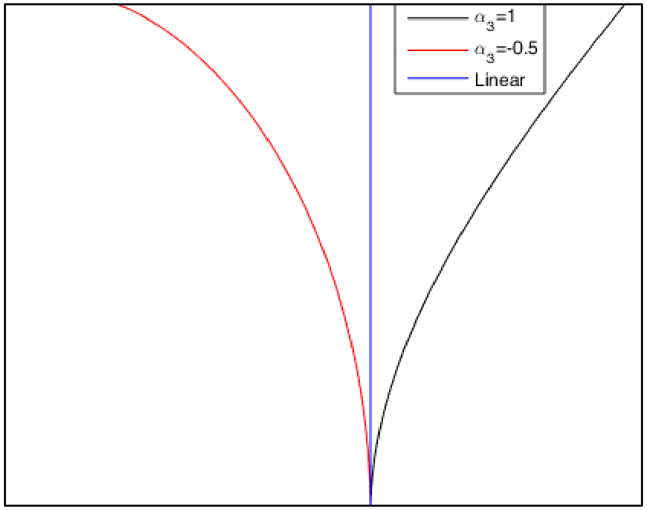
Nonlinear Vibrations of Aerospace Structures

University of Liège, Belgium

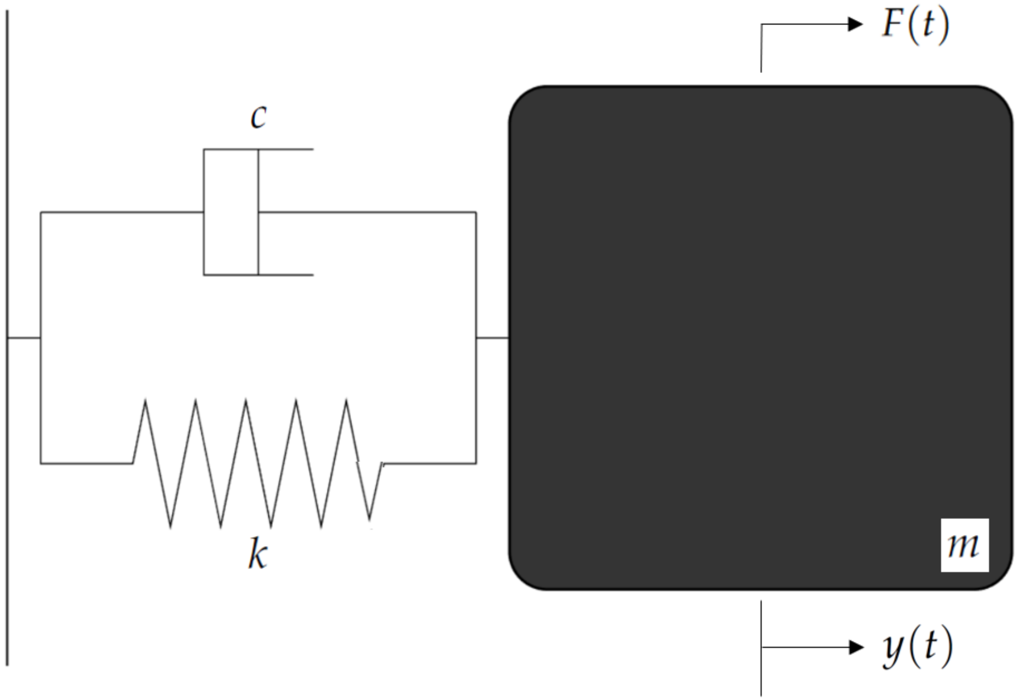
L01b 1DOF oscillators

Free vibration:

exact solutions



The spring-mass-damper oscillator



$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = F(t), \quad \dot{y}(0) = \dot{y}_0,\ y(0) = y_0$$

Important dynamical quantities

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = F(t), \quad \dot{y}(0) = \dot{y}_0,\ y(0) = y_0$$

Divide by m

$$\ddot{y}(t) + 2\xi\omega_0\dot{y}(t) + \omega_0^2 y(t) = f(t), \quad \dot{y}(0) = \dot{y}_0,\ y(0) = y_0$$

$\omega_0 = \sqrt{k/m},$	$\xi = \dfrac{c}{2\sqrt{km}},$	$f(t) = \dfrac{F(t)}{m}$
Natural frequency	*Damping ratio*	*Mass-normalized forcing*

Free vibration of 1DOF oscillators

$$\ddot{y}(t) + 2\xi\omega_0\dot{y}(t) + \omega_0^2 y(t) = 0, \quad \dot{y}(0) = \dot{y}_0,\ y(0) = y_0$$

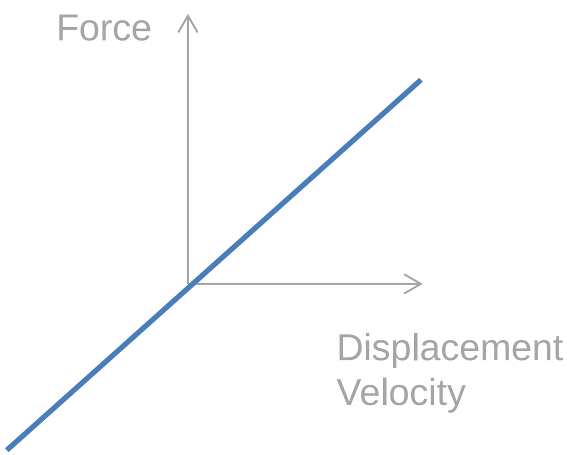
LINEAR

$$\ddot{y}(t) + f(y(t), \dot{y}(t)) = 0, \quad \dot{y}(0) = \dot{y}_0,\ y(0) = y_0$$

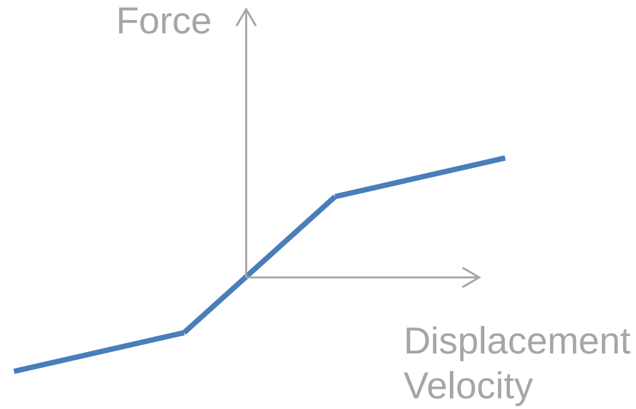
NONLINEAR

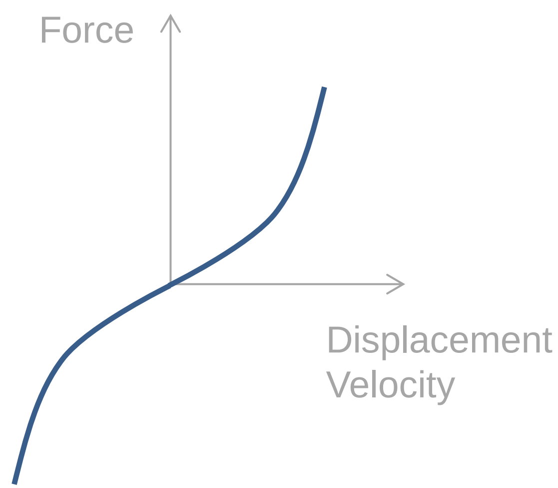
Nonlinear functions for mechanical systems

LINEAR



NONLINEAR





Historical perspective (from Kovacic and Brennan)

Leonhard Euler (1707–1783) was the first person to write down the equation of motion of a harmonically forced, undamped linear oscillator.

Euler formally introduced the nondimensional driving frequency Ω and noted that the response becomes infinite when Ω=1. He was the first person to explain the phenomenon of resonance.

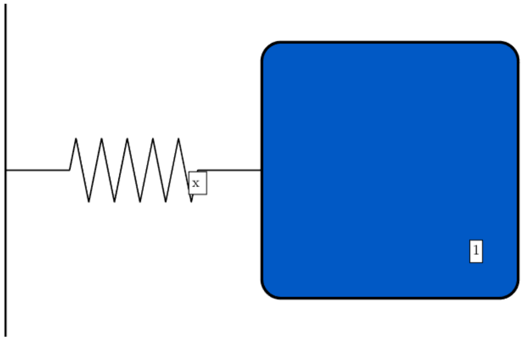
More than 100 years later, Helmholtz (1821–1894) was the first person to add a nonlinear stiffness term to Euler's equation of motion. He postulated that the eardrum behaved as an asymmetric oscillator, such that the restoring force was $f = kx + k_2x^2$.

Rayleigh investigated a symmetrical force–deflection characteristic given by $f = kx + k_3x^3$ (Duffing systems)

Exact solutions of undamped, unforced oscillators

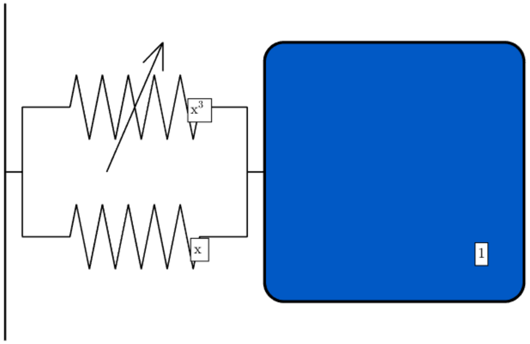
1. The linear (harmonic) oscillator:

Motivation
Exact solution
Key findings



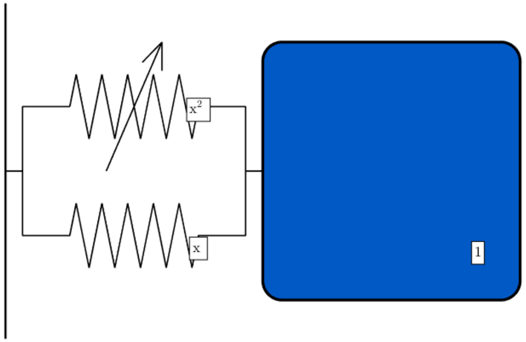
2. The Duffing oscillator

Motivation
Exact solution
Key findings



3. The Helmholtz oscillator

Motivation
Exact solution
Key findings



Colored boxes

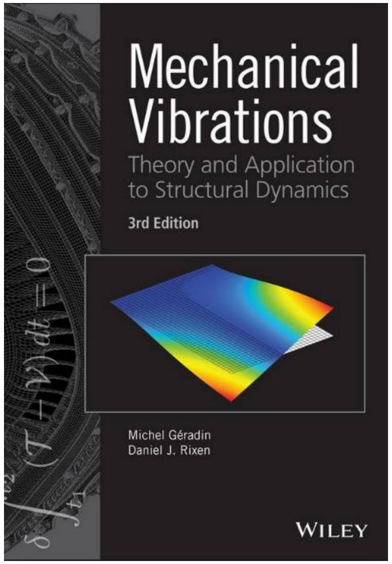
Important take-away message

I need your opinion

1. The linear (harmonic) oscillator

$$\ddot{y}(t) + \omega_0^2 y(t) = 0$$

Motivation for linear systems



2

Undamped Vibrations of n-Degree-of-Freedom Systems

From the theory and examples discussed in Chapter 1 one should remember that the description of the dynamics of a mechanical system as obtained from Hamilton's principle, from the Lagrange equations or from the principle of virtual work generally leads to a set of nonlinear equations. Solving such nonlinear equations usually requires applying time-integration techniques as those described later in Chapter 7. Nevertheless, for many practical applications, dynamical behaviour manifests itself only as time-varying perturbation around a static solution. Indeed systems can often be described as being in an statical equilibrium configuration, around which they undergo only small dynamic motion, namely *vibrations*.

In that case, the description of the system can be significantly simplified and a linearization around an equilibrium position of the generally nonlinear dynamic equations is possible.

Small displacements and rotations

$$\boxed{\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right)}$$

Green's strain tensor

A particular case: linear deformation

The *geometric linearity* assumption may be split into two parts:

1. The extension strains remain infinitesimal:

$$\frac{\partial u_i}{\partial x_i} \ll 1 \qquad i = 1, 2, 3 \tag{4.4}$$

2. The rotations have small amplitudes:

$$\frac{\partial u_i}{\partial x_j} \ll 1 \qquad i \neq j \tag{4.5}$$

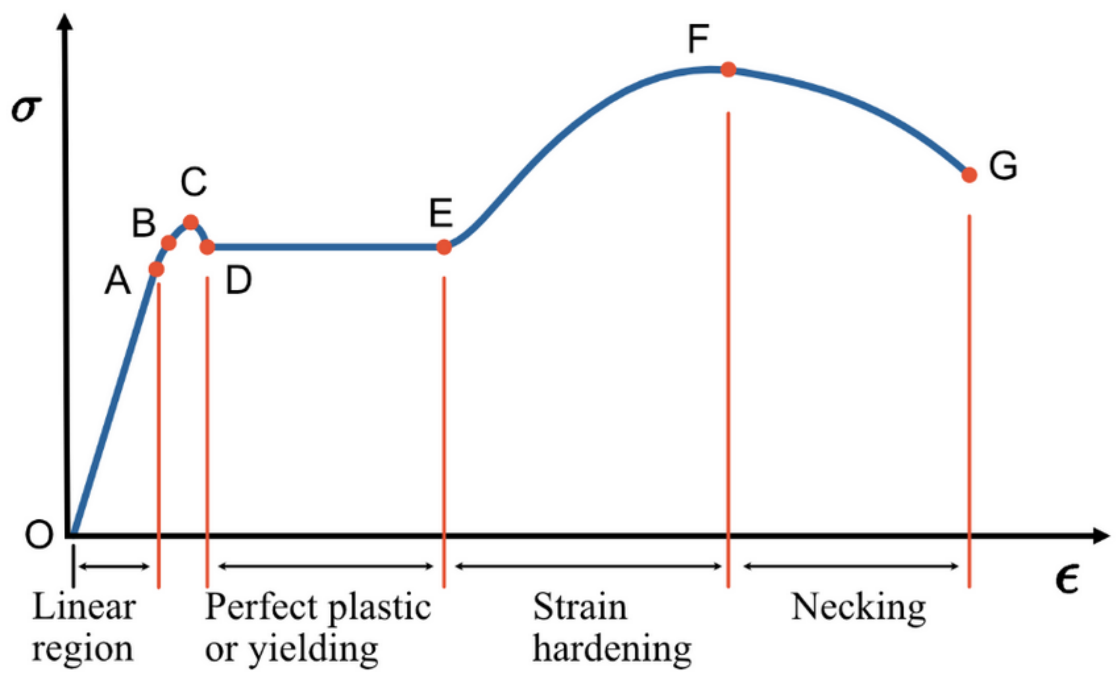
This leads to the linear expression of the infinitesimal strain tensor

$$\boxed{\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)} \tag{4.6}$$

Linear material: steel, aluminum, titanium

A material has linear elastic properties when the stress state remains strictly proportional to the strain state. Owing to the symmetry of the tensors σ_{ij} and ε_{kl}, such a material is characterized in the general case by 21 distinct coefficients C_{ijkl} so that:

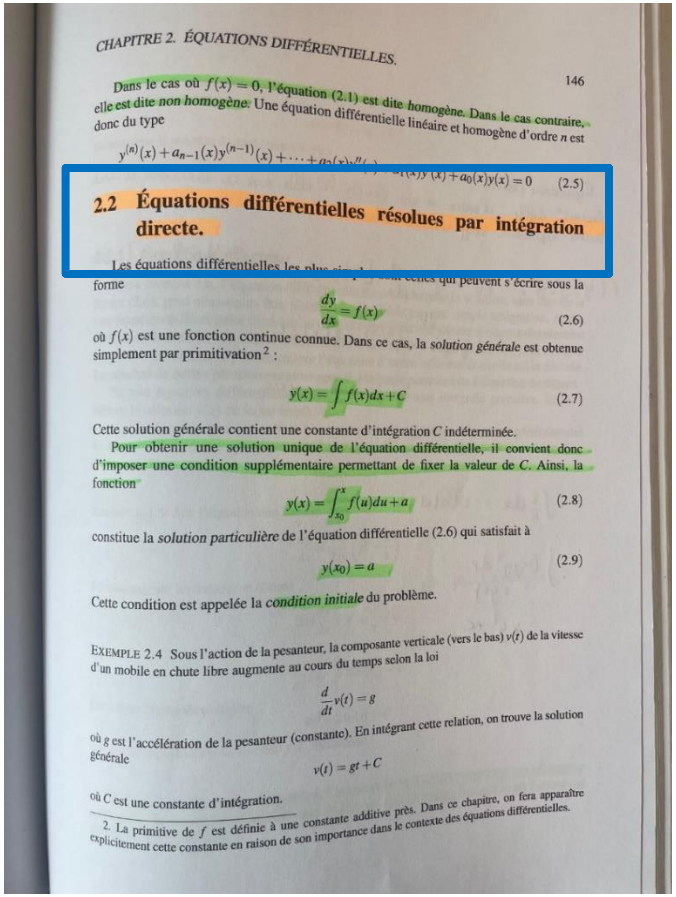
$$\sigma_{ij} = C_{ijkl}\, \varepsilon_{kl} \tag{4.14}$$



Mild steel

How to find the exact solution ?

$$\ddot{y}(t) + \omega_0^2 y(t) = 0, \quad \dot{y}(0) = \dot{y}_0, \; y(0) = y_0$$



CHAPITRE 2. ÉQUATIONS DIFFÉRENTIELLES 146

Dans le cas où $f(x) = 0$, l'équation (2.1) est dite *homogène*. Dans le cas contraire, elle est dite *non homogène*. Une équation différentielle linéaire et homogène d'ordre n est donc du type

$$y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_0(x)y(x) = 0 \qquad (2.5)$$

2.2 Équations différentielles résolues par intégration directe.

Les équations différentielles les plus simples sont celles qui peuvent s'écrire sous la forme

$$\frac{dy}{dx} = f(x) \qquad (2.6)$$

où $f(x)$ est une fonction continue connue. Dans ce cas, la *solution générale* est obtenue simplement par primitivation[2] :

$$y(x) = \int f(x)dx + C \qquad (2.7)$$

Cette solution générale contient une constante d'intégration C indéterminée.

Pour obtenir une solution unique de l'équation différentielle, il convient donc d'imposer une condition supplémentaire permettant de fixer la valeur de C. Ainsi, la fonction

$$y(x) = \int_{x_0}^{x} f(u)du + a \qquad (2.8)$$

constitue la *solution particulière* de l'équation différentielle (2.6) qui satisfait à

$$y(x_0) = a \qquad (2.9)$$

Cette condition est appelée la *condition initiale* du problème.

EXEMPLE 2.4 Sous l'action de la pesanteur, la composante verticale (vers le bas) $v(t)$ de la vitesse d'un mobile en chute libre augmente au cours du temps selon la loi

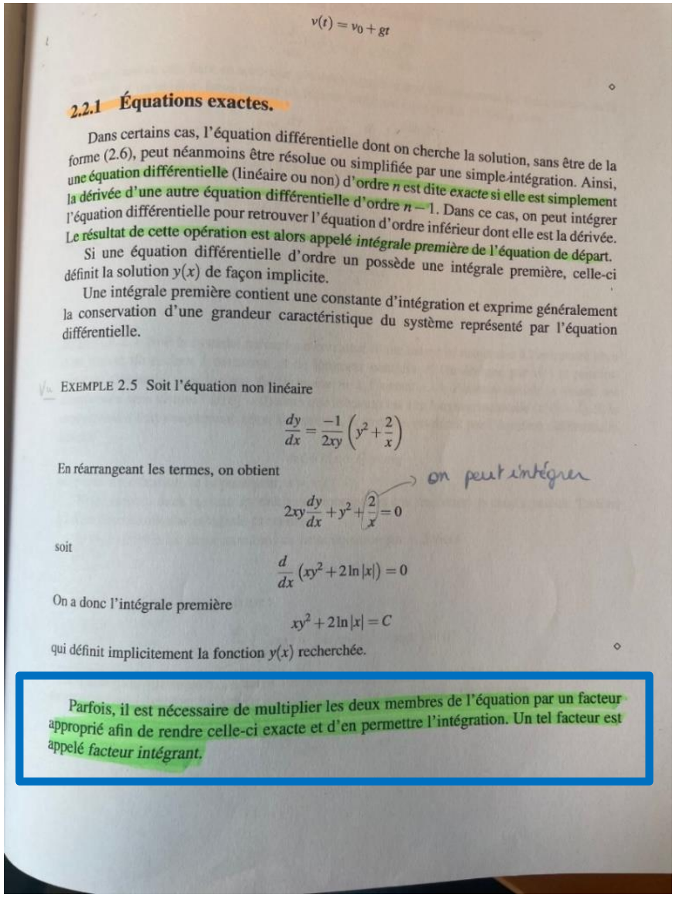
$$\frac{d}{dt}v(t) = g$$

où g est l'accélération de la pesanteur (constante). En intégrant cette relation, on trouve la solution générale

$$v(t) = gt + C$$

où C est une constante d'intégration.

2. La primitive de f est définie à une constante additive près. Dans ce chapitre, on fera apparaître explicitement cette constante en raison de son importance dans le contexte des équations différentielles.



$$v(t) = v_0 + gt$$

2.2.1 Équations exactes.

Dans certains cas, l'équation différentielle dont on cherche la solution, sans être de la forme (2.6), peut néanmoins être résolue ou simplifiée par une simple intégration. Ainsi, une équation différentielle (linéaire ou non) d'ordre n est dite *exacte* si elle est simplement la dérivée d'une autre équation différentielle d'ordre $n-1$. Dans ce cas, on peut intégrer l'équation différentielle pour retrouver l'équation d'ordre inférieur dont elle est la dérivée. Le résultat de cette opération est alors appelé *intégrale première* de l'équation de départ.

Si une équation différentielle d'ordre un possède une intégrale première, celle-ci définit la solution $y(x)$ de façon implicite.

Une intégrale première contient une constante d'intégration et exprime généralement la conservation d'une grandeur caractéristique du système représenté par l'équation différentielle.

EXEMPLE 2.5 Soit l'équation non linéaire

$$\frac{dy}{dx} = \frac{-1}{2xy}\left(y^2 + \frac{2}{x}\right)$$

En réarrangeant les termes, on obtient

$$2xy\frac{dy}{dx} + y^2 + \frac{2}{x} = 0$$

on peut intégrer

soit

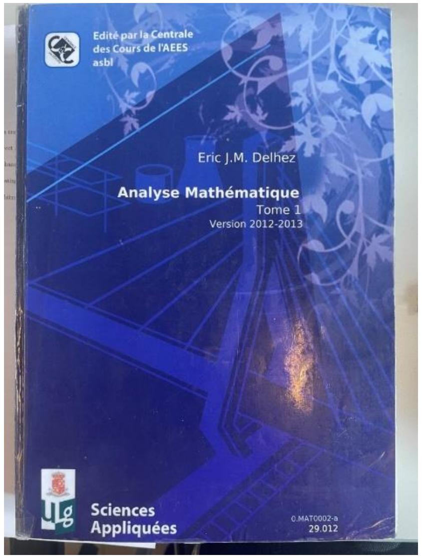
$$\frac{d}{dx}\left(xy^2 + 2\ln|x|\right) = 0$$

On a donc l'intégrale première

$$xy^2 + 2\ln|x| = C$$

qui définit implicitement la fonction $y(x)$ recherchée. ◇

Parfois, il est nécessaire de multiplier les deux membres de l'équation par un facteur approprié afin de rendre celle-ci exacte et d'en permettre l'intégration. Un tel facteur est appelé *facteur intégrant*.



