Nonlinear Vibrations of Aerospace Structures

University of Liège, Belgium

L01 Introduction

Course objectives
Review of linear theory
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Course details:

► http://www.s3l.be
What Is a Nonlinearity?

LINEAR

NONLINEARITY

Force

Displacement
Velocity

Force

Displacement
Velocity

Force

Displacement
Velocity
What Is a Nonlinear Vibration?
Course Motivation

THEORETICAL
STANDPOINT

PRACTICAL
STANDPOINT

Studied extensively!
It Is Nonlinear, So What?
It Is Nonlinear, So What?

REGIME 1: weakly nonlinear effects

REGIME 2: strongly nonlinear effects
It Is Nonlinear, So What?

REGIME 1: weakly nonlinear effects

REGIME 2: strongly nonlinear effects

Linear

Nonlinear
Course Objectives

At the end of this course, you will

► Understand the impact of nonlinearity on system dynamics.
► Master the concepts of mode shape, resonance frequency and frequency response function of nonlinear systems.
► Be familiar with new nonlinear concepts including stability and bifurcations.
► Recognize nonlinearity in real-world (aerospace) structures.
► Know how to use the NI2D software.

You will be exposed to new theoretical concepts, advanced computational methods and practical experimental techniques.
The Nonlinear Identification to Design Software
Course Outline

1. Brief review of linear theory
2. Impact of nonlinearity, nonlinear FRFs and 4 new concepts
3. Mathematical modeling and numerical computation
4. Nonlinear modes
5. Introduction to system identification and nonlinearity detection
6. Nonlinearity characterization
7. Nonlinear parameter estimation
8. Advanced concepts: bifurcations, modal interactions, isolas.
9. Industrial case study
Nonlinear Vibrations of Aerospace Structures

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L01 Introduction

Course objectives

Review of linear theory
How To Write the Governing Equations?

\[-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) + \frac{\partial T}{\partial q_s} - \frac{\partial V}{\partial q_s} - \frac{\partial D}{\partial q_s} + Q_s(t) = 0, s = 1, \ldots n.\]

Lagrange equations for \( n \) generalized coordinates

\[T = \frac{1}{2} mL^2 \dot{\theta}^2, \quad V = mgL(1 - \cos \theta)\]

\[\ddot{\theta} + \frac{g}{L}\sin \theta = 0\]
Let’s Calculate the Period of the Motion

\[ T + V = E \]

\[
\frac{1}{2} mL^2 \dot{\theta}^2 + mgL(1 - \cos \theta) = mgL(1 - \cos \theta_0)
\]

\[
\dot{\theta} = \frac{d\theta}{dt} = \pm \sqrt{\frac{2g}{L}} (\cos \theta - \cos \theta_0)
\]

\[
dt = \frac{d\theta}{\dot{\theta}} = \sqrt{\frac{L}{2g(\cos \theta - \cos \theta_0)}} d\theta
\]

\[
Period = 4 \sqrt{\frac{L}{2g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}} = 2\pi \sqrt{\frac{L}{g}} \left[ 1 + \frac{\theta_0^2}{16} + \cdots \right]
\]
The Linearization Around an Equilibrium

\[ V(q) = V(0) + \sum_{s=1}^{n} \left( \frac{\partial V}{\partial q_s} \right)_{q=0} + \frac{1}{2} \sum_{s=1}^{n} \sum_{r=1}^{n} \left( \frac{\partial^2 V}{\partial q_s \partial q_r} \right)_{q=0} q_r q_s + \ldots \]

\[ V(q) = \frac{1}{2} \sum_{s=1}^{n} \sum_{r=1}^{n} k_{rs} q_r q_s = \frac{1}{2} q^T K q \]

\[ T(\dot{q}) = \frac{1}{2} \sum_{s=1}^{n} \sum_{r=1}^{n} m_{rs} \dot{q}_r \dot{q}_s = \frac{1}{2} \dot{q}^T M \dot{q} \]

\[ M \ddot{q} + K q = 0 \]
The Linearization of the Pendulum

\[ \ddot{\theta} + \frac{g}{L} \sin \theta = 0 \]

\[ \ddot{\theta} + \frac{g}{L} \left( \theta - \frac{\theta^3}{6} + \cdots \right) = 0 \]

\[ \theta \ll 1 \]

\[ \ddot{\theta} + \frac{g}{L} \theta = 0 \]

