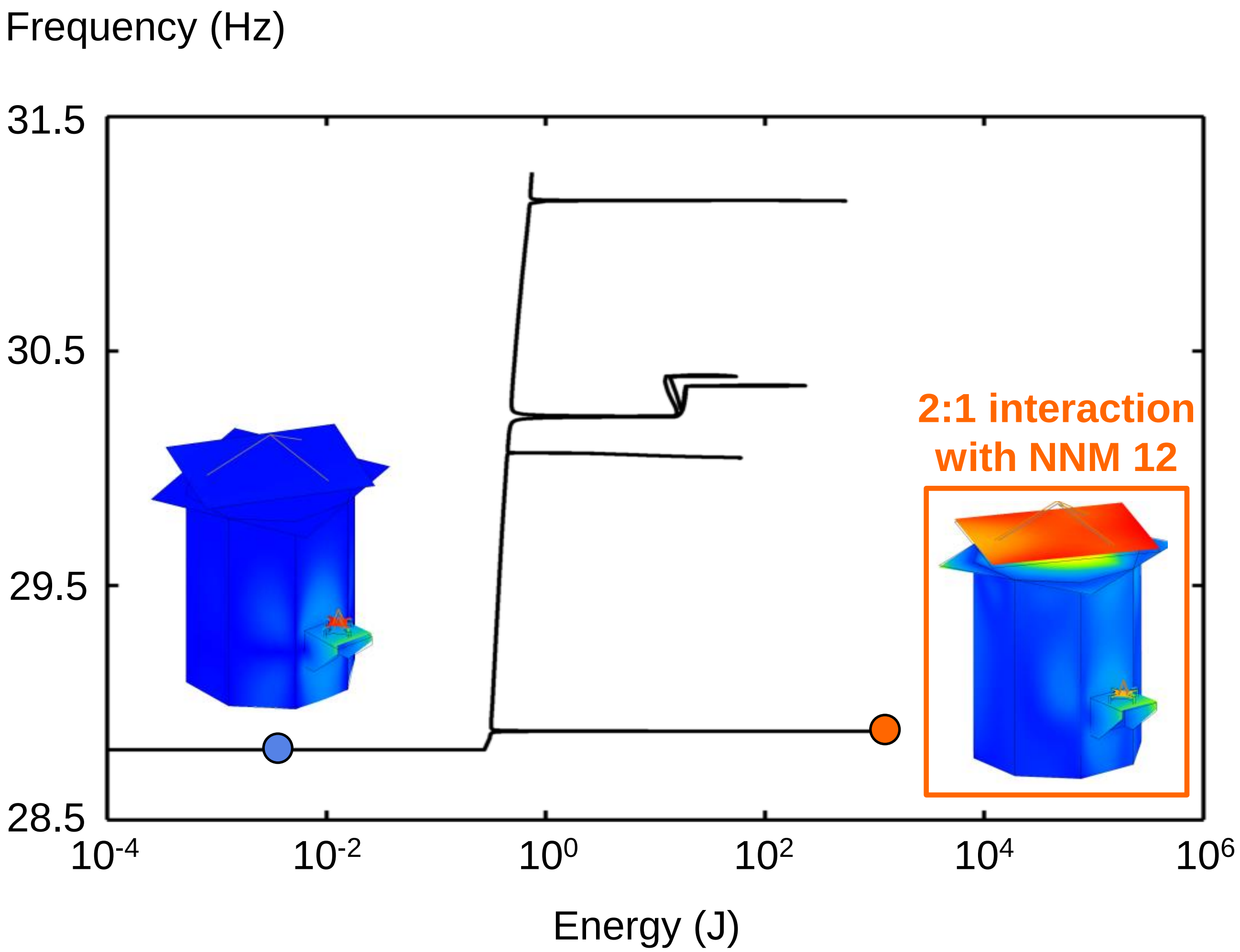


A dangerous nonlinear resonance





It can be predicted using an updated FEM



In summary

Clear physical meaning	LNMs	NNMs
Structural deformation at resonance	**YES**	**YES**
Synchronous vibration of the structure	**YES**	**YES, BUT…**
Important mathematical properties		
Orthogonality	**YES**	**NO**
Modal superposition	**YES**	**NO**

Key lessons learned

1. Frequency-amplitude/energy dependence

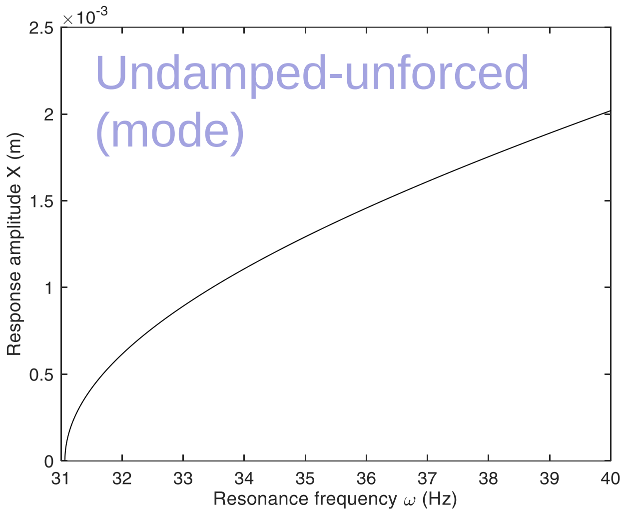
2. Harmonics

3. Bifurcations (additional resonances)

4. Stability

! Both for FRCs and modes !

Nonlinear modes and FRFs



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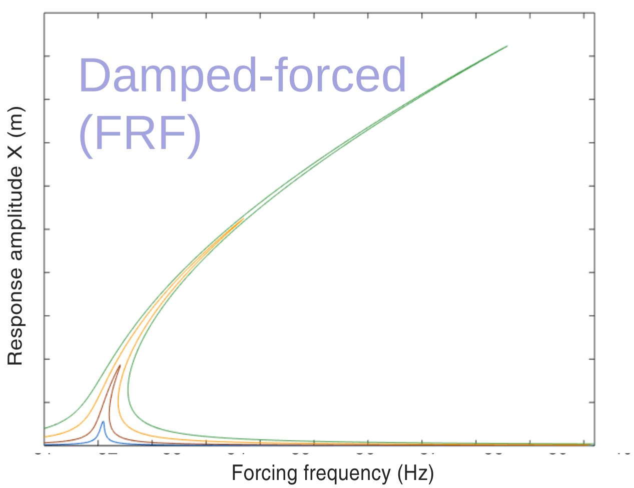

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