Exact solution by integration

$$\ddot{y}(t) + \omega_0^2 y(t) = 0, \quad \dot{y}(0) = \dot{y}_0,\ y(0) = y_0$$

Multiply by velocity & integrate

$$\frac{\dot{y}^2(t)}{2} + \frac{\omega_0^2 y^2(t)}{2} = \frac{\dot{y}_0^2}{2} + \frac{\omega_0^2 y_0}{2}$$

$$\omega_0\, t = \pm \int_{y_0}^{y} \frac{dy}{\sqrt{\left(y_0^2 + \frac{\dot{y}_0^2}{\omega_0^2}\right) - y^2}}$$

$$\omega_0\, t = \pm \int_{y_0}^{y} \frac{dy}{\sqrt{a^2 - y^2}} \qquad a^2 = y_0^2 + \frac{\dot{y}_0^2}{\omega_0^2}$$

Exact solution by integration

$$\omega_0 \, t = \pm \int_{y_0}^{y} \frac{dy}{\sqrt{a^2 - y^2}}$$

Handbook of Mathematical Functions

With

Formulas, Graphs, and Mathematical Tables

Edited by
Milton Abramowitz and Irene A. Stegun

3.3.44 $$\int \frac{dx}{(a^2 - x^2)^{\frac{1}{2}}} = \arcsin \frac{x}{a}$$

https://personal.math.ubc.ca/~cbm/aands/abramowitz_and_stegun.pdf

The exact solution of the linear oscillator

$$\omega_0 t = \pm \int_{y_0}^{y} \frac{dy}{\sqrt{a^2 - y^2}}$$

$$a^2 = y_0^2 + \frac{\dot{y}_0^2}{\omega_0^2}$$

$$\int \frac{dx}{(a^2 - x^2)^{\frac{1}{2}}} = \arcsin \frac{x}{a}$$

$$\omega_0 t = \pm \left[\sin^{-1} \left(\frac{y}{\sqrt{y_0^2 + \frac{\dot{y}_0^2}{\omega_0^2}}} \right) - \sin^{-1} \left(\frac{y_0}{\sqrt{y_0^2 + \frac{\dot{y}_0^2}{\omega_0^2}}} \right) \right]$$

$$\sin^{-1}(x) = \tan^{-1} \frac{x}{\sqrt{1 - x^2}}$$

$$\omega_0 t = \pm \left[\sin^{-1} \left(\frac{y}{\sqrt{y_0^2 + \frac{\dot{y}_0^2}{\omega_0^2}}} \right) - \tan^{-1} \frac{\omega_0 y_0}{\dot{y}_0} \right]$$

$$\boxed{y(t) = \sqrt{y_0^2 + \frac{\dot{y}_0^2}{\omega_0^2}} \sin \left(\omega_0 t + \tan^{-1} \frac{\omega_0 y_0}{\dot{y}_0} \right)}$$

EXACT SOLUTION

Newmark's method for linear systems

$$\mathbf{M}, \mathbf{C}, \mathbf{K} \quad \mathbf{q}_0, \dot{\mathbf{q}}_0$$

Compute $\ddot{\mathbf{q}}_0$

Time incrementation

$$t_{n+1} = t_n + h$$

Prediction

$$\dot{\mathbf{q}}^*_{n+1} = \dot{\mathbf{q}}_n + (1-\gamma)\, h\, \ddot{\mathbf{q}}_n$$

$$\mathbf{q}^*_{n+1} = \mathbf{q}_n + h\,\dot{\mathbf{q}}_n + (0.5-\beta)\, h^2\, \ddot{\mathbf{q}}_n$$

Computation of accelerations

$$\mathbf{S} = \mathbf{M} + h\,\gamma\,\mathbf{C} + h^2\,\beta\,\mathbf{K}$$

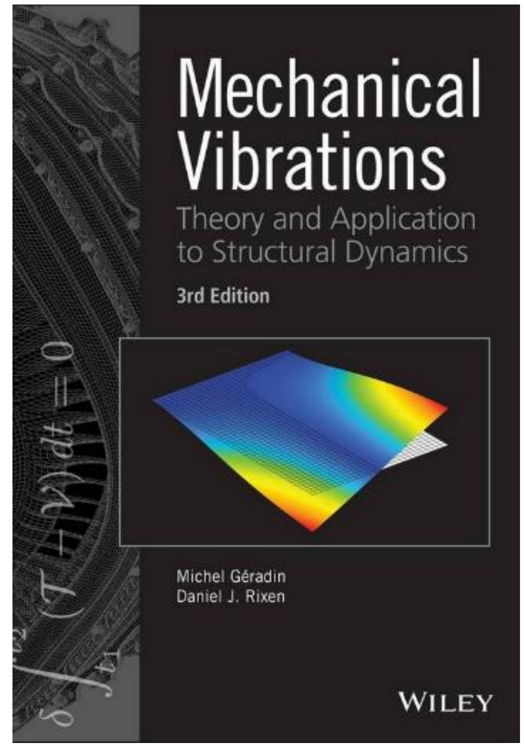
$$\mathbf{S}\,\ddot{\mathbf{q}}_{n+1} = \mathbf{p}_{n+1} - \mathbf{C}\,\dot{\mathbf{q}}^*_{n+1} - \mathbf{K}\,\mathbf{q}^*_{n+1}$$

Correction

$$\dot{\mathbf{q}}_{n+1} = \dot{\mathbf{q}}^*_{n+1} + h\,\gamma\,\ddot{\mathbf{q}}_{n+1}$$

$$\mathbf{q}_{n+1} = \mathbf{q}^*_{n+1} + h^2\,\beta\,\ddot{\mathbf{q}}_{n+1}$$

Newmark integration scheme for linear systems



Verification using numerical simulation (Newmark)

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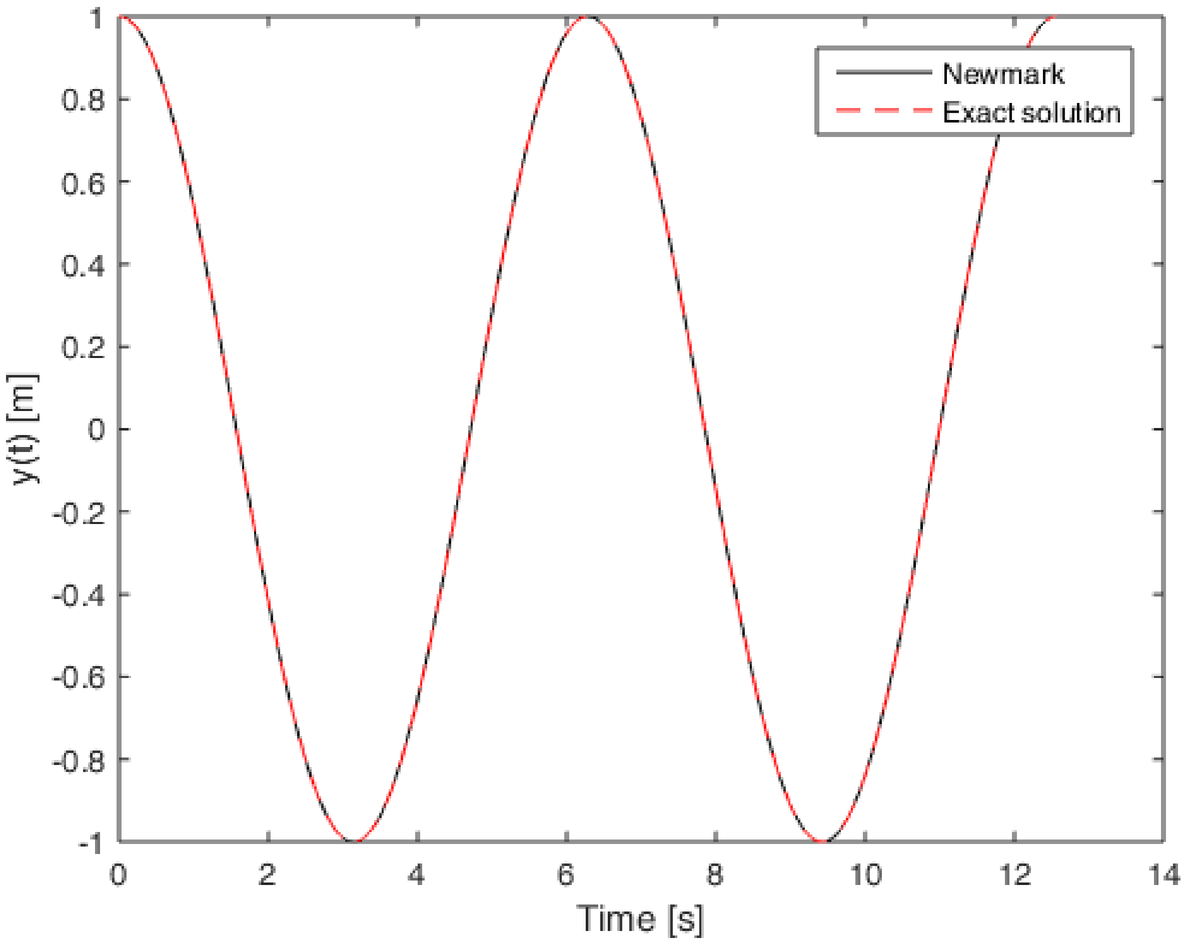


First important findings

$$y(t) = \sqrt{y_0^2 + \frac{\dot{y}_0^2}{\omega_0^2}} \sin\left(\omega_0 t + \tan^{-1}\frac{\omega_0 y_0}{\dot{y}_0}\right) \qquad \omega_0 = \sqrt{k/m}$$

Amplitude Frequency Phase

The response of a linear oscillator takes the form of harmonic motion at the natural frequency ω_0

The natural frequency ω_0 does not depend on the initial conditions but depends only on k and m.

Second important finding

Response to y_0 and $\dot{y}_0$

$$y(t) = \boxed{\sqrt{y_0^2 + \frac{\dot{y}_0^2}{\omega_0^2}} \sin\left(\omega_0 t + \tan^{-1}\frac{\omega_0 y_0}{\dot{y}_0}\right)}$$

$$= \sqrt{y_0^2 + \frac{\dot{y}_0^2}{\omega_0^2}} \left(\sin\omega_0 t \cos\left(\tan^{-1}\frac{\omega_0 y_0}{\dot{y}_0}\right) + \cos\omega_0 t \sin\left(\tan^{-1}\frac{\omega_0 y_0}{\dot{y}_0}\right)\right)$$

$$= \sqrt{y_0^2 + \frac{\dot{y}_0^2}{\omega_0^2}} \left(\sin\omega_0 t \frac{1}{\sqrt{1+\frac{\omega_0^2 y_0^2}{\dot{y}_0^2}}} + \cos\omega_0 t \frac{\frac{\omega_0 y_0}{\dot{y}_0}}{\sqrt{1+\frac{\omega_0^2 y_0^2}{\dot{y}_0^2}}}\right)$$

$$= \boxed{\frac{\dot{y}_0}{\omega_0}\sin\omega_0 t} + \boxed{y_0 \cos\omega_0 t}$$

Response to $\dot{y}_0$ only

Response to y_0 only

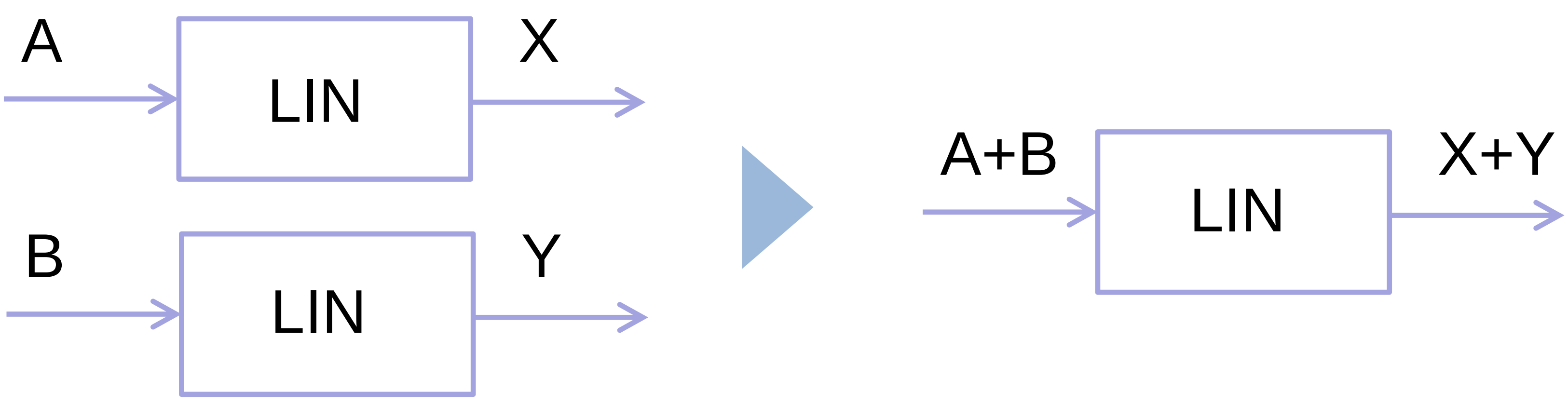
$$\cos(\mathrm{Arctan}(x)) = \frac{1}{\sqrt{1+x^2}}$$

$$\sin(\mathrm{Arctan}(x)) = \frac{x}{\sqrt{1+x^2}}$$

The principle of superposition

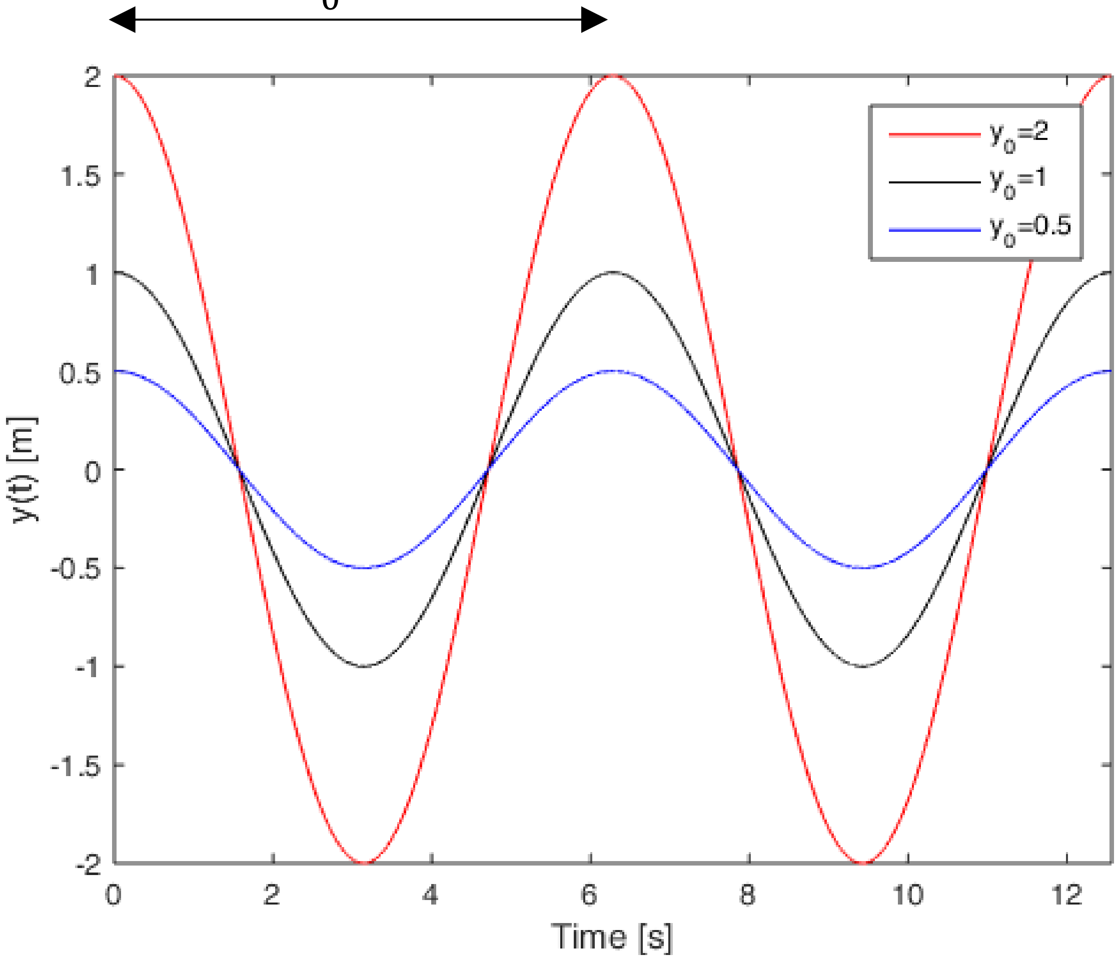
The principle of superposition is the cornerstone of linear theory:

The response caused by two or more inputs is the sum of the responses that would have been caused by each input individually.



Response to different initial displacements ($\omega_0 = 1$)

$$T = \frac{2\pi}{\omega_0} = 6.28s$$



Key findings for a linear oscillator

$$\ddot{y}(t) + \omega_0^2 y(t) = 0, \quad \dot{y}(0) = \dot{y}_0,\ y(0) = y_0$$

$$y(t) = \sqrt{y_0^2 + \frac{\dot{y}_0^2}{\omega_0^2}} \sin\left(\omega_0 t + \tan^{-1}\frac{\omega_0 y_0}{\dot{y}_0}\right)$$

The response of a linear oscillator takes the form of harmonic motion at the natural frequency $\omega_0 = \sqrt{k/m}$.

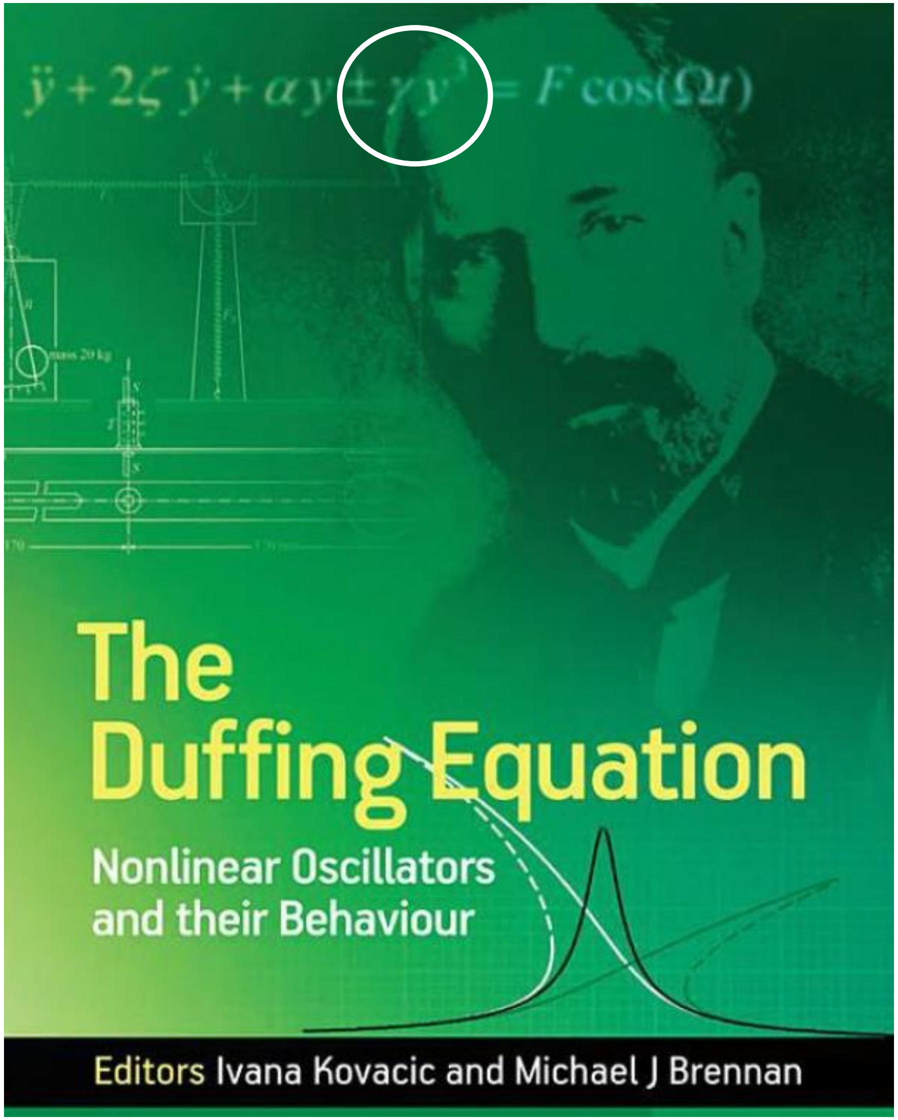
The natural frequency depends only on k and m.

The principle of superposition is the cornerstone of linear theory.

2. Undamped, unforced cubic oscillator

$$\ddot{y}(t) + \omega_0^2 y(t) + \alpha_3 y^3(t) = 0$$

The so-called Duffing oscillator



Why focus on cubic stiffness ?

Motivation for cubic nonlinearity (positive coefficient)

To illustrate the concept of nonlinear effects, let us consider the one-dimensional system consisting of a central mass particle fixed to a cable (Figure 4.7). If the cable is not stretched, we will see that the transverse motion is conditioned only by nonlinear effects.