Period = \(2\pi \sqrt{\frac{L}{g}}\)
3 Main Assumptions in Linear Structural Dynamics

\[ M\ddot{q} + C\dot{q} + Kq = 0 \]

- Linear elasticity → nonlinear materials
- Small displ. and rotations → geometrical nonlinearity → nonlinear boundary conditions
- Viscous damping → nonlinear damping mechanisms
Assumption 1: Nonlinear Materials

Hyperelastic material (e.g., rubber)

Shape memory alloy
Assumption 1: Ligament in Your Knee Joint

Load

1. Toe region: normal range
2. Linear
3. Yield
4. Human cadaveric anterior cruciate ligament in knee joint (Dr. Ziv, MAE, Buffalo)

Extension
Assumption 2: Geometrical Nonlinearities

\[ F = 2k (l_1 - l_0) \frac{x}{l_1} = 2kx \left( 1 - \frac{l_0}{\sqrt{x^2 + l_0^2}} \right) \]

Green's strain tensor

\[ \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i} \frac{\partial u_m}{\partial x_j} \right) \]
Assumption 2: A Geometrical Beam in Our Lab
Assumption 3: Nonlinear Damping

Viscous damping but also…

Coulomb friction

Aerodynamic damping
The Concept of Mode Shape

\[ M \ddot{q} + Kq = 0 \quad + \quad q = x \varphi(t) \]

Synchronous vibration of the structure

\[ \text{det}(K - \omega^2 M) = 0 \]

\[ (K - \omega_{(r)}^2 M)x_{(r)} = 0 \]

\[ \varphi_r(t) = \alpha_r \cos(\omega_r t) + \beta_r \sin(\omega_r t) \]

\( n \) natural frequencies

\( n \) normal modes

normal coordinates
Normal Modes: Important Properties

Clear physical meaning:

► Structural deformation at resonance
► Synchronous vibration of the structure

Important mathematical properties:

► Orthogonality
► Decoupling of the equations of motion (modal superposition)
► Invariance
The Concept of FRF

\[ M\ddot{q} + Kq = s \cos(\omega t) \quad + \quad q = x \cos(\omega t) \]

\[ x = (K - \omega^2 M)^{-1} s = H(\omega) s \]

\[ H(\omega) = \sum \frac{x(s)x(s)^T}{(\omega_s^2 - \omega^2)\mu_s} \]

Clear link between the FRF and the modal parameter
FRF: Important Properties

- The FRF is a constant system properties for a linear system.
- FRF can be easily estimated from measured data.
- Very convenient way of locating resonance frequencies.
Time Response: Mode Displacement Method

\[ M\ddot{q} + Kq = p(t) = g \varphi(t) \]

\[
q(t) = \sum_{s=1}^{n} \frac{x(s)x(s)^T g}{\mu_s \omega_s} \int_{0}^{t} \sin\omega_s(t - \tau) \varphi(t) d\tau \quad \text{Exact}
\]

\[
q(t) = \sum_{s=1}^{k<n} \frac{x(s)x(s)^T g}{\mu_s \omega_s} \int_{0}^{t} \sin\omega_s(t - \tau) \varphi(t) d\tau \quad \text{Approximate}
\]
Time Response: Numerical Integration

\[ M, C, K \quad q_0, \dot{q}_0 \]

Compute \( \ddot{q}_0 \)

Time incrementation
\[ t_{n+1} = t_n + h \]