To analyze the transverse motion of the mass particle M in the (x, y) plane, let us assume that the cable is massless so that it remains straight on both sides of the mass. The cable is subjected to no initial stress and the diameter of the mass is negligible compared to the cable length ℓ. We may thus write:

$$\begin{aligned} \frac{\partial v}{\partial x} &= \frac{2v_M}{\ell} && \text{when} \quad 0 < x < \frac{\ell}{2} \\ \frac{\partial v}{\partial x} &= \frac{-2v_M}{\ell} && \text{when} \quad \frac{\ell}{2} < x < \ell \end{aligned} \qquad \text{(E4.2.a)}$$

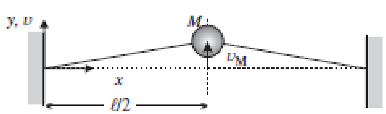


Figure 4.7 Cable with central mass particle.

where v_M, the transverse displacement of mass M, is the only independent variable of the problem. If the analysis is limited to the transverse motion, the axial strain can be expressed by:

$$\varepsilon_x = \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 \qquad \text{(E4.2.b)}$$

The kinetic and potential energies take the form:

$$\begin{aligned} \mathcal{T} &= \frac{1}{2} M \dot{v}_M^2 \\ \mathcal{V}_{int} &= \frac{1}{2}\int_0^{\ell} EA\varepsilon_x^2 \, dx \end{aligned} \qquad \text{(E4.2.c)}$$

Let us now apply Hamilton's principle to obtain the equation of motion in the absence of external forces:

$$\int_{t_1}^{t_2} \left\{ M\dot{v}_M \delta\dot{v}_M + \int_0^{\ell} EA\varepsilon_x \delta\varepsilon_x \, dx \right\} dt = 0$$

and next, by making use of Equation (E4.2.b),

$$\int_{t_1}^{t_2} \left\{ M\dot{v}_M \delta\dot{v}_M + \int_0^{\ell} \frac{EA}{2}\left(\frac{\partial v}{\partial x}\right)^3 \delta\frac{\partial v}{\partial x} \, dx \right\} dt = 0$$

Taking account of the application rules of the principle, integrating by parts the first term and making use of relationships (E4.2.a) *yields:*

$$M\ddot{v}_M \delta v_M + \int_0^{\frac{\ell}{2}} \frac{EA}{2}\left(\frac{2v_M}{\ell}\right)^3 \frac{2}{\ell}\delta v_M \, dx + \int_{\frac{\ell}{2}}^{\ell} \frac{EA}{2}\left(\frac{2v_M}{\ell}\right)^3 \frac{2}{\ell}\delta v_M \, dx = 0$$

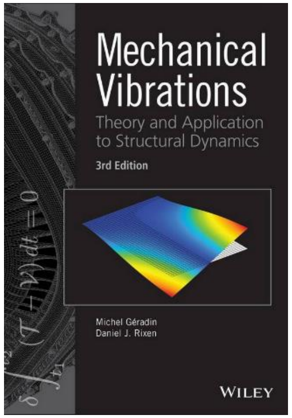
and thus:

$$\left\{ M\ddot{v}_M + \ell\frac{EA}{\ell}\left(\frac{2v_M}{\ell}\right)^3 \right\} \delta v_M = 0$$

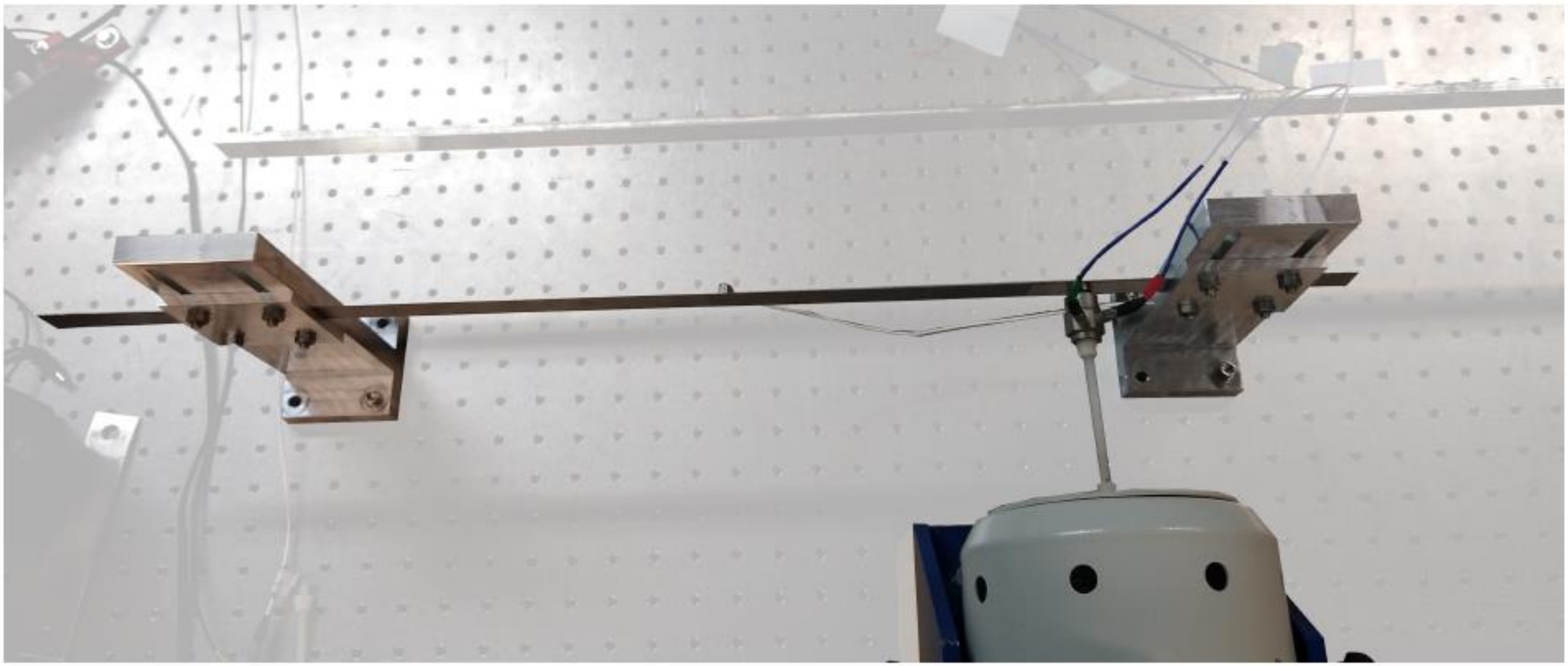
Since δv_M is arbitrary, the equation governing the free transverse motion takes the final form:

$$\boxed{M\ddot{v}_M + EA\left(\frac{2v_M}{\ell}\right)^3 = 0} \qquad \text{(E4.2.d)}$$

The relationship above expresses equilibrium of mass M when the restoring force is due to non-linear effects only. In other words, when the mass particle M moves in the perpendicular direction to x, the cable is stretched. This is commonly called the cable effect. *Let us notice the nonlinear form of the restoring force versus v_M.*

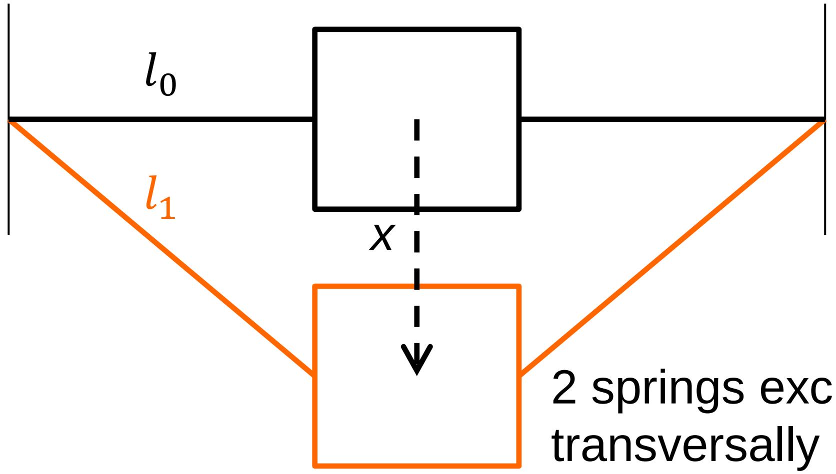


Motivation for cubic nonlinearity (positive coefficient)

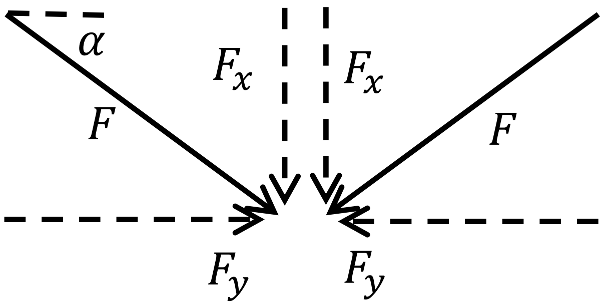


Clamped-clamped beam
(length: 46cm, width: 2cm, thickness: 0.08cm)

Motivation for cubic nonlinearity (positive coefficient)



2 springs excited transversally



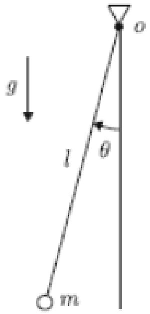
$$Force = 2F_x = 2Fsin\alpha = 2k(l_1 - l_0)\frac{x}{l_1} = 2kx\left(1 - \frac{l_0}{l_1}\right) = 2kx\left(1 - \frac{l_0}{\sqrt{x^2 + {l_0}^2}}\right)$$

$$\frac{l_0}{\sqrt{x^2 + {l_0}^2}} = 1 - \frac{x^2}{2{l_0}^2} + \frac{3x^4}{8{l_0}^4} + O(x^6)$$

$$\boxed{Force \approx 2kx\left(1 - 1 + \frac{x^2}{2{l_0}^2}\right) \approx \frac{kx^3}{{l_0}^2}}$$

Hardening

Motivation for cubic nonlinearity (negative coefficient)



$$T = \frac{1}{2} m v_m^2 = \frac{1}{2} m l^2 \dot{\theta}^2$$

$$V = -mgy = -mgl\cos\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

$$L = T - V = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k$$

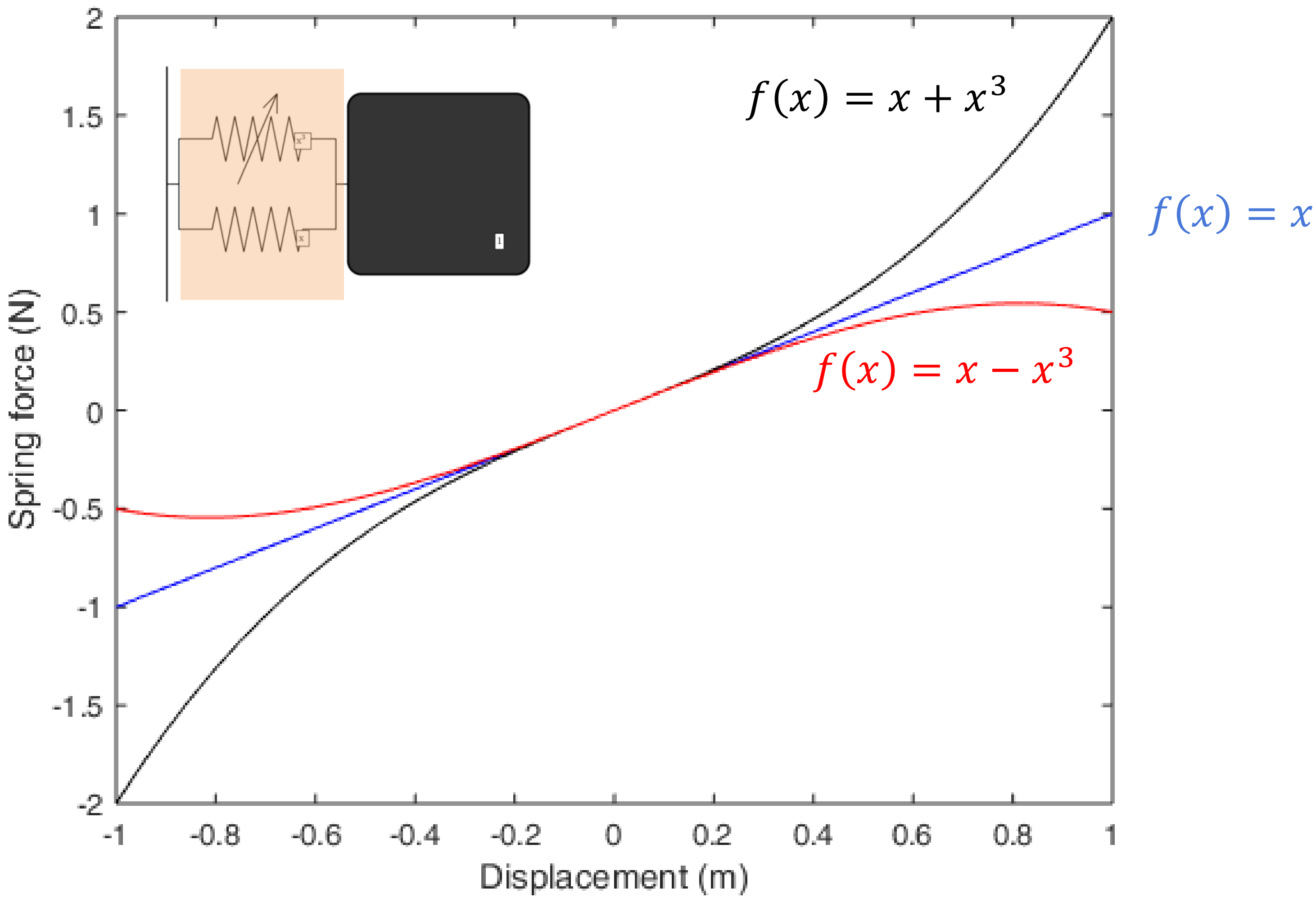
$$\longrightarrow \quad ml^2\ddot{\theta} + mgl\sin\theta = 0$$

$$\boxed{\ddot{\theta} + \frac{g}{l}\left(\theta - \frac{\theta^3}{6} + \cdots\right) = 0}$$

Softening

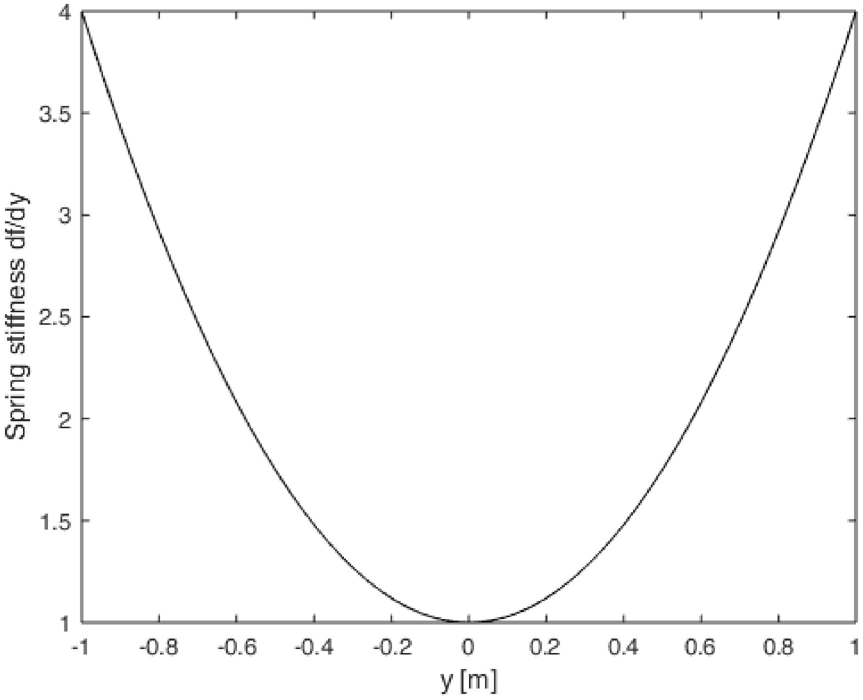
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

Springs with positive/negative stiffness



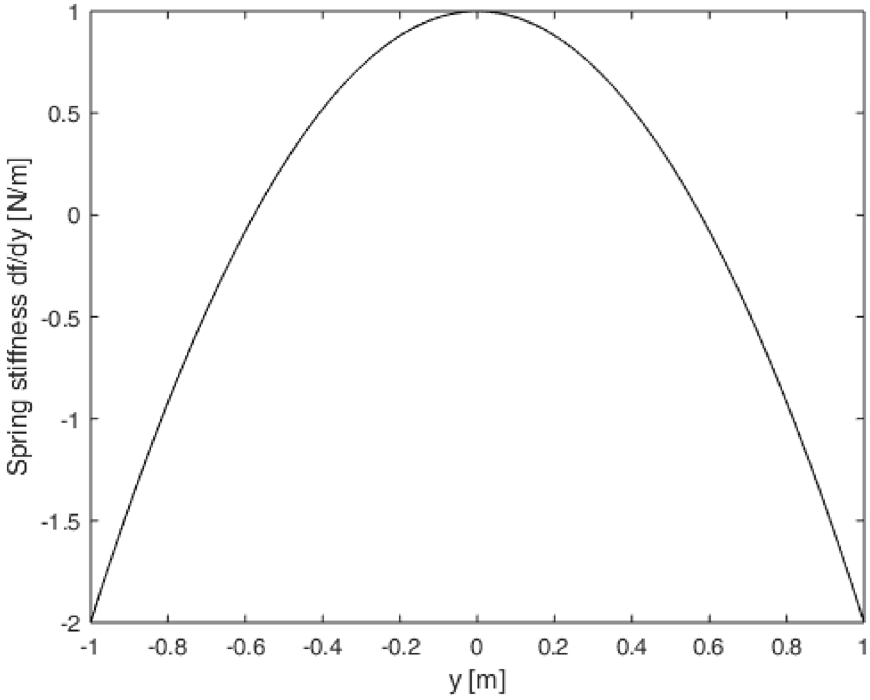
Spring stiffness: df/dy

$f(x) = x + x^3$



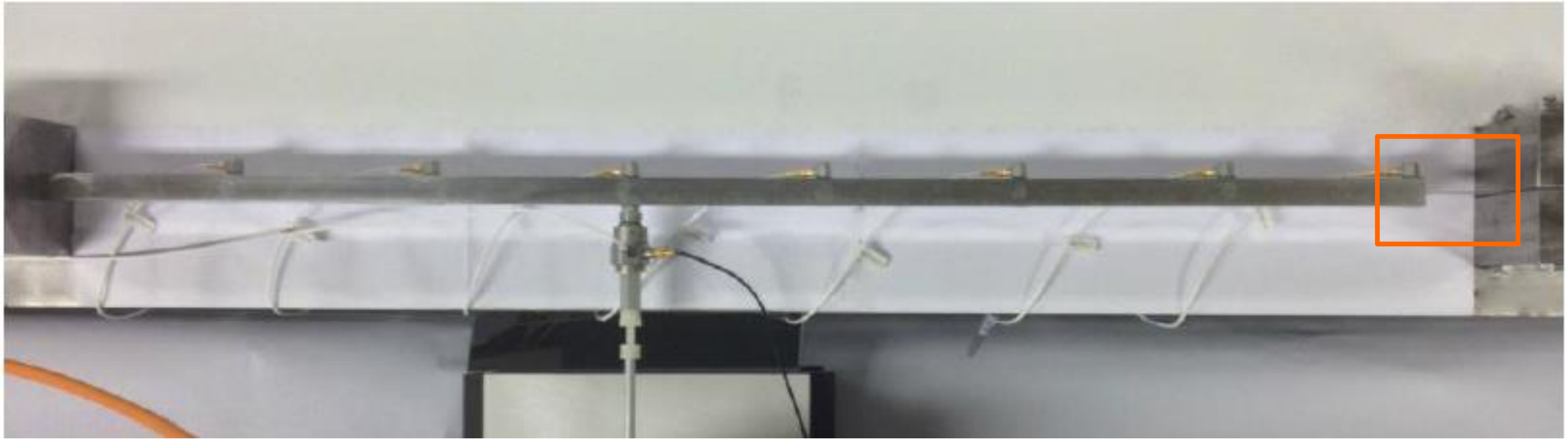
The stiffness increases:
hardening spring

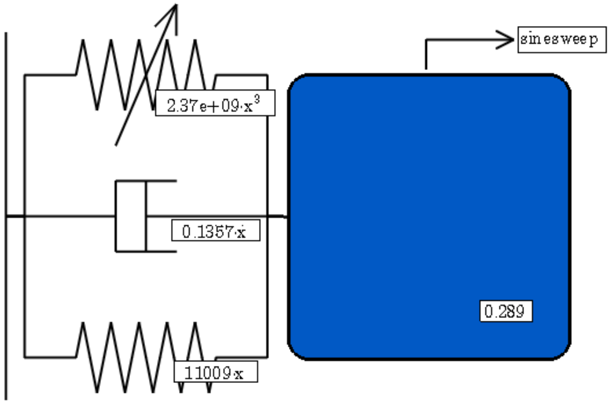
$f(x) = x - x^3$



The stiffness decreases:
softening spring

Small vs. large displacements





x=1mm, F_{NL}=2.4 N ~ F_{LIN} =11 N

x=1µm, F_{NL} =2.4e-9 N <<< F_{LIN} =1.1e-2 N

Exact solution by integration

$$\ddot{y}(t) + \omega_0^2 y(t) + \alpha_3 y^3(t) = 0 \qquad \dot{y}(0) = 0\ ,\ y(0) = y_0$$

Multiply by the velocity and integrate

$$\frac{\dot{y}^2(t)}{2} + \frac{\omega_0^2 y^2(t)}{2} + \frac{\alpha_3 y^4(t)}{4} = \frac{\omega_0^2 y_0^2(t)}{2} + \frac{\alpha_3 y_0^4(t)}{4}$$

$$t = \int_{y_0}^{y} \frac{-dy}{\sqrt{\left(y_0^2 - y^2\right)\left(\omega_0^2 + \frac{\alpha_3}{2}\left(y_0^2 + y^2\right)\right)}}$$

Next two pages

$$y = y_0 \cos\phi$$

$$\Omega t = \int_0^{\phi} \frac{d\phi'}{\sqrt{1 - m\sin^2\phi'}}$$

$$\Omega = \sqrt{\omega_0^2 + \alpha_3 y_0^2}$$

$$m = \alpha_3 y_0^2 / 2\Omega^2$$

Details of the proof

$$\left(\frac{dy}{dt}\right)^2 = \omega_0^2 y_0^2 - \omega_0^2 y^2 + \frac{\alpha_3}{2} y_0^4 - \frac{\alpha_3}{2} y^4 \;\rightarrow\; dt = \sqrt{\frac{dy^2}{\omega_0^2\left(y_0^2 - y^2\right) + \frac{\alpha_3}{2}\left(y_0^2 - y^2\right)\left(y_0^2 + y^2\right)}}$$

$$t = \int_{y_0}^{y} \frac{-dy}{\sqrt{\left(y_0^2 - y^2\right)\left(\omega_0^2 + \frac{\alpha_3}{2}\left(y_0^2 + y^2\right)\right)}}$$

$$y = y_0 \cos\phi$$

$$t = \int_0^{\phi} \frac{y_0 \sin\phi}{\sqrt{y_0^2 \sin^2\phi \left(\omega_0^2 + \frac{\alpha_3}{2}\left(y_0^2 + y_0^2\cos^2\phi\right)\right)}}$$

$$t = \int_0^{\phi} \frac{d\phi}{\sqrt{\omega_0^2 + \frac{\alpha_3}{2}\left(y_0^2 + y_0^2\cos^2\phi\right)}}$$

Details of the proof

$$\omega_0^2 + \frac{y_0^2\alpha_3}{2} + \frac{y_0^2\alpha_3}{2} + \frac{y_0^2\alpha_3}{2}\left(\cos^2\phi - 1\right) = \Omega^2 - \frac{y_0^2\alpha_3}{2}\sin^2\phi = \Omega^2\left(1 - \frac{y_0^2\alpha_3}{2\Omega^2}\sin^2\phi\right)$$

$$t = \int_0^\phi \frac{d\phi}{\sqrt{\omega_0^2 + \frac{\alpha_3}{2}\left(y_0^2 + y_0^2\cos^2\phi\right)}} \quad \rightarrow \quad t = \int_0^\phi \frac{d\phi}{\Omega\sqrt{1 - \frac{y_0^2\alpha_3}{2}\sin^2\phi}}$$

$$\rightarrow \quad \Omega t = \int_0^\phi \frac{d\phi}{\sqrt{1 - m\sin^2\phi}}$$

$$\Omega = \sqrt{\omega_0^2 + \alpha_3 y_0^2}$$

$$m = \alpha_3 y_0^2 / 2\Omega^2$$

Let’s compare the linear and nonlinear cases

$$\ddot{y}(t) + \omega_0^2 y(t) + \alpha_3 y^3(t) = 0 \qquad \dot{y}(0) = 0\ ,\ y(0) = y_0$$

LINEAR

$$\omega_0\, t = \int_{y_0}^{y} \frac{dy}{\sqrt{\left(y_0^2 + \frac{\dot{y}_0^2}{\omega_0^2}\right) - y^2}}$$

$$\omega_0 = \sqrt{k/m}$$

NONLINEAR

$$\Omega t = \int_0^{\phi} \frac{d\phi'}{\sqrt{1 - m \sin^2 \phi'}}$$

$$\Omega = \sqrt{\omega_0^2 + \alpha_3 y_0^2}$$

What can you conclude ?
Compare ω_0 and Ω

The solution is expressed as an elliptic cosine

$$u = F(\phi|m) = \int_0^{\phi} \frac{d\phi'}{\sqrt{1 - m\sin^2\phi'}}$$

$$= \Omega t$$

$$\mathrm{cn}(u|m) = \cos\phi$$

$$y = y_0 \cos\phi$$

$$y(t) = y_0\,\mathrm{cn}\left(\Omega t|m\right)$$

Handbook of Mathematical Functions

With

Formulas, Graphs, and Mathematical Tables

Edited by
Milton Abramowitz and Irene A. Stegun

The Jacobian elliptic functions can also be defined with respect to certain integrals. Thus if

16.1.3 $$u=\int_0^{\varphi} \frac{d\theta}{(1-m\sin^2\theta)^{1/2}},$$

the angle φ is called the *amplitude*

16.1.4 $\varphi=\mathrm{am}\ u$

and we define

16.1.5

sn $u=\sin\varphi$, cn $u=\cos\varphi$,

See Appendix A

The exact solution of the Duffing oscillator

$$\ddot{y}(t) + \omega_0^2 y(t) + \alpha_3 y^3(t) = 0 \qquad \dot{y}(0) = \dot{y}_0,\ y(0) = y_0$$

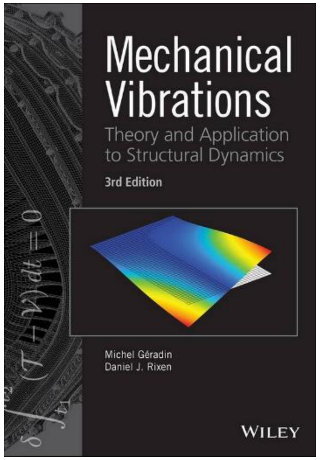
$$\boxed{y(t) = y_0 \,\mathrm{cn}\left(\Omega t | m\right) \quad \text{with} \quad \Omega = \sqrt{\omega_0^2 + \alpha_3 y_0^2}}$$

EXACT SOLUTION

The response does not take the form of a harmonic function.

The frequency depends on k and m but also on the initial displacement and nonlinear coefficient.

Newmark's method for nonlinear systems



Newmark integration scheme for nonlinear systems

$$\mathbf{M}, \mathbf{f}, \mathbf{p}, \mathbf{S} \quad \mathbf{q}_0, \dot{\mathbf{q}}_0$$

Compute $\ddot{\mathbf{q}}_0$

$$\ddot{\mathbf{q}}_0 = \mathbf{M}^{-1}\left(\mathbf{g}_0 - \mathbf{f}(\dot{\mathbf{q}}_0, \mathbf{q}_0)\right)$$

Time incrementation

$$t_{n+1} = t_n + h$$

Prediction

$$\dot{\mathbf{q}}_{n+1} = \dot{\mathbf{q}}_n + (1-\gamma)\, h\, \ddot{\mathbf{q}}_n$$

$$\mathbf{q}_{n+1} = \mathbf{q}_n + h\, \dot{\mathbf{q}}_n + (0.5-\beta)\, h^2\, \ddot{\mathbf{q}}_n$$

$$\ddot{\mathbf{q}}_{n+1} = 0$$

Residual vector evaluation

$$\mathbf{r}_{n+1} = \mathbf{M}\, \ddot{\mathbf{q}}_{n+1} + \mathbf{f}_{n+1} - \mathbf{g}_{n+1}$$

Convergence ?

$$\left\|\mathbf{r}_{n+1}\right\| < \varepsilon \left\|\mathbf{f}_{n+1}\right\|$$

Yes

No

Calculation of the correction

$$\mathbf{S}(\mathbf{q}_{n+1})\, \Delta\mathbf{q} = -\mathbf{r}_{n+1}$$

Correction

$$\mathbf{q}_{n+1} = \mathbf{q}_{n+1} + \Delta\mathbf{q}$$

$$\dot{\mathbf{q}}_{n+1} = \dot{\mathbf{q}}_{n+1} + \frac{\gamma}{\beta\, h}\, \Delta\mathbf{q}$$

$$\ddot{\mathbf{q}}_{n+1} = \ddot{\mathbf{q}}_{n+1} + \frac{1}{\beta\, h^2}\, \Delta\mathbf{q}$$

Verification using numerical simulation

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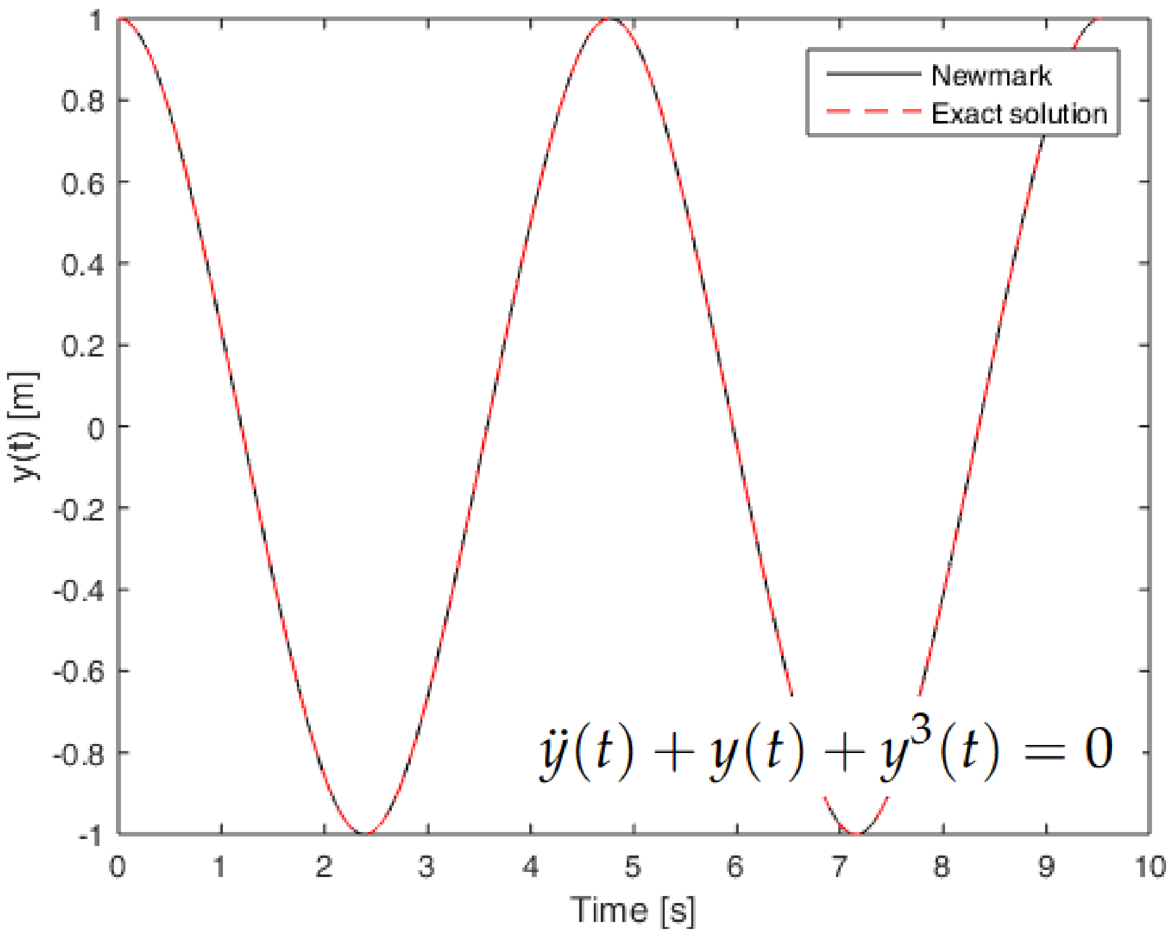
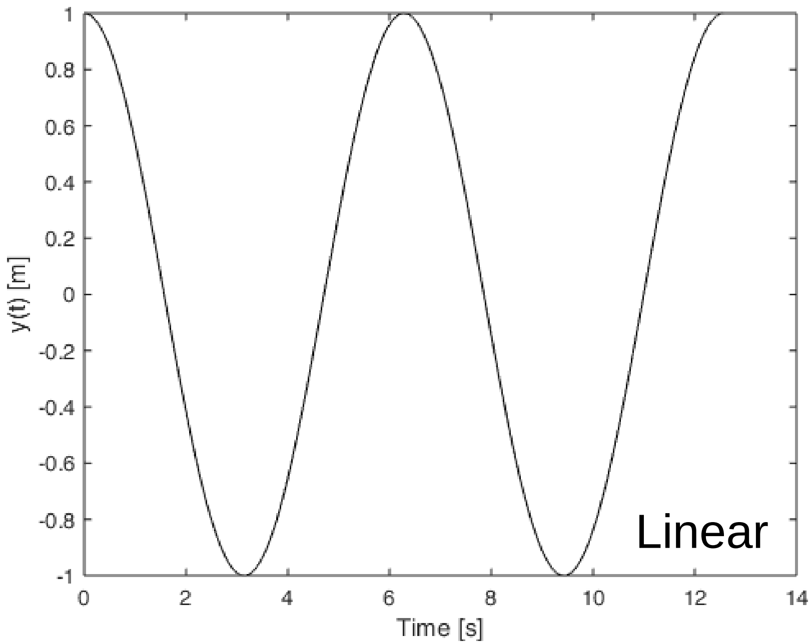
Function ellipj.m

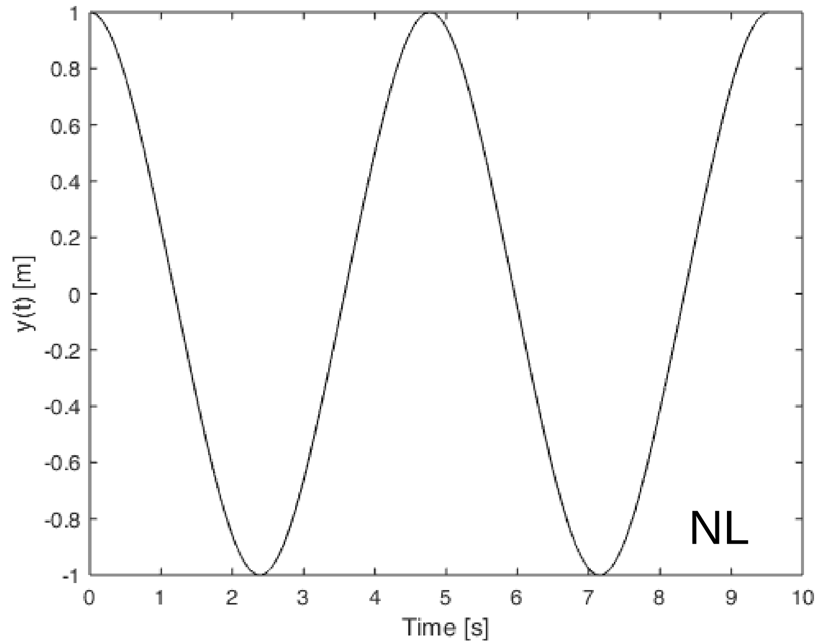


Let's compare the linear and nonlinear cases

$$\ddot{y}(t) + y(t) = 0$$



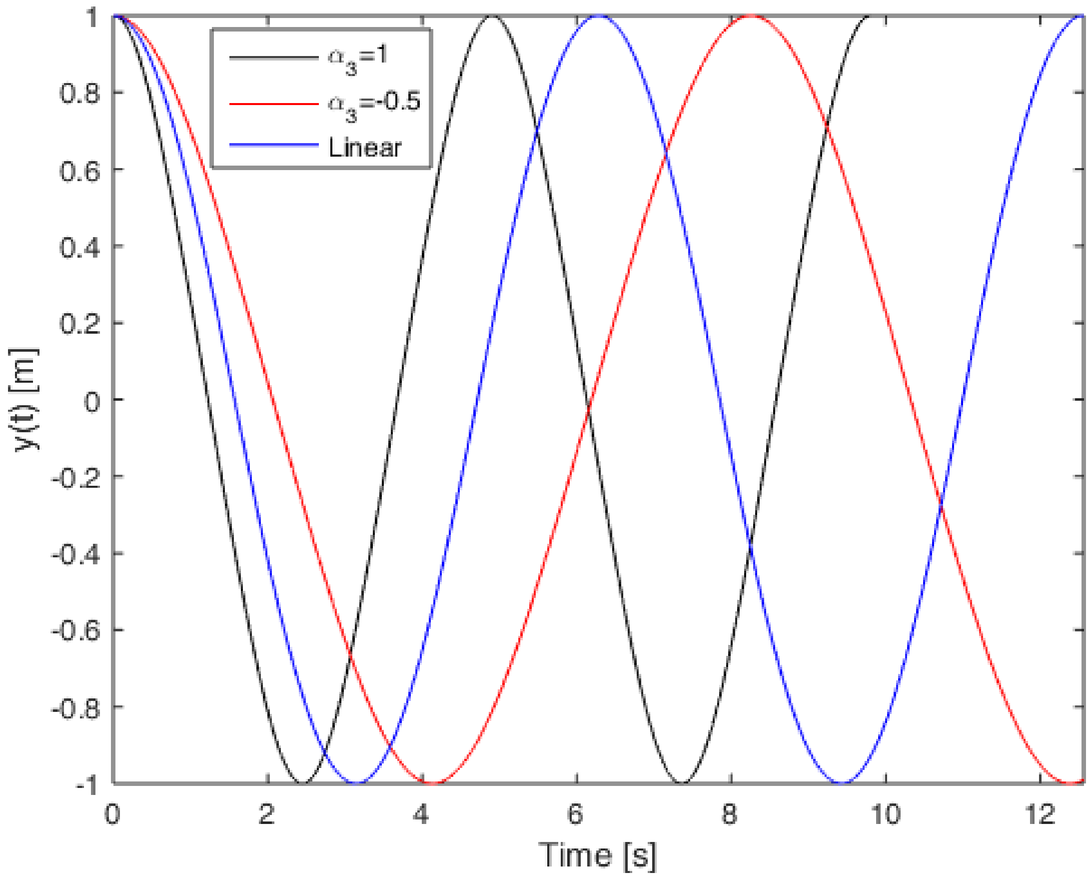
$$\ddot{y}(t) + y(t) + y^3(t) = 0$$



What do you observe ?

The period depends on the nonlinear coefficient

$$\ddot{y}(t) + y(t) + \alpha_3 y^3(t) = 0$$



Hardening case,

« More stiffness »

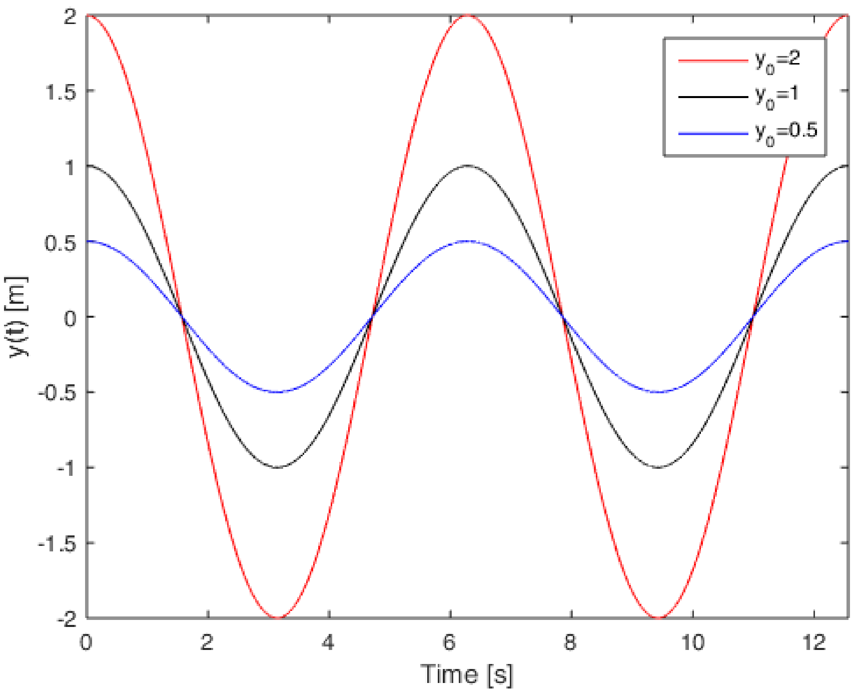
Softening case

« Less stiffness »

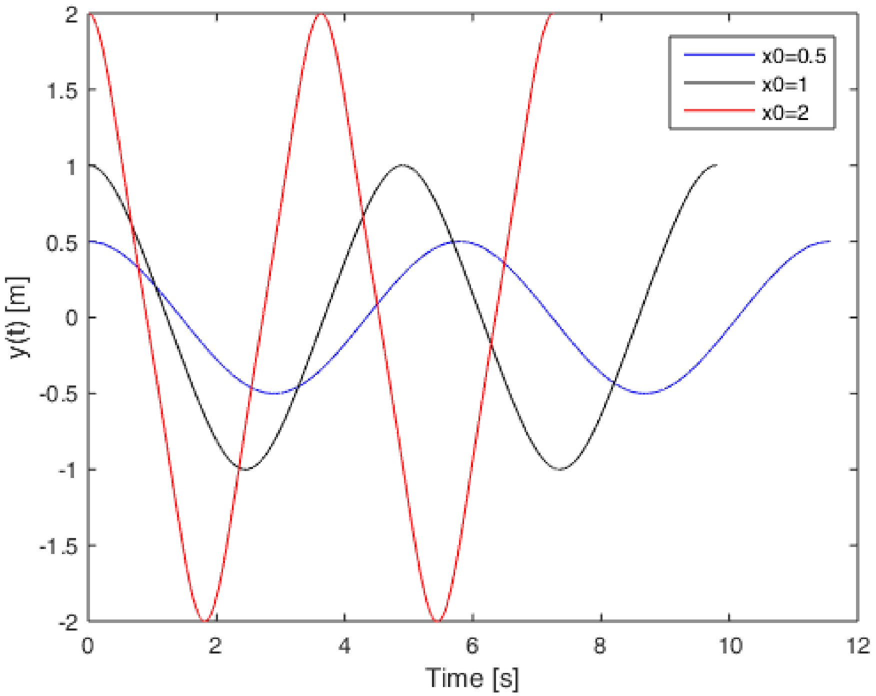
Linear case

The period depends on the initial displacement

$$\ddot{y}(t) + y(t) = 0$$



$$\ddot{y}(t) + y(t) + y^3(t) = 0$$



What can you conclude ?

Failure of superposition principle

The period can be calculated explicitly

The period is equal to 4 times the time to move from the initial position to the equilibrium position. The corresponding variation of ϕ is between 0 and $\pi/2$.

$$u = F(\phi|m) = \int_0^{\phi} \frac{d\phi'}{\sqrt{1 - m\sin^2\phi'}}$$
$$= \Omega t$$

$$\boxed{T = \frac{4F\left(\frac{\pi}{2}|m\right)}{\Omega} = \frac{4K(m)}{\Omega}}$$

EXACT EXPRESSION

17.3. Complete Elliptic Integrals of the First and Second Kinds

Referred to the canonical forms of **17.2**, the elliptic integrals are said to be *complete* when the amplitude is $\frac{1}{2}\pi$ and so $x=1$. These complete integrals are designated as follows

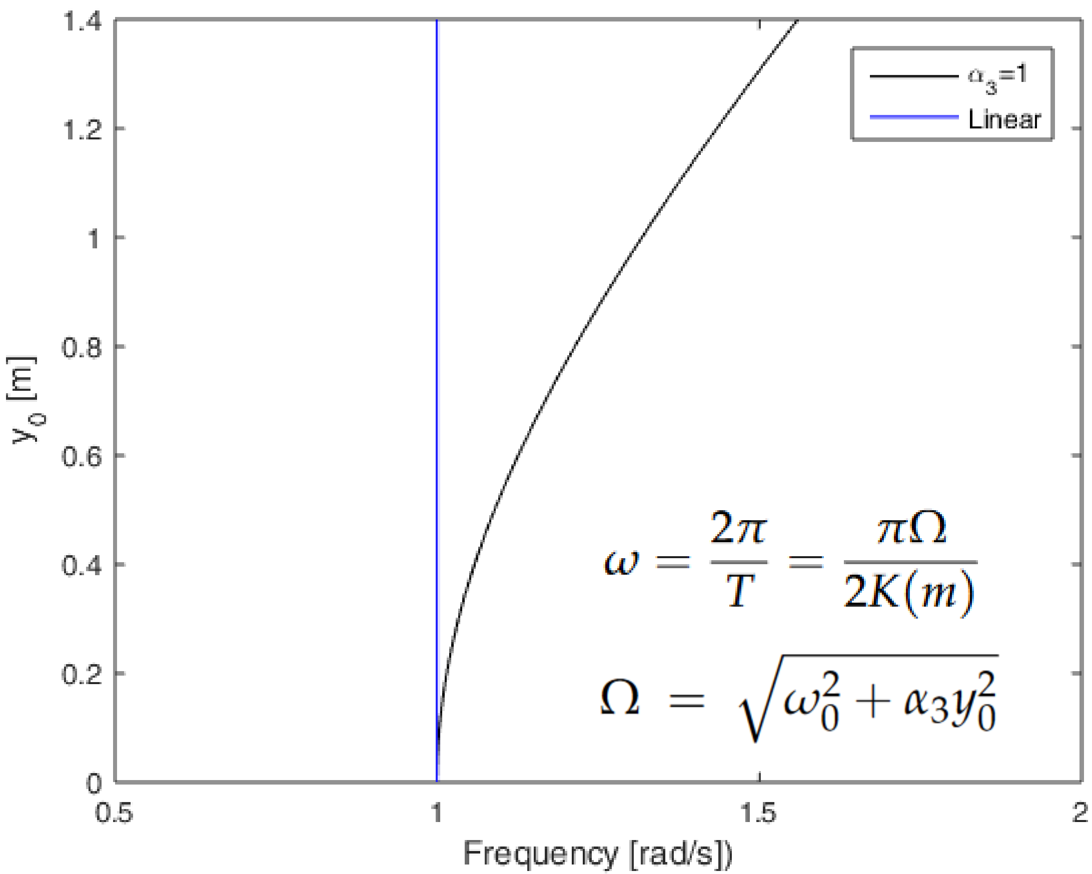
17.3.1

$$[K(m)]=K=\int_0^1 [(1-t^2)(1-mt^2)]^{-1/2}dt$$
$$=\int_0^{\pi/2}(1-m\sin^2\theta)^{-1/2}d\theta$$

17.3.2 $\quad K=F(\frac{1}{2}\pi|m)=F(\frac{1}{2}\pi\backslash\alpha)$

Function ellipke.m

The natural frequency of the Duffing oscillator

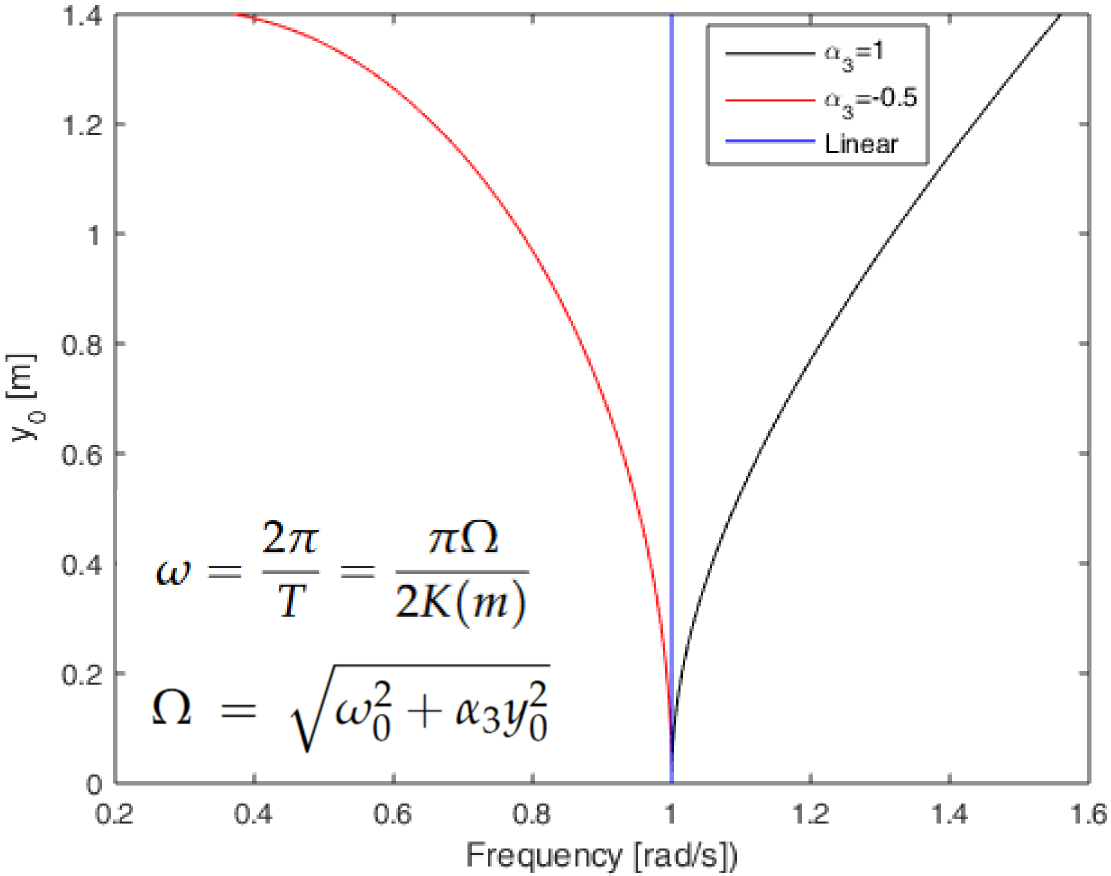


Frequency-amplitude dependence of nonlinear ocillations

Very important concept: the backbone curve

What if the cubic coefficient is negative ?