Prediction
\[
\begin{align*}
\ddot{q}_{n+1} &= \dot{q}_n + (1 - \gamma) h \ddot{q}_n \\
q_{n+1}^* &= q_n + h \dot{q}_n + (0.5 - \beta) h^2 \ddot{q}_n
\end{align*}
\]

Computation of accelerations
\[
S = M + h \gamma C + h^2 \beta K
\]
\[
S \ddot{q}_{n+1} = p_{n+1} - C \dot{q}_{n+1}^* - K q_{n+1}^*
\]

Correction
\[
\begin{align*}
\dot{q}_{n+1} &= \dot{q}_{n+1}^* + h \gamma \ddot{q}_{n+1} \\
q_{n+1} &= q_{n+1}^* + h^2 \beta \ddot{q}_{n+1}
\end{align*}
\]

Newmark integration scheme for linear systems
Time Response: Numerical Integration

\[ M, f, p, S \quad q_0, \dot{q}_0 \]

Compute \( \ddot{q}_0 \)
\[ \ddot{q}_0 = M^{-1} (g_0 - f(q_0, q_0)) \]

Time incrementation
\[ t_{n+1} = t_n + h \]

Prediction
\[ \ddot{q}_{n+1} = \dot{q}_n + (1 - \gamma) h \ddot{q}_n \]
\[ q_{n+1} = q_n + h \dot{q}_n + (0.5 - \beta) h^2 \ddot{q}_n \]
\[ \ddot{q}_{n+1} = 0 \]

Residual vector evaluation
\[ r_{n+1} = M \ddot{q}_{n+1} + f_{n+1} - g_{n+1} \]

Newmark integration scheme for nonlinear systems

Convergence ?
\[ \| r_{n+1} \| < \varepsilon \| f_{n+1} \| \]

Calculation of the correction
\[ S(q_{n+1}) \Delta q = -r_{n+1} \]

Correction
\[ q_{n+1} = q_{n+1} + \Delta q \]
\[ \dot{q}_{n+1} = \dot{q}_{n+1} + \frac{\gamma}{\beta h} \Delta q \]
\[ \ddot{q}_{n+1} = \ddot{q}_{n+1} + \frac{1}{\beta h^2} \Delta q \]
Very Important: Sampling Frequency

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>$\gamma$</th>
<th>$\beta$</th>
<th>$\omega h$</th>
<th>$\rho - 1$</th>
<th>$\frac{\Delta T}{T}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Purely explicit</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{\omega^2 h^2}{4}$</td>
<td>–</td>
</tr>
<tr>
<td>Central difference</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>$-\frac{\omega^2 h^2}{24}$</td>
</tr>
<tr>
<td>Fox &amp; Goodwin</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{12}$</td>
<td>2.45</td>
<td>0</td>
<td>$O(h^3)$</td>
</tr>
<tr>
<td>Linear acceleration</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{6}$</td>
<td>3.46</td>
<td>0</td>
<td>$\frac{\omega^2 h^2}{24}$</td>
</tr>
<tr>
<td>Average constant acceleration</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{4}$</td>
<td>$\infty$</td>
<td>0</td>
<td>$\frac{\omega^2 h^2}{12}$</td>
</tr>
<tr>
<td>Average constant acceleration (modified)</td>
<td>$\frac{1}{2} + \alpha$</td>
<td>$\frac{(1+\alpha)^2}{4}$</td>
<td>$\infty$</td>
<td>$-\alpha \frac{\omega^2 h^2}{2}$</td>
<td>$\frac{\omega^2 h^2}{12}$</td>
</tr>
</tbody>
</table>
In Summary: Common Sources of Nonlinearity

- **Bolts, joints and gaps**
- **Elastomers and composites**
- **Friction and contact**
- **Large amplitudes**
In Summary: Important Linear Concepts/Methods

Mode shapes, resonance frequencies, damping ratios

Frequency response functions (FRFs)

Modal superposition/numerical integration

**OPEN QUESTION:**
Will they remain valid/useful for nonlinear systems?