The natural frequency of the Duffing oscillator

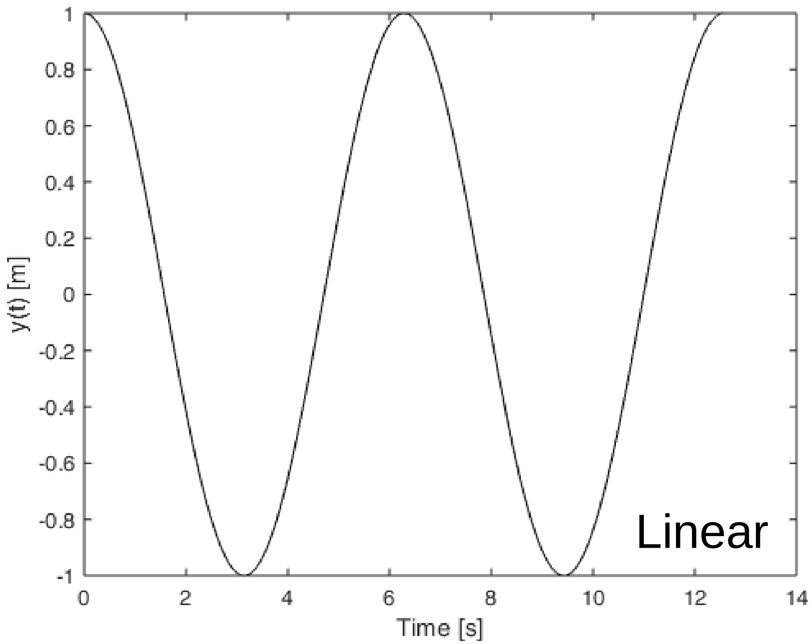


$$\omega = \frac{2\pi}{T} = \frac{\pi\Omega}{2K(m)}$$

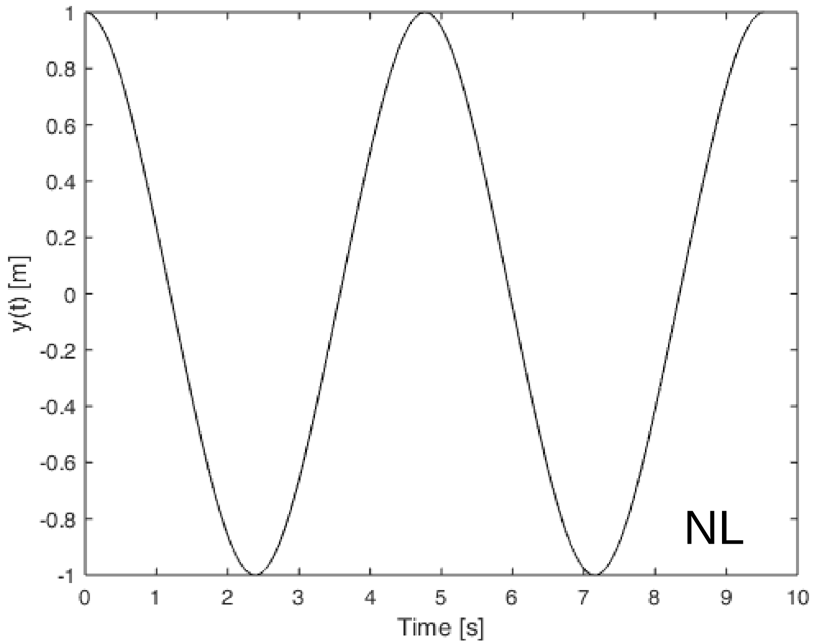
$$\Omega = \sqrt{\omega_0^2 + \alpha_3 y_0^2}$$

Let's compare the linear and nonlinear cases

$$\ddot{y}(t) + y(t) = 0$$



$$\ddot{y}(t) + y(t) + y^3(t) = 0$$



What do you observe ?

Does the elliptic cosine look like a pure cosine ?

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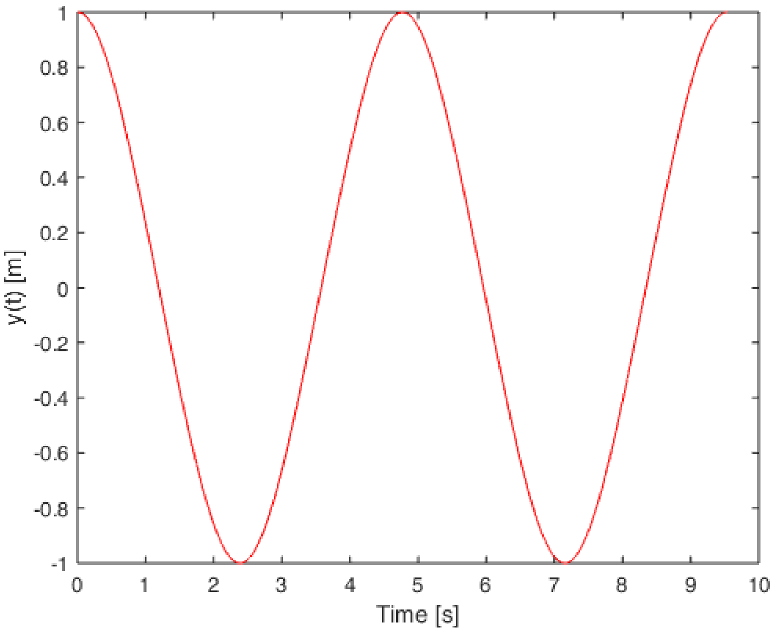
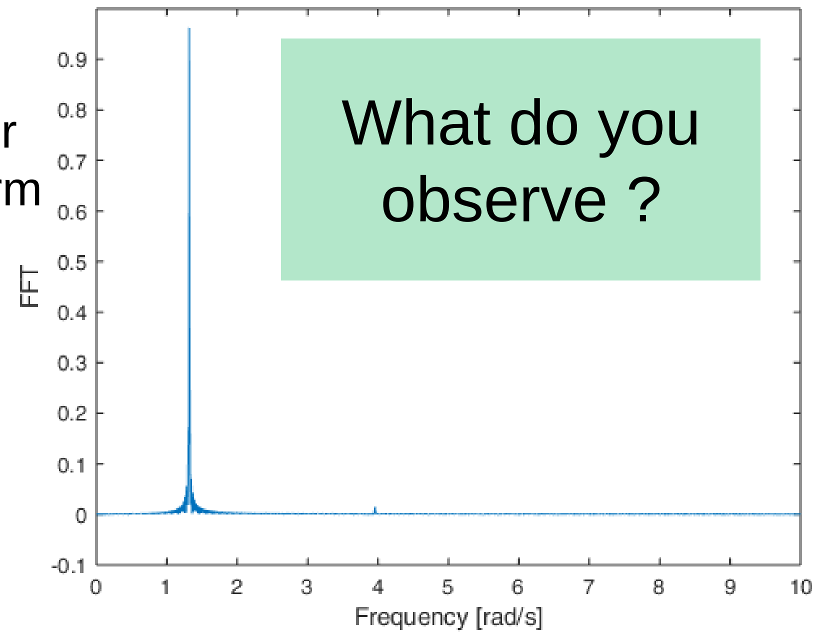


Fourier transform



What do you observe ?

Fourier expansion of elliptic functions

16.23. Series Expansions in Terms of the Nome $q=e^{-\pi K'/K}$ and the Argument $v=\pi u/(2K)$

16.23.1 $$\operatorname{sn}(u|m)=\frac{2\pi}{m^{1/2}K}\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1-q^{2n+1}}\sin\,(2n+1)v$$

16.23.2 $$\operatorname{cn}(u|m)=\frac{2\pi}{m^{1/2}K}\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1+q^{2n+1}}\cos\,(2n+1)v$$

Infinite number of harmonics to represent an elliptic cosine/sine.

Harmonics more visible in the acceleration signal

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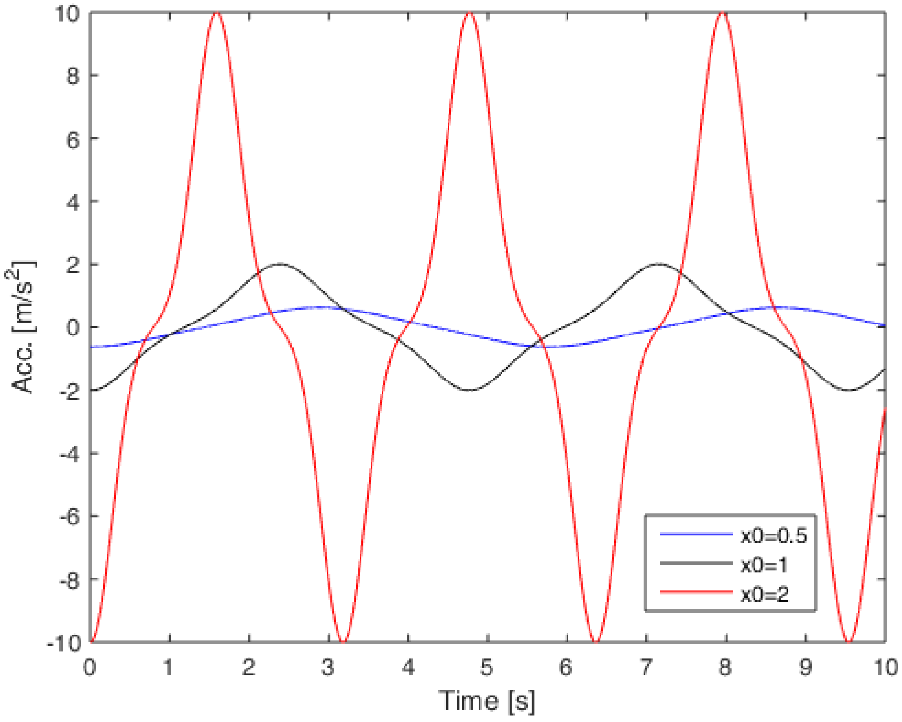
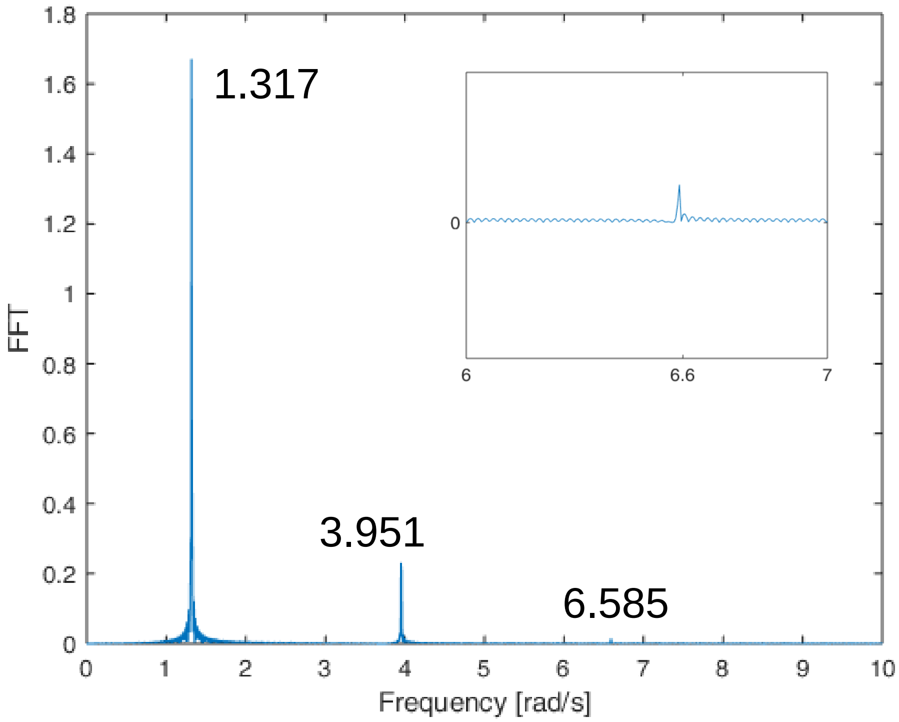




Key findings for a Duffing oscillator

	LINEAR	NONLINEAR
$y(0) = y_0$	$y(t) = y_0 \cos(\omega_0 t)$	$y(t) = y_0 \, \mathrm{cn}\, (\Omega t \| m)$
$\dot{y}(0) = \dot{y}_0$	$y(t) = \dfrac{\dot{y}_0}{\omega_0} \sin(\omega_0 t)$	$y(t) = \dfrac{\dot{y}_0}{\Omega} \mathrm{sn}\, (\Omega t \| m)$

1. The response is no longer purely harmonic

Key findings for a Duffing oscillator

	LINEAR	NONLINEAR
$y(0) = y_0$	$y(t) = y_0 \cos(\omega_0 t)$	$y(t) = y_0 \operatorname{cn}(\Omega t \| m)$ $\operatorname{cn}(u \| m) = \frac{2\pi}{m^{1/2}K} \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1+q^{2n+1}} \cos(2n+1)v$
$\dot{y}(0) = \dot{y}_0$	$y(t) = \frac{\dot{y}_0}{\omega_0} \sin(\omega_0 t)$	$y(t) = \frac{\dot{y}_0}{\Omega} \operatorname{sn}(\Omega t \| m)$ $\operatorname{sn}(u \| m) = \frac{2\pi}{m^{1/2}K} \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1-q^{2n+1}} \sin(2n+1)v$

2. Nonlinear systems generate harmonics

Key findings for a Duffing oscillator

	LINEAR	NONLINEAR
$y(0) = y_0$	$y(t) = y_0 \cos(\omega_0 t)$	$y(t) = y_0 \,\mathrm{cn}(\Omega t \| m)$ $\Omega = \sqrt{\omega_0^2 + \alpha_3 y_0^2}$
$\dot{y}(0) = \dot{y}_0$	$y(t) = \frac{\dot{y}_0}{\omega_0} \sin(\omega_0 t)$	$y(t) = \frac{\dot{y}_0}{\Omega} \mathrm{sn}(\Omega t \| m)$ $\Omega = \sqrt{\frac{\omega_0^2 + \sqrt{\omega_0^4 + 2\alpha_3 \dot{y}_0^2}}{2}}$

3. No superposition principle

Key findings for a Duffing oscillator

	LINEAR	NONLINEAR
$y(0) = y_0$	$y(t) = y_0 \cos(\omega_0 t)$	$y(t) = y_0 \, \mathrm{cn}(\Omega t \vert m)$ $\Omega = \sqrt{\omega_0^2 + \alpha_3 y_0^2}$
$\dot{y}(0) = \dot{y}_0$	$y(t) = \dfrac{\dot{y}_0}{\omega_0} \sin(\omega_0 t)$	$y(t) = \dfrac{\dot{y}_0}{\Omega} \mathrm{sn}(\Omega t \vert m)$ $\Omega = \sqrt{\dfrac{\omega_0^2 + \sqrt{\omega_0^4 + 2\alpha_3 \dot{y}_0^2}}{2}}$

4. Frequency-amplitude dependence: backbone curve

Appendix A: elliptic functions

We recall that trigonometric functions can be defined in terms of the functional inverse of specific integrals. For example,

$$\theta = \int_0^y \frac{dy'}{\sqrt{1-y'^2}} = \arcsin y \;\; \to \sin\theta = y \qquad (3.63)$$

Similarly, Jacobi elliptic functions result from the inversion of the elliptic integral of the first kind. For instance,

$$u = \int_0^y \frac{dy'}{\sqrt{(1-y'^2)\,(1-k^2y'^2)}} \;\to \mathrm{sn}(u,k) = y \qquad (3.64)$$

or, if $y = \sin\phi$,

$$F(\phi,k) = \int_0^\phi \frac{d\phi'}{\sqrt{1-k^2\sin^2\phi'}} \;\to \mathrm{sn}(u,k) = \sin\phi \qquad (3.65)$$

While trigonometric functions are defined with reference to a circle, the previous section has shown that the Jacobi elliptic functions refer to the ellipse. But their geometrical interpretation is similar:

$$\cos\theta = \frac{x}{r}, \ \sin\theta = \frac{y}{r} \tag{3.66}$$

$$\mathrm{cn}(u,k) = \frac{x}{r}, \ \mathrm{sn}(u,k) = \frac{y}{r} \tag{3.67}$$

with $r = 1$ on the unit circle whereas r varies along the unit ellipse.

Finally, Jacobi elliptic functions include trigonometric and hyperbolic functions as special cases

$$k = 0 \quad : \quad \mathrm{sn}(u,0) = \sin(u), \ \mathrm{cn}(u,0) = \cos(u) \tag{3.68}$$

$$k = 1 \quad : \quad \mathrm{sn}(u,1) = \tanh(u), \ \mathrm{cn}(u,1) = \mathrm{sech}(u) \tag{3.69}$$

and

$$\mathrm{cn}(0,k) = 1, \quad \mathrm{sn}(0,k) = 0 \tag{3.70}$$

$$\mathrm{cn}^2(u,k) + \mathrm{sn}^2(u,k) = 1 \tag{3.71}$$

$$\frac{\mathrm{d}}{\mathrm{d}u}\mathrm{cn} = -\mathrm{sn}\,\mathrm{dn}, \ \ \frac{\mathrm{d}}{\mathrm{d}u}\mathrm{sn} = \mathrm{cn}\,\mathrm{dn}, \ \ \frac{\mathrm{d}}{\mathrm{d}u}\mathrm{dn} = -k\,\mathrm{cn}\,\mathrm{sn} \tag{3.72}$$

Appendix B: Analytical expression of the acceleration

$$\frac{\mathrm{d}}{\mathrm{du}}\mathrm{cn} = -\mathrm{sn}\,\mathrm{dn}, \quad \frac{\mathrm{d}}{\mathrm{du}}\mathrm{sn} = \mathrm{cn}\,\mathrm{dn}, \quad \frac{\mathrm{d}}{\mathrm{du}}\mathrm{dn} = -k\,\mathrm{cn}\,\mathrm{sn}$$

$$y(t) = y_0\,\mathrm{cn}\left(\Omega t | m\right)$$

$$\ddot{y}(t) = y_0\Omega^2\mathrm{cn}\left(\Omega t|m\right)\left[m\,\mathrm{sn}^2\left(\Omega t|m\right) - \mathrm{dn}^2\left(\Omega t|m\right)\right]$$

Appendix C: response to an initial velocity

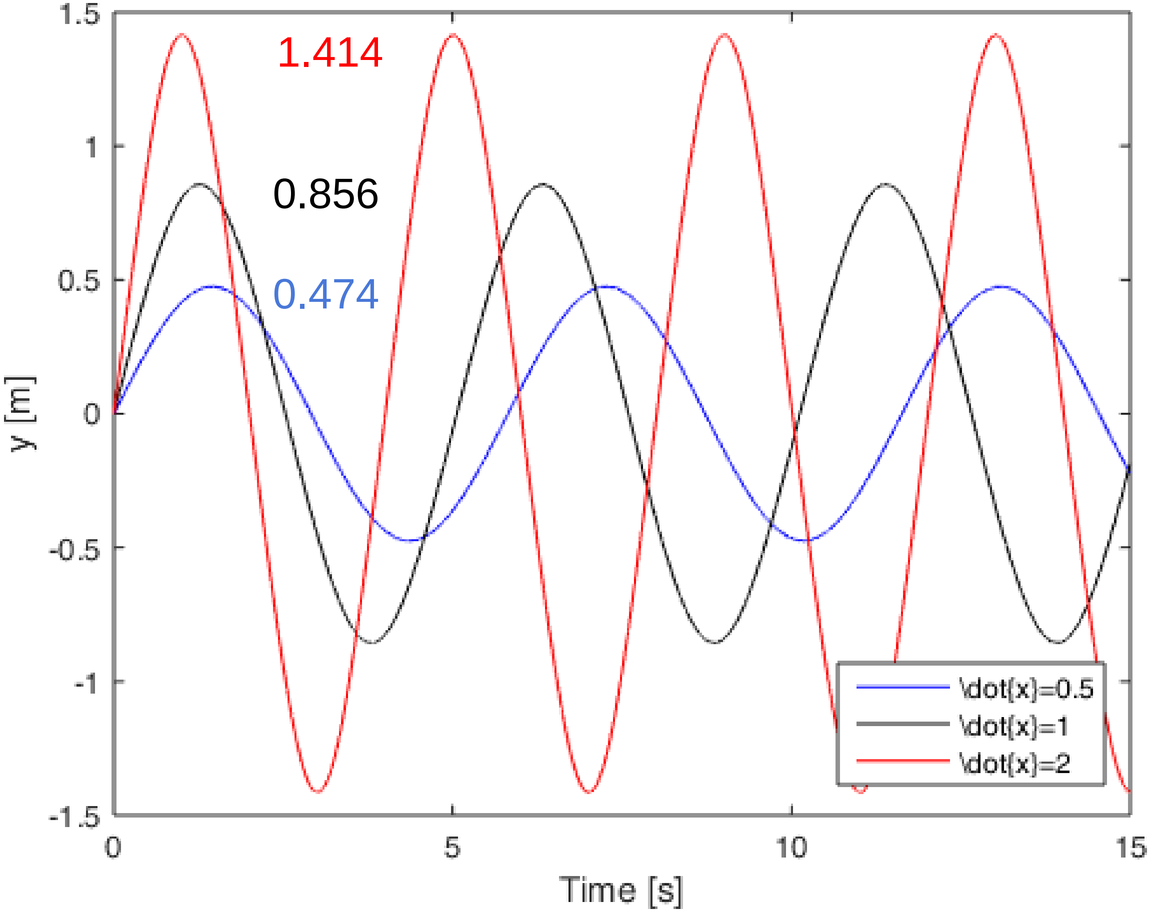
$$\ddot{y}(t) + \omega_0^2 y(t) + \alpha_3 y^3(t) = 0 \qquad \dot{y}(0) = \dot{y}_0,\; y(0) = 0$$

$$y(t) = \frac{\dot{y}_0}{\Omega} \mathrm{sn}\left(\Omega t | m\right)$$

$$\begin{aligned} \Omega &= \sqrt{\frac{\omega_0^2 + \sqrt{\omega_0^4 + 2\alpha_3 \dot{y}_0^2}}{2}} \\ m &= \frac{-\alpha_3 \dot{y}_0^2}{\omega_0^4 + \alpha_3 \dot{y}_0^2 + \omega_0^2 \sqrt{\omega_0^4 + 2\alpha_3 \dot{y}_0^2}} \end{aligned}$$

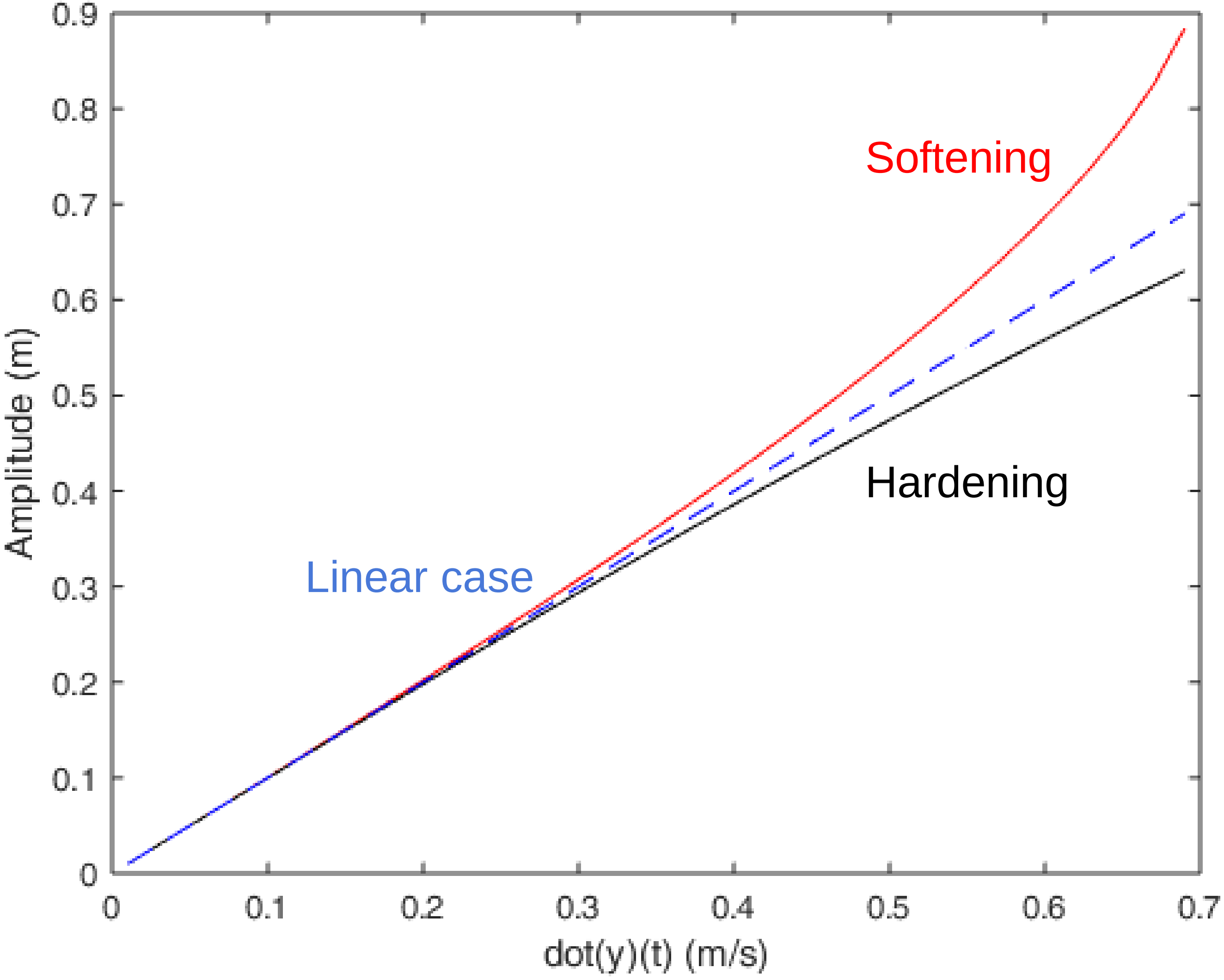
Appendix C: response to an initial velocity

$$\ddot{y}(t) + y(t) + y^3(t) = 0$$



Failure of superposition principle

Appendix D: beneficial/detrimental effect of nonlinearity



3. Undamped, unforced Helmholtz oscillator

$$\ddot{y}(t) + \omega_0^2 y(t) + \alpha_2 y^2(t) = 0$$

Motivation for quadratic nonlinearity

When we apply Taylor series to an odd function, it has nonzero coefficients only for odd degree terms. The first nonlinear term is a cubic term (see the pendulum example).

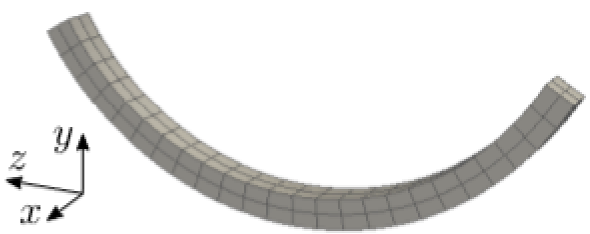
When we apply Taylor series to a function which is not odd, the first term is a quadratic term.

A function which is not odd: absence of symmetry

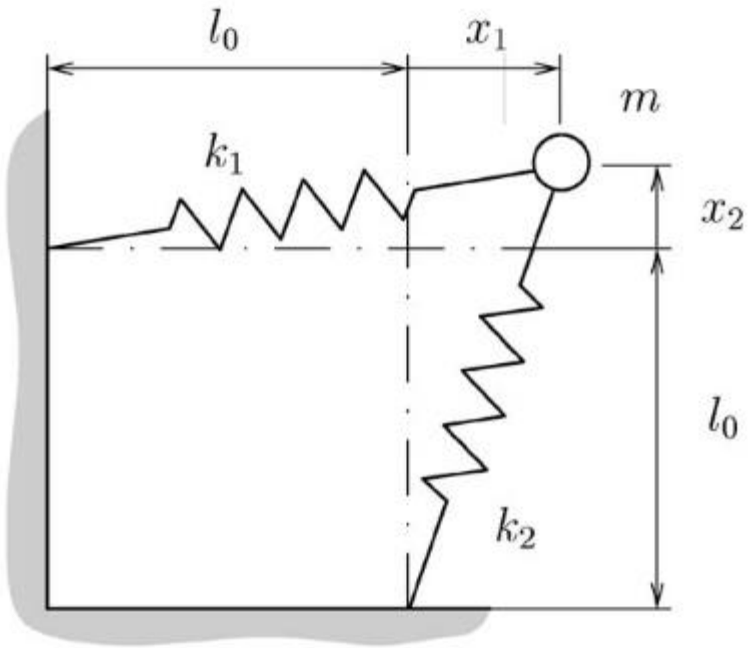
Motivation for quadratic nonlinearity

Helmholtz postulated that the eardrum (prestressed membrane) behaved as an asymmetric oscillator, with $f = kx + k_2x^2$.

symmetric beam. The two other examples, as shown in Fig. 8b and c, are arches: the first one is shallow, while the third one is non-shallow. Adding curvature has two important effects. First, flexural and in-plane modes are no longer linearly uncoupled. Second, the curvature renders the restoring force asymmetric and an important quadratic nonlinearity appears between the bending modes. This example illustrates the fact



Motivation for quadratic nonlinearity



$$\ddot{X}_1 + \omega_1^2 X_1 + \frac{\omega_1^2}{2}(3X_1^2 + X_2^2) + \omega_2^2 X_1 X_2 + \frac{\omega_1^2 + \omega_2^2}{2} X_1(X_1^2 + X_2^2) = 0,$$

$$\ddot{X}_2 + \omega_2^2 X_2 + \frac{\omega_2^2}{2}(3X_2^2 + X_1^2) + \omega_1^2 X_1 X_2 + \frac{\omega_1^2 + \omega_2^2}{2} X_2(X_1^2 + X_2^2) = 0,$$

Exact solution by integration

Closed-form solutions for the quadratic mixed-parity nonlinear oscillator

A Beléndez[1,2]*, A Hernández[1,2], T Beléndez[1,2], E Arribas[3] and M L Álvarez[1,2]

$$\frac{d^2x}{dt^2} + a_1 x + a_2 x^2 = 0,$$

with initial conditions

$$x(0) = x_0, \quad \frac{dx}{dt}(0) = 0.$$

Main steps of the proof

$$\mathrm{d}t = \pm \frac{\mathrm{d}x}{\sqrt{a_1\left(x_0^2 - x^2\right) + \frac{2}{3}a_2\left(x_0^3 - x^3\right)}},$$

$$t(x) = \sqrt{\frac{3}{2a_2}} \int_x^{x_0} \frac{\mathrm{d}x}{\sqrt{(x_0 - x)(x - x_1)(x - x_2)}}.$$

Factorize the cubic polynomial

$$\int_x^a \frac{\mathrm{d}u}{\sqrt{(a-u)(u-b)(u-c)}} = \frac{2}{\sqrt{a-c}} F(\lambda, p)$$

F is the incomplete elliptic integral of the first kind

$$F(\varphi, m) = \int_0^{\varphi} \frac{\mathrm{d}\theta}{\sqrt{1 - m\sin^2\theta}}.$$

$$\lambda = \arcsin\left(\sqrt{\frac{a-x}{a-b}}\right),$$

$$p = \frac{a-b}{a-c}.$$

The exact solution of the Helmholtz oscillator

$$\ddot{y}(t) + \omega_0^2 y(t) + \alpha_2 y^2(t) = 0 \qquad \dot{y}(0) = 0\ ,\ y(0) = y_0$$

$$\boxed{y(t) = y_0 - a\,\mathrm{sn}^2(\Omega t|m)}$$

EXACT SOLUTION

$$a = \frac{3\omega_0^2 + 6\alpha_2 y_0 - \phi}{4\alpha_2}$$

$$\phi = \sqrt{3}\sqrt{3\omega_0^2 - 4\omega_0\alpha_2 y_0 - 4\alpha_2^2 y_0^2}$$

$$\Omega = \frac{\sqrt{3\omega_0^2 + 6\alpha_2 y_0 + \phi}}{2\sqrt{6}}$$

$$m = \frac{3\omega_0^2 + 6\alpha_2 y_0 - \phi}{3\omega_0^2 + 6\alpha_2 y_0 + \phi}$$

Verification using numerical simulation

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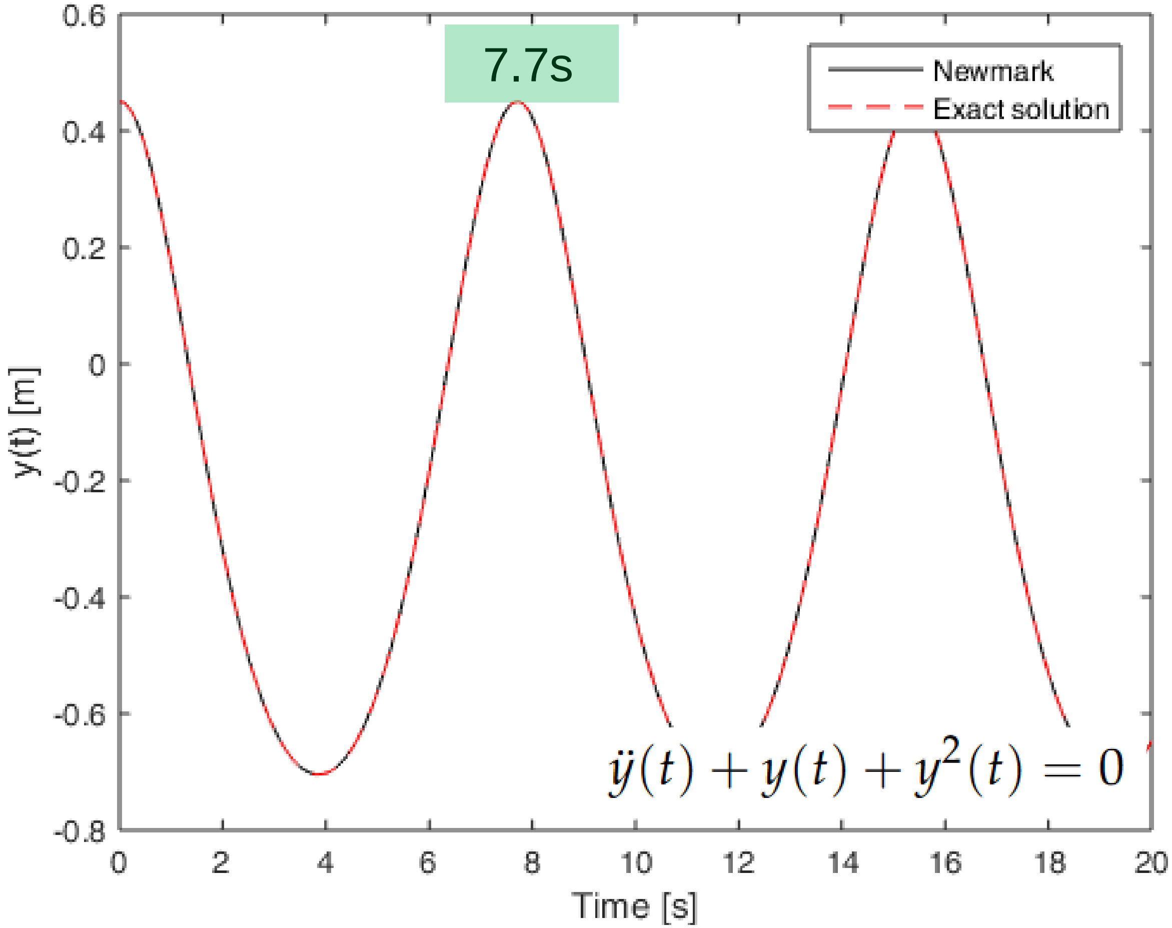
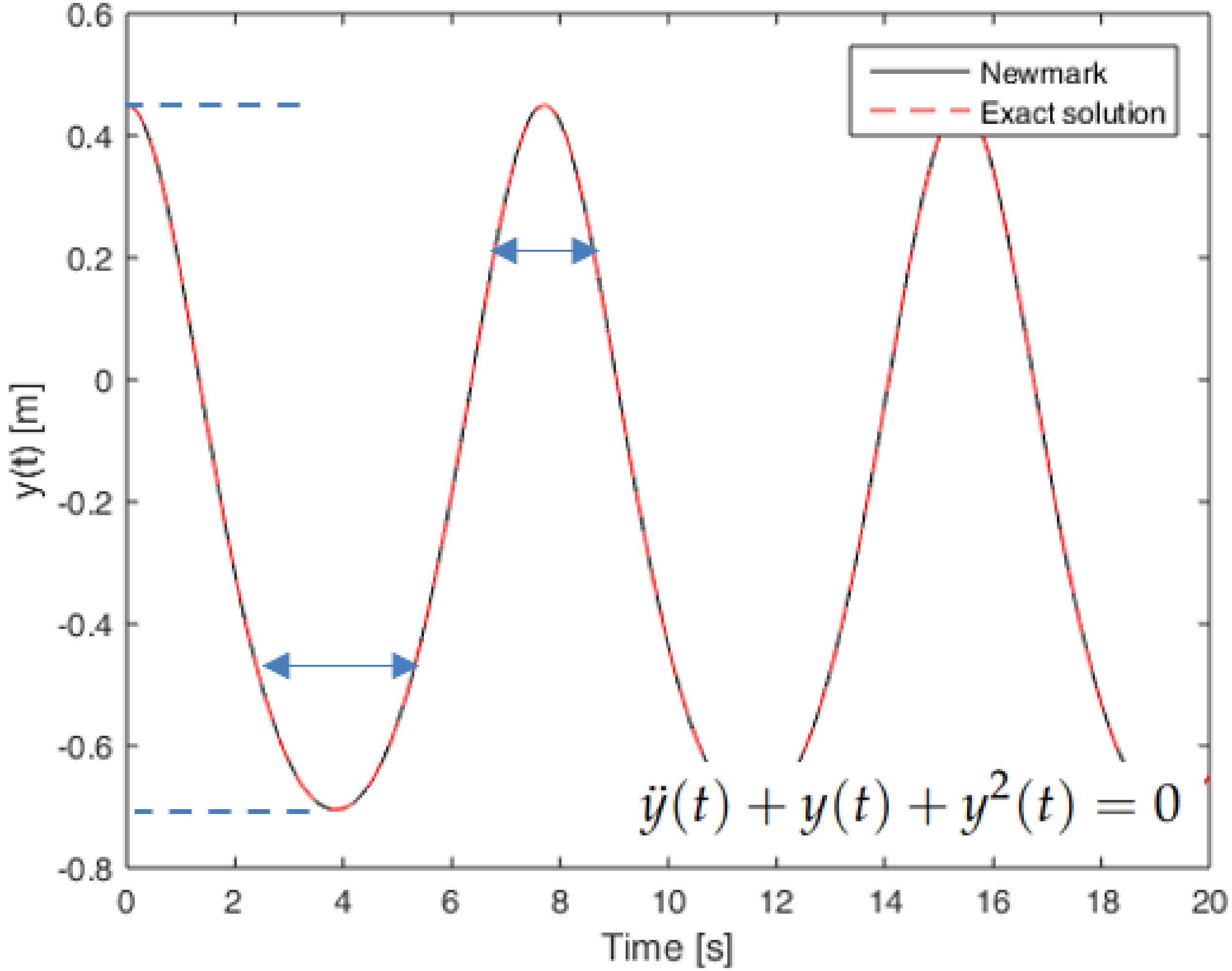
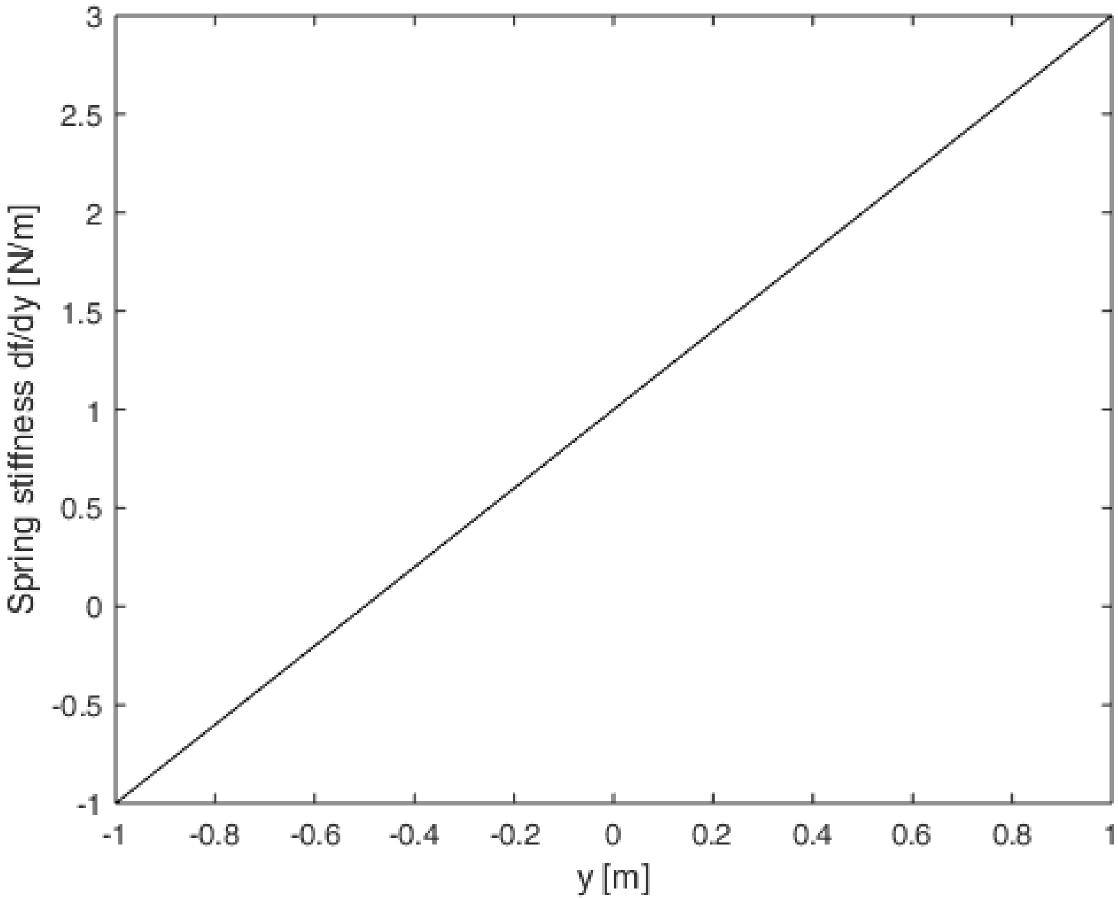


Something special with the time series ?

Asymmetry of the time series



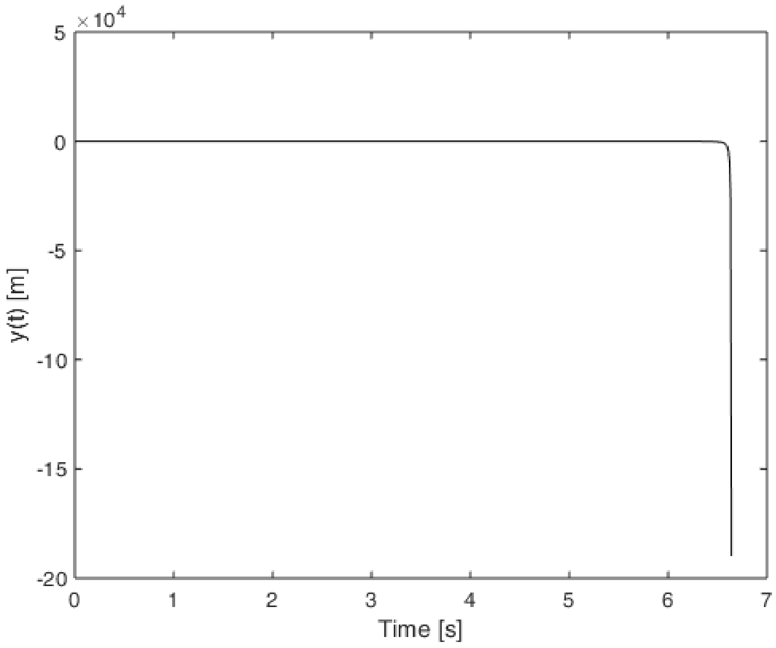
The stiffness decreases for negative displacements



A greater initial displacement

$$\ddot{y}(t) + y(t) + y^2(t) = 0 \qquad y(0)=0.6$$

Newmark



Exact solution

```
Error using ellipj (line 36)
Input arguments must be real.

Error in CalculateEllipticHelmholtz (line 3)
[SN,CN,DN] = ellipj(Omeg*temps,EllipParam);

Error in ICdep_ValidationExactSolutionHelmholtz (line 34)
SN=CalculateEllipticHelmholtz(Alph,m,temps);
```

WHY ?

Mathematical explanation

$$a = \frac{3\omega_0^2 + 6\alpha_2 y_0 - \phi}{4\alpha_2}$$

$$\phi = \sqrt{3}\sqrt{3\omega_0^2 - 4\omega_0\alpha_2 y_0 - 4\alpha_2^2 y_0^2}$$

$$\Omega = \frac{\sqrt{3\omega_0^2 + 6\alpha_2 y_0 + \phi}}{2\sqrt{6}}$$

$$m = \frac{3\omega_0^2 + 6\alpha_2 y_0 - \phi}{3\omega_0^2 + 6\alpha_2 y_0 + \phi}$$

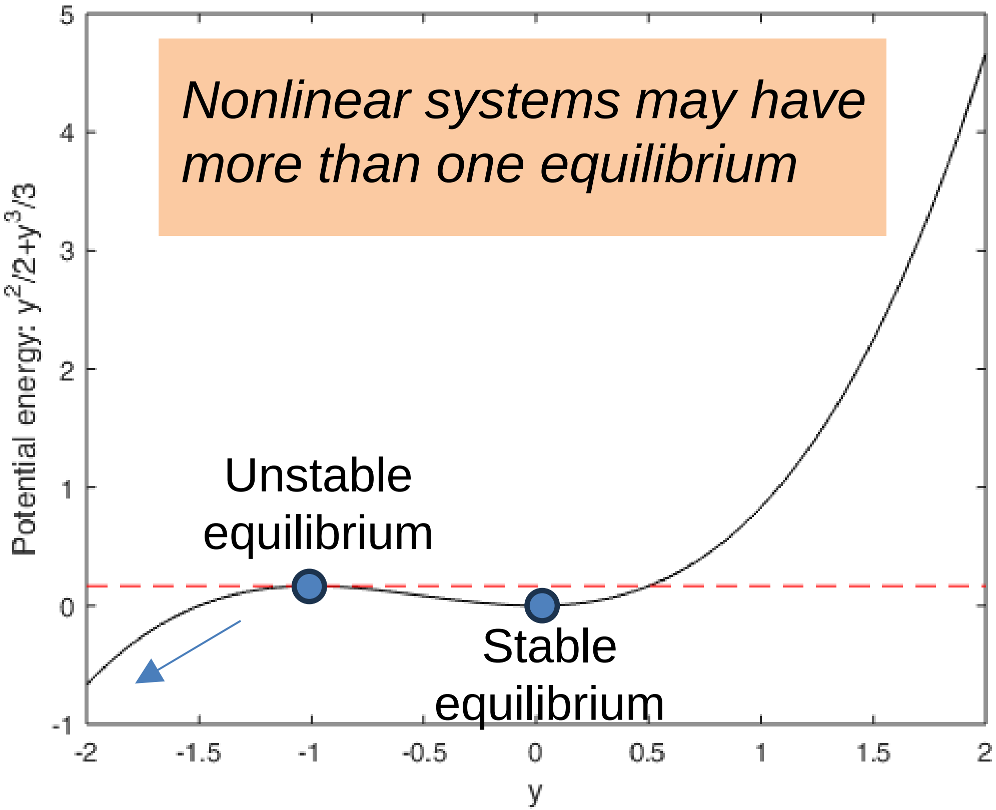
Existence condition
for the square root:

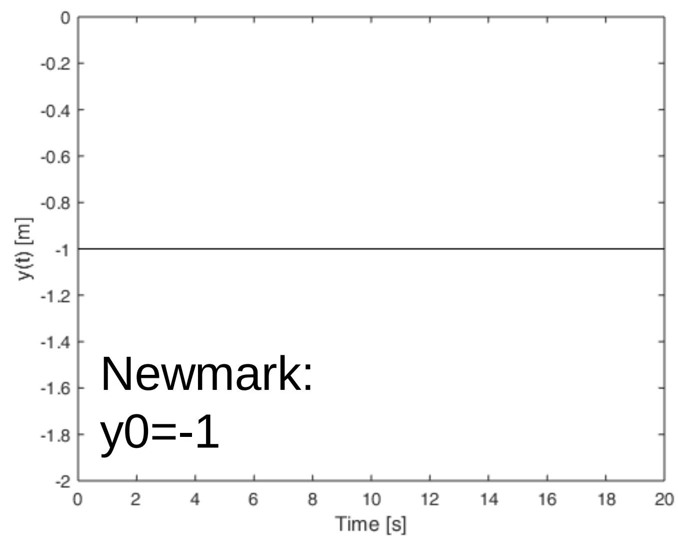
$$3\omega_0^2 - 4\omega_0\alpha_2 y_0 - 4\alpha_2^2 y_0^2 > 0$$

$$\Downarrow$$

$$y_0 < \frac{1}{2\alpha_2}$$

Physical explanation: potential energy

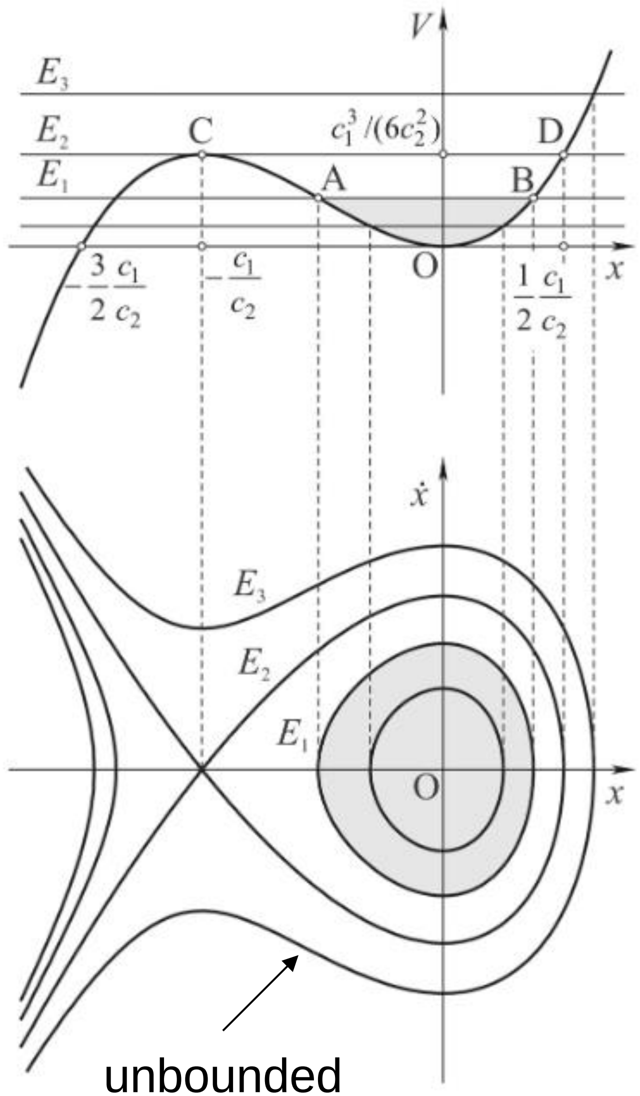




When y_0 >0.5 or y_0 <-1,
unbounded motion

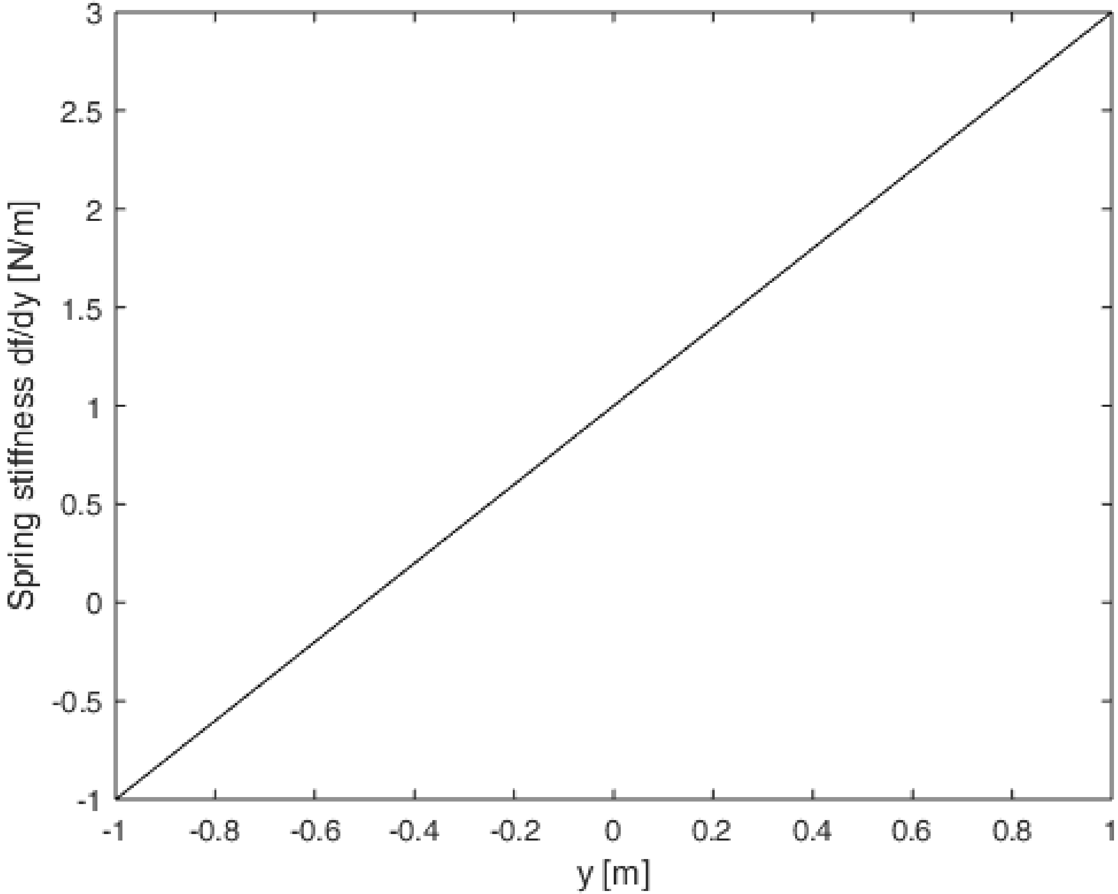
When y_0 =1,
equilibrium position

Potential energy and associated phase space



unbounded
motion

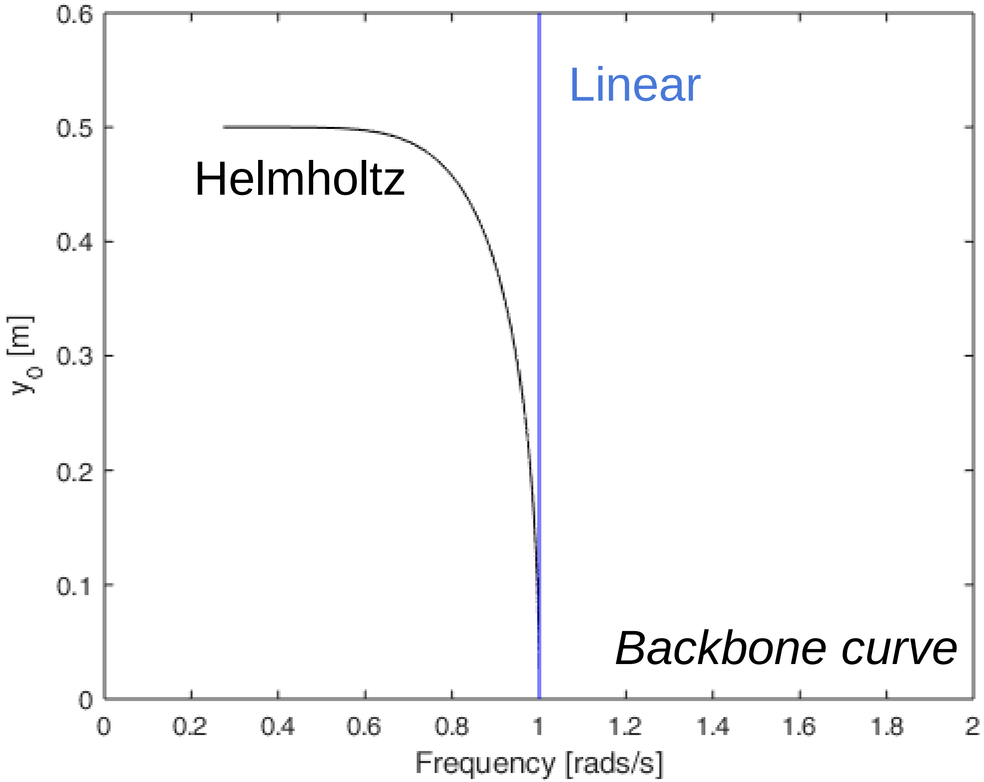
The stiffness is negative when y(t)<-0.5m



The period can be calculated explicitly

$$T = \frac{4\sqrt{6}}{\sqrt{3\omega_0^2 + 6\alpha_2 y_0 + \phi}} K(m)$$

Complete elliptic integral of the first kind



The natural frequency tends to 0 as y_0 tends to 0.5

The Helmholtz oscillator with positive coefficient is softening

Frequency content

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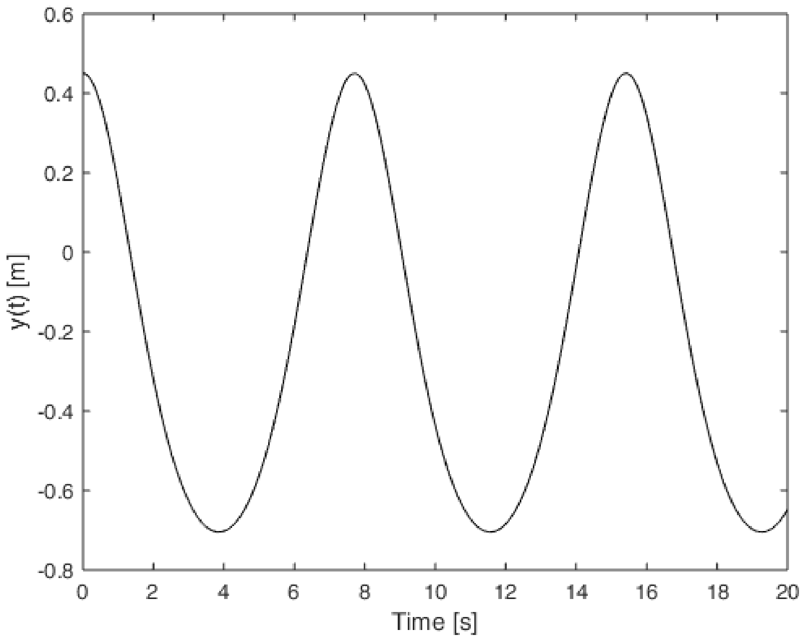
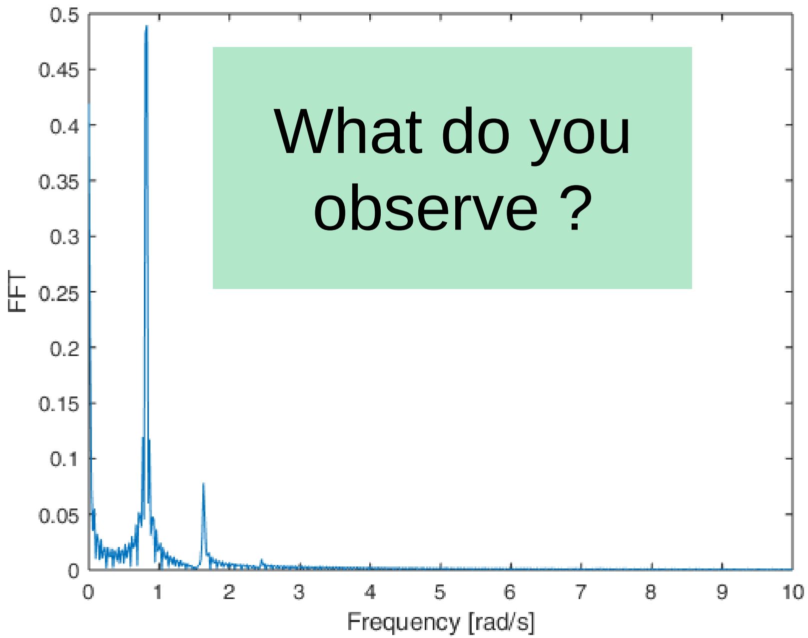
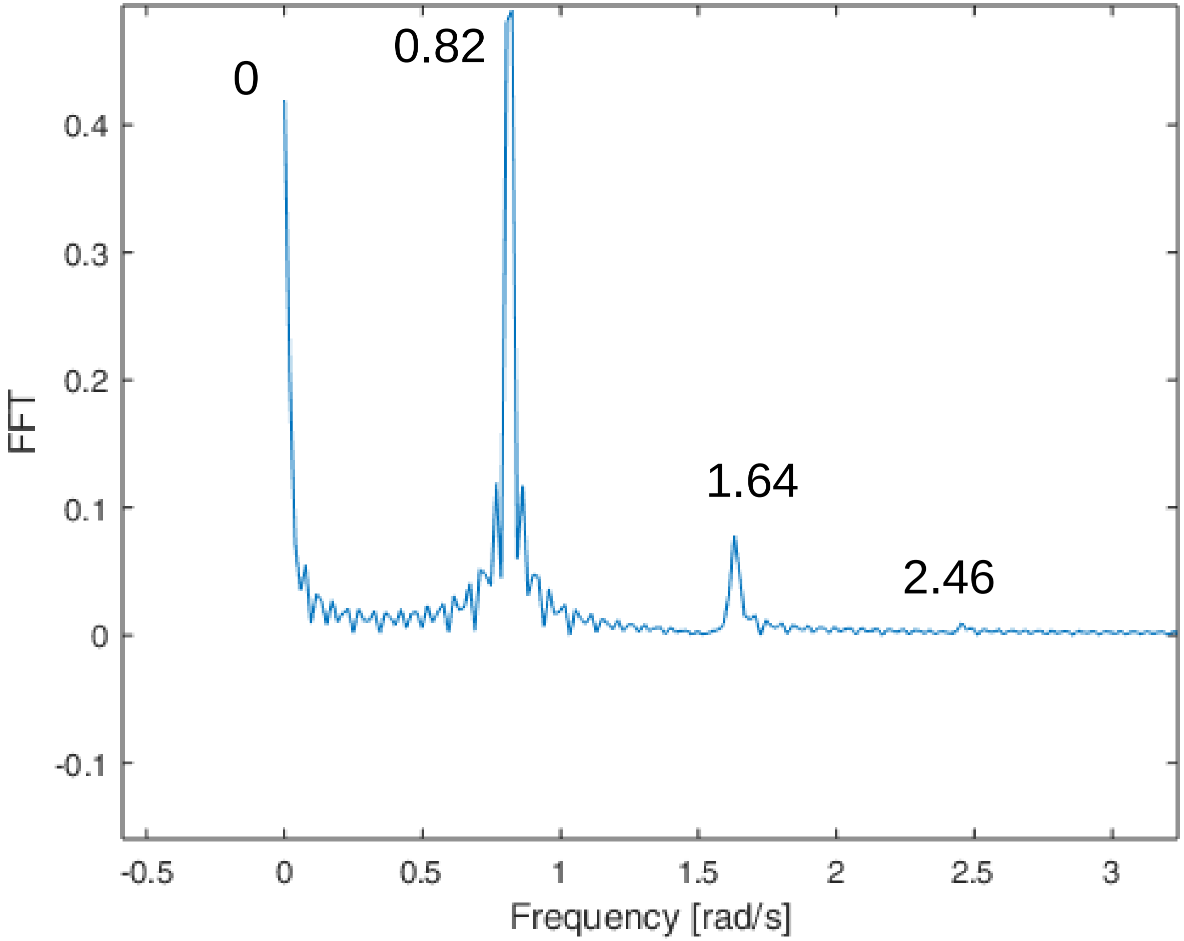


Fourier transform



What do you observe ?

Zero, first, second, third harmonics generated



Key findings

$$\ddot{y}(t) + \omega_0^2 y(t) + \alpha_2 y^2(t) = 0 \qquad \dot{y}(0) = 0\ ,\ y(0) = y_0$$

$$y(t) = y_0 - a\,\mathrm{sn}^2(\Omega t|m)$$

A nonlinear system can possess multiple equilibrium positions.

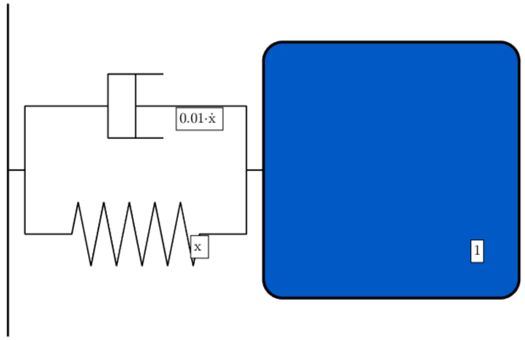
From a theoretical perspective, a nonlinear system can exhibit unbounded motion.

Quadratic nonlinearity gives rise to even and odd harmonics whereas cubic nonlinearity gives rise to odd harmonics.

Exact solutions of damped, unforced oscillators

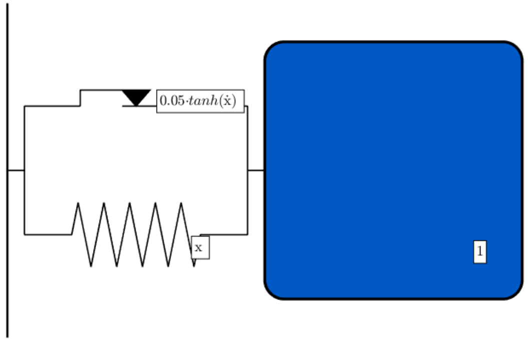
4. The linear damped oscillator:

Motivation
Exact solution
Key findings



5. Coulomb friction

Motivation
Exact solution
Key findings



4. The linear damped oscillator

$$\ddot{y}(t) + 2\xi\omega_0\dot{y}(t) + \omega_0^2 y(t) = 0$$

Motivation for linear damping

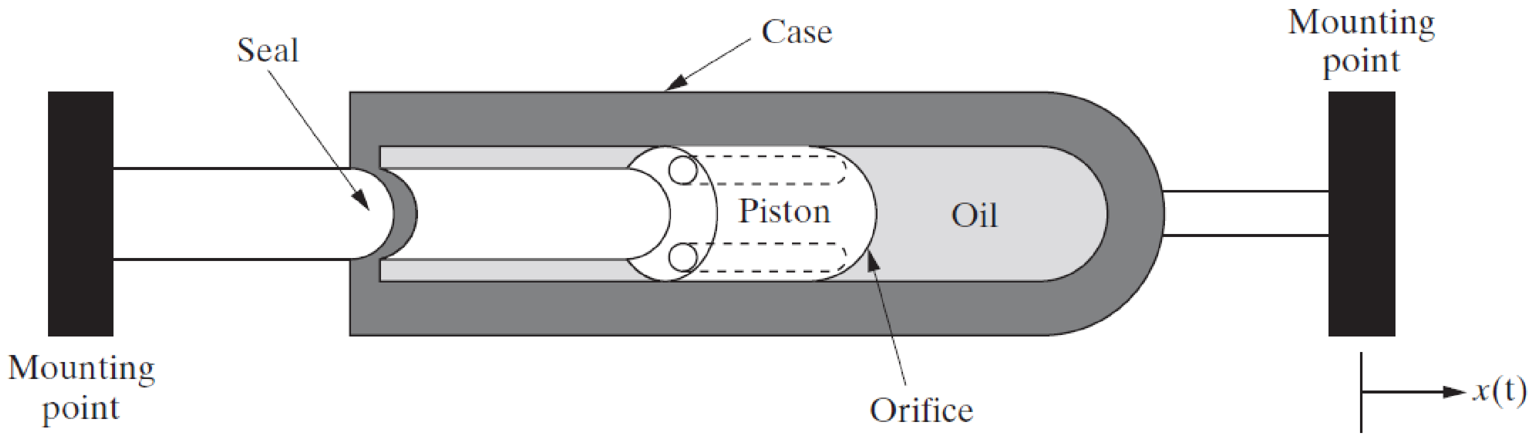


Figure 1.8 A schematic of a dashpot that produces a damping force $f_c(t) = c\dot{x}(t)$, where $x(t)$ is the motion of the case relative to the piston.

Motivation for linear damping

A simple way to construct a damping matrix $\boldsymbol{C}$ that guarantees diagonal modal damping consists of making a weighted sum of the mass and stiffness matrices:

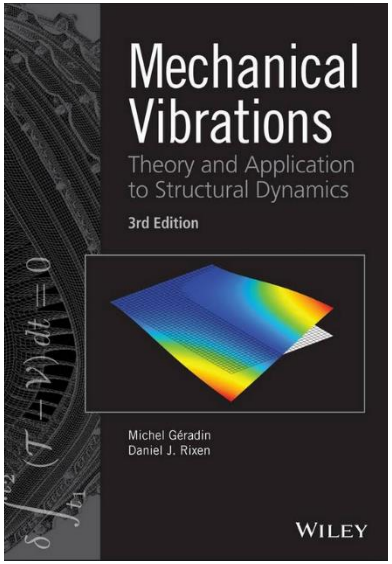
$$\boldsymbol{C} = a\boldsymbol{K} + b\boldsymbol{M} \tag{3.19}$$

This matrix is commonly known as a *proportional damping* (or *Rayleigh damping*) matrix and results in a diagonal modal damping matrix with coefficients:

$$\beta_r = a\gamma_r + b\mu_r$$

and the associated modal damping ratios are:

$$\varepsilon_r = \frac{1}{2}\left(a\omega_{0r} + \frac{b}{\omega_{0r}}\right)$$



Exact solution by integration

$$\ddot{y}(t) + 2\xi\omega_0\dot{y}(t) + \omega_0^2 y(t) = 0, \quad \dot{y}(0) = \dot{y}_0,\ y(0) = y_0$$

$$\tan^{-1}\frac{\frac{\dot{y}}{y} + \xi\omega_0}{\omega_d} + \omega_d t = \phi$$

$$\omega_d = \omega_0\sqrt{1-\xi^2}$$

Find again a constant of motion called ϕ (no longer the energy !)

$$y(t) = Y e^{-\xi\omega_0 t}\cos\left(\omega_d t - \phi\right)$$

$$\boxed{y(t) = \sqrt{y_0^2 + \left(\frac{\dot{y}_0 + \xi\omega_0 y_0}{\omega_d}\right)^2}\, e^{-\xi\omega_0 t}\sin\left(\omega_d t + \tan^{-1}\frac{y_0\omega_d}{\dot{y}_0 + \xi\omega_0 y_0}\right)}$$

EXACT SOLUTION

The response of a damped linear oscillator is an exponentially-damped sine wave

Details of the proof

As for the undamped case, the exact analytical solution can be derived by finding a constant of motion, i.e., without assuming a trial solution. Posing $u = \frac{\dot{y}}{y}$, we have $\ddot{y} = \dot{u}y + \dot{y}u$, and Equation (2.6) is transformed into

$$\dot{u} + u^2 + 2\xi\omega_0 u + \omega_0^2 = 0 \tag{2.22}$$

or equivalently into

$$\frac{du}{u^2 + 2\xi\omega_0 u + \omega_0^2} = -dt \tag{2.23}$$

Upon integration[4] and provided that $\xi < 1$, the following conservation law is obtained:

$$\tan^{-1}\frac{u + \xi\omega_0}{\omega_d} + \omega_d t = \phi \tag{2.24}$$

where $\omega_d = \omega_0\sqrt{1-\xi^2}$ is the *damped natural frequency*, and ϕ is an integration constant. The cases $\xi > 1$ and $\xi = 1$ correspond to aperiodic motions; they can also be tackled [5], but they are of less practical interest and are not detailed herein.

Details of the proof

Recalling that $u = \dot{y}/y$, we have

$$\frac{dy}{y} = \left[-\xi\omega_0 - \omega_d \tan\left(\omega_d t - \phi\right)\right] dt \tag{2.25}$$

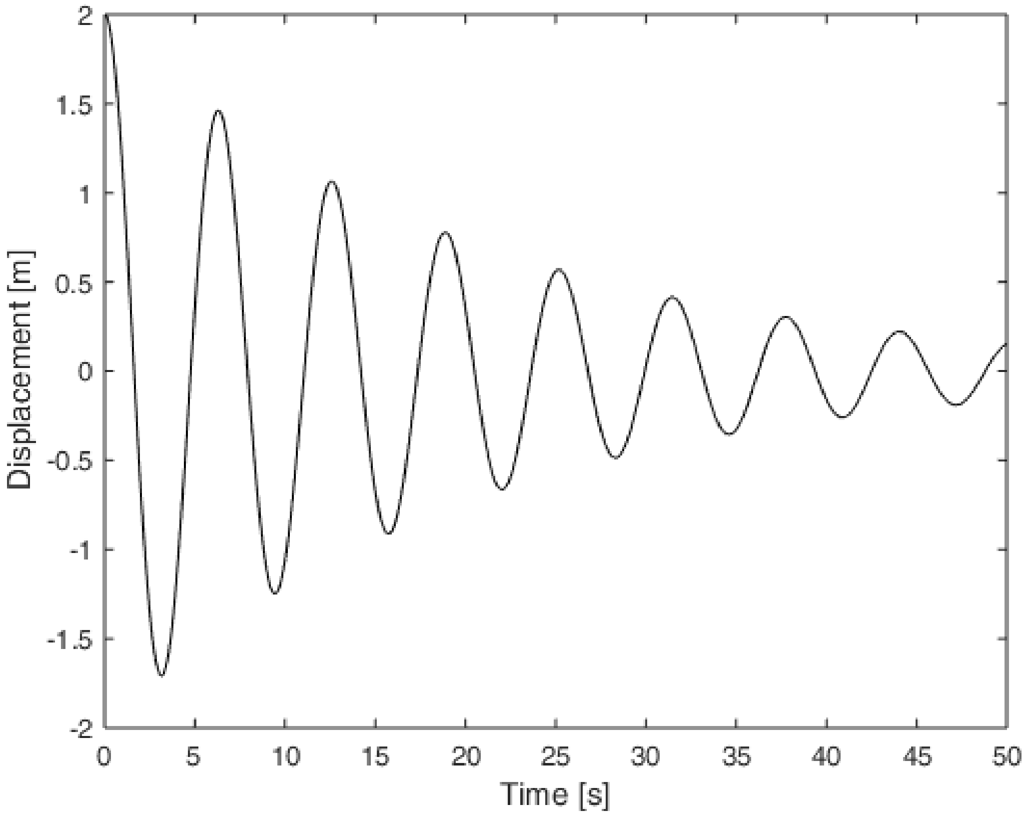
Equation (2.25) can be further integrated to provide the general solution to Equation (2.6)

$$y(t) = Ye^{-\xi\omega_0 t} \cos\left(\omega_d t - \phi\right) \tag{2.26}$$

where Y is the second integration constant. Both Y and ϕ can be determined from the knowledge of the initial conditions y_0 and $\dot{y}_0$. Equation (2.26) expresses that the free response of a viscously-damped oscillator takes the form of harmonic motion of frequency ω_d with exponentially decaying amplitude which can be seen as a damped sine wave.

Exponentially-damped sine wave

$$y(t) = \sqrt{y_0^2 + \left(\frac{\dot{y}_0 + \xi\omega_0 y_0}{\omega_d}\right)^2} e^{-\xi\omega_0 t} \sin\left(\omega_d t + \tan^{-1} \frac{y_0\omega_d}{\dot{y}_0 + \xi\omega_0 y_0}\right)$$



$$\ddot{y} + 0.1\dot{y} + y = 0, \quad y(0) = 2, \dot{y}(0) = 0$$

$$\xi = 0.05, \omega_0 = 1\,rad/s, \omega_d = 0.999\,rad/s,$$

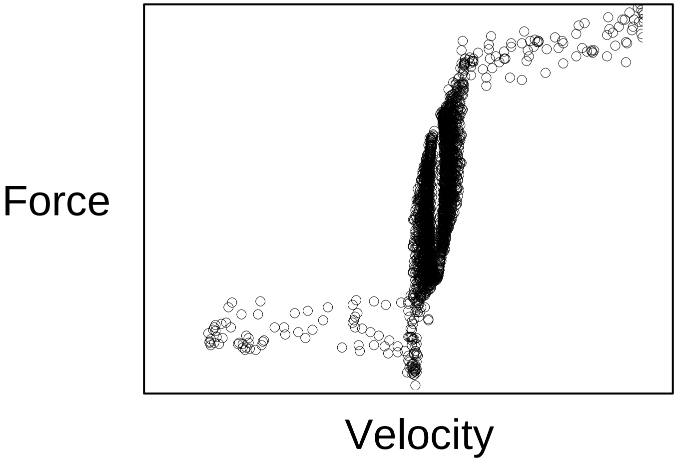
5. Coulomb friction

$$\ddot{y}(t) + \omega_0^2 y(t) + \mu g \,\mathrm{sign}(\dot{y}(t)) = 0$$

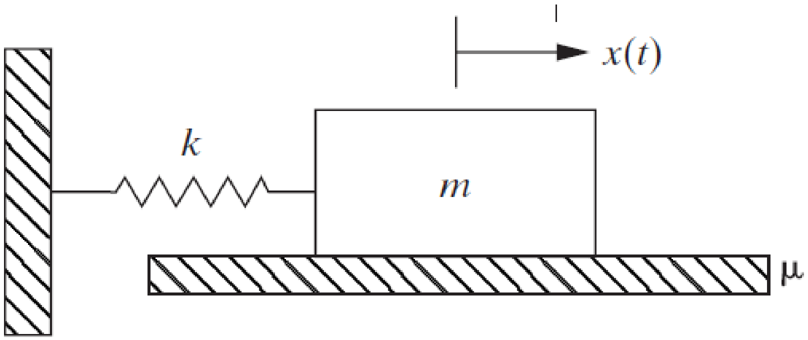
Motivation for Coulomb friction: interfacial damping


Right missile


Connection with the wing



System with Coulomb friction



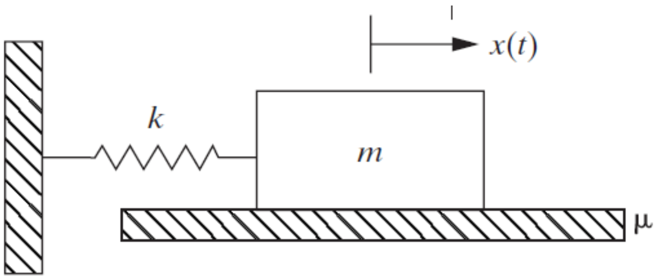
$$f_c = F_c(\dot{x}) = \begin{Bmatrix} -\mu N & \dot{x} > 0 \\ 0 & \dot{x} = 0 \\ \mu N & \dot{x} < 0 \end{Bmatrix}$$

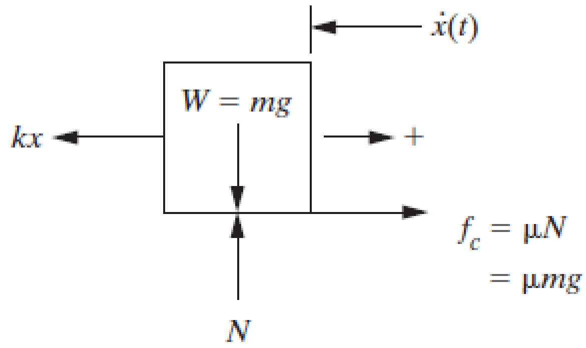
$$m\ddot{y}(t) + ky(t) + \mu mg\,\mathrm{sign}(\dot{y}(t)) = 0 \qquad \dot{y}(0) = \dot{y}_0,\ y(0) = y_0$$

Divide by m

$$\ddot{y}(t) + \omega_0^2 y(t) + \mu g\,\mathrm{sign}(\dot{y}(t)) = 0$$

System with Coulomb friction





$$f_c = F_c(\dot{x}) = \begin{Bmatrix} -\mu N & \dot{x} > 0 \\ 0 & \dot{x} = 0 \\ \mu N & \dot{x} < 0 \end{Bmatrix}$$

$$m\ddot{y}(t) + ky(t) + \mu mg\,\mathrm{sign}(\dot{y}(t)) = 0 \qquad \dot{y}(0) = \dot{y}_0,\ y(0) = y_0$$

Divide by m

$$\ddot{y}(t) + \omega_0^2 y(t) + \mu g\,\mathrm{sign}(\dot{y}(t)) = 0$$

Exact solution

$$\dot{y}(0) = \dot{y}_0,\ y(0) = y_0$$

$$\dot{y} < 0$$

$$\ddot{y}(t) + \omega_0^2 y(t) - \mu g = 0$$

Superposition principle

$$y(t) = A\cos\omega_0 t + \frac{\mu g}{\omega_0^2}$$

$$y(t) = \left(y_0 - \frac{\mu g}{\omega_0^2}\right)\cos\omega_0 t + \frac{\mu g}{\omega_0^2}$$

Valid until

$$\dot{y} < 0$$

$$\dot{y}(t) = -\omega_0\left(y_0 - \frac{\mu g}{\omega_0^2}\right)\sin\omega_0 t = 0 \text{ when } t = \frac{\pi}{\omega_0}$$

Exact solution

When $t = \frac{\pi}{\omega_0}$, $y = -y_0 + \frac{2\mu g}{\omega_0^2}$ and $\dot{y} = 0$

$$\dot{y} > 0$$

$$\ddot{y}(t) + \omega_0^2 y(t) + \mu g = 0$$

$$y(t) = A \cos \omega_0 t - \frac{\mu g}{\omega_0^2}$$

$$y(t) = \left(y_0 - \frac{3\mu g}{\omega_0^2} \right) \cos \omega_0 t - \frac{\mu g}{\omega_0^2}$$

Valid until

$$\dot{y} > 0$$

$$\dot{y}(t) = -\omega_0 \left(y_0 - \frac{3\mu g}{\omega_0^2} \right) \sin \omega_0 t = 0 \text{ when } t = \frac{2\pi}{\omega_0}$$

In summary

$[0, \frac{\pi}{\omega_0}]$ $$y(t) = \left(y_0 - \frac{\mu g}{\omega_0^2}\right)\cos\omega_0 t + \frac{\mu g}{\omega_0^2}$$

$[\frac{\pi}{\omega_0}, \frac{2\pi}{\omega_0}]$ $$y(t) = \left(y_0 - \frac{3\mu g}{\omega_0^2}\right)\cos\omega_0 t - \frac{\mu g}{\omega_0^2}$$

$[\frac{2\pi}{\omega_0}, \frac{3\pi}{\omega_0}]$ $$y(t) = \left(y_0 - \frac{5\mu g}{\omega_0^2}\right)\cos\omega_0 t + \frac{\mu g}{\omega_0^2}$$

$[\frac{3\pi}{\omega_0}, \frac{4\pi}{\omega_0}]$ $$y(t) = \left(y_0 - \frac{7\mu g}{\omega_0^2}\right)\cos\omega_0 t - \frac{\mu g}{\omega_0^2}$$

This procedure is repeated until the motion stops, i.e., when the velocity is zero and the spring force is insufficient to overcome the frictional force.

The motion can thus stop at a potentially different equilibrium position than the rest position. There are thus multiple equilibrium positions !

Verification using numerical simulation

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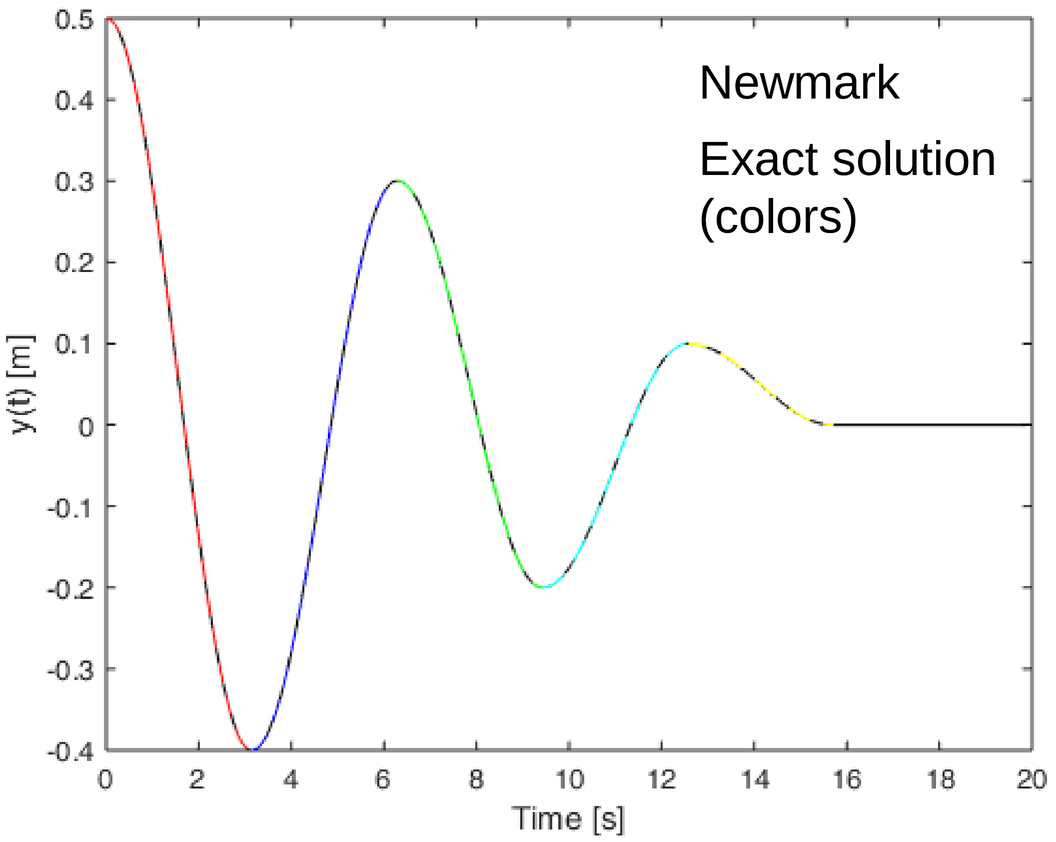


What is the period of the motion ?

$[0, \frac{\pi}{\omega_0}]$ $$y(t) = \left(y_0 - \frac{\mu g}{\omega_0^2}\right)\cos\omega_0 t + \frac{\mu g}{\omega_0^2}$$

$[\frac{\pi}{\omega_0}, \frac{2\pi}{\omega_0}]$ $$y(t) = \left(y_0 - \frac{3\mu g}{\omega_0^2}\right)\cos\omega_0 t - \frac{\mu g}{\omega_0^2}$$

$[\frac{2\pi}{\omega_0}, \frac{3\pi}{\omega_0}]$ $$y(t) = \left(y_0 - \frac{5\mu g}{\omega_0^2}\right)\cos\omega_0 t + \frac{\mu g}{\omega_0^2}$$

$[\frac{3\pi}{\omega_0}, \frac{4\pi}{\omega_0}]$ $$y(t) = \left(y_0 - \frac{7\mu g}{\omega_0^2}\right)\cos\omega_0 t - \frac{\mu g}{\omega_0^2}$$

One half of a cycle every π/ω_0; the full period is $2\pi/\omega_0$.

Linear frequency

Linear decay of the amplitude

$[0, \frac{\pi}{\omega_0}]$ $\qquad y(t) = \left(y_0 - \frac{\mu g}{\omega_0^2}\right)\cos\omega_0 t + \frac{\mu g}{\omega_0^2}$

$[\frac{\pi}{\omega_0}, \frac{2\pi}{\omega_0}]$ $\qquad y(t) = \left(y_0 - \frac{3\mu g}{\omega_0^2}\right)\cos\omega_0 t - \frac{\mu g}{\omega_0^2}$

$[\frac{2\pi}{\omega_0}, \frac{3\pi}{\omega_0}]$ $\qquad y(t) = \left(y_0 - \frac{5\mu g}{\omega_0^2}\right)\cos\omega_0 t + \frac{\mu g}{\omega_0^2}$

$[\frac{3\pi}{\omega_0}, \frac{4\pi}{\omega_0}]$ $\qquad y(t) = \left(y_0 - \frac{7\mu g}{\omega_0^2}\right)\cos\omega_0 t - \frac{\mu g}{\omega_0^2}$

Every cycle, the decrease in amplitude is $4\mu g/\omega_0^2$.

Contrast with the exponential decay of the linear oscillator.

In summary

Multiple equilibria

Unbounded motion

Elliptic functions

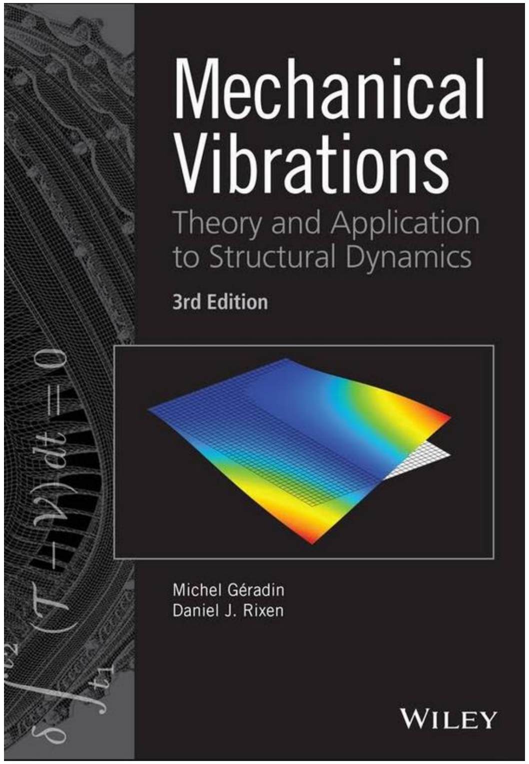
Even/odd harmonics

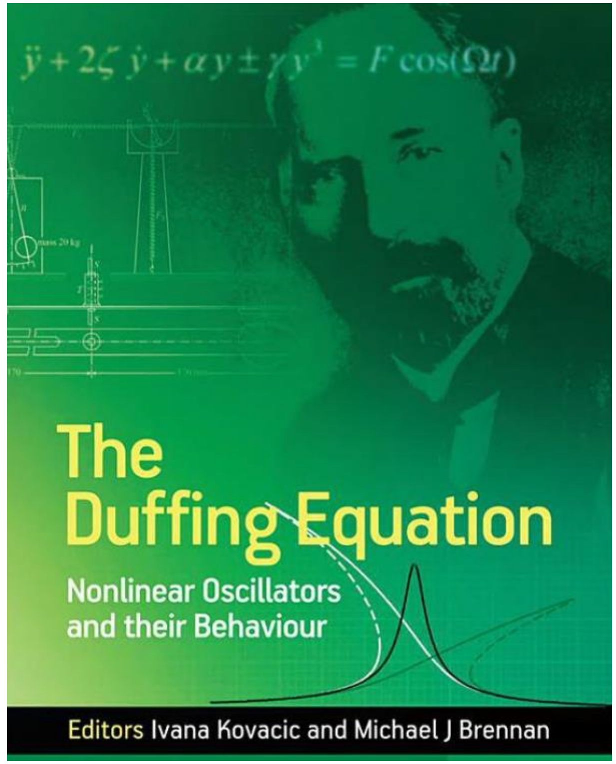
Hardening/softening

Backbone curve (frequency-amplitude dependence)

Decay rate change

References





ORDINARY NON-LINEAR DIFFERENTIAL EQUATIONS IN ENGINEERING AND PHYSICAL SCIENCES

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Closed-form solutions for the quadratic mixed-parity nonlinear oscillator

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References: Airbus A-400M, satellite, F-16 aircraft

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