Nonlinear Vibrations of Aerospace Structures

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L02 Impact of NL

Nonlinear FRFs
4 new concepts
The Frequency Response Function

\[ X(\omega) = H(\omega)F(\omega) \]

Output \hspace{1cm} FRF \hspace{1cm} Input
Frequency Response Function: Bode Plot

Bode plot of amplitude and phase of a FRF function. Amplitude has peaks corresponding to resonances. Phase has shift at resonant frequency.
How Do We Calculate the FRF?

\[ m\ddot{x} + kx = F\sin \omega t \]

\[ x(t) = X\sin \omega t \]

\[ -m\omega^2 X + kX = F \]

\[ X = \frac{F}{k - \omega^2 m} \]

**Linear system**: output directly proportional to input.
Superposition Principle

Cornerstone of linear theory:

\[ A \rightarrow \text{LIN} \rightarrow X \]
\[ B \rightarrow \text{LIN} \rightarrow Y \]

\[ A + B \rightarrow \text{LIN} \rightarrow X + Y \]
MS760 Aircraft Has Nonlinear Bolted Joints
Impact on Measured FRFs

- Decrease in stiffness
- Increase in damping
F-16 Aircraft Has A Nonlinear Sliding Joint
Impact on Measured FRFs

- Decrease in stiffness
- Increase in damping

FRF vs. Frequency (Hz)
So, No Superposition Principle for a NL System!
Let’s Go Back to Basics

Cantilever beam with at very thin beam at its tip:

$l=70$ cm
$t=1.4$ cm

$l=4$ cm
$t=0.05$ cm

Beam @ ULiège
A 1DOF Model of the First Beam Mode Identified

\[0.289 \ddot{x} + 0.1357 \dot{x} + 11009x = F \sin \omega t\]
However, the thin beam is geometrically nonlinear.
A 1DOF Nonlinear Model of the First Beam Mode

\[ 0.289\ddot{x} + 0.1357\dot{x} + 11009x + 2.37 \times 10^9 x^3 = F \sin \omega t \]
How Do We Calculate the Nonlinear FRF?

\[ m\ddot{x} + kx + k_3x^3 = F\sin \omega t \]

\[ x = x(F, \omega, t)? \]
1. Harmonics
The Harmonic Balance Method

\[ m\ddot{x} + kx + k_3x^3 = F\sin \omega t \]

\[ x(t) = X\sin \omega t \]

\[-m\omega^2 X\sin \omega t + kX\sin \omega t + k_3X^3\sin^3 \omega t = F\sin \omega t \]

\[ \sin^3 \omega t = (3\sin \omega t - \sin 3\omega t)/4 \]

Nonlinear relation between \( X \) and \( F \)

**OPTION 1: EXACT**

\[ x(t) = X\sin \omega t + X_3\sin 3\omega t \]

Solution: infinite series of harmonics

**OPTION 2: APPROXIMATION**

\[-m\omega^2 X + kX + \frac{3}{4}k_3X^3 = F \]

Solve a 3\textsuperscript{rd} order polynomial in \( X \)

(! BIFURCATIONS !)
The Harmonic Balance Method (with Damping)

\[ m\ddot{x} + c\dot{x} + kx + k_3x^3 = F\sin \omega t \]

\[ x(t) = X\sin \omega t + Y\cos \omega t \]

Same machinery
HB for the Linear System in NI2D

- Harmonics = 1
- F = 0.06N
- Stepsize = 30
- Scaling = 1e-6

0.0022 OK!

# harmonics=1
F=0.06N
Stepsize=30
Scaling=1e-6
HB for the Nonlinear System in NI2D

BIFURCATIONS

# harmonics=1
F=0.06N
Stepsize=30
Scaling=1e-6

< 0.0022 NICE!

NL Frequency Response of dof nº1
Very Fast Convergence of HB in this Case
Importance of Harmonics

Harmonics for dof n°1

Amplitude

Frequency (Hz)

Harmonic 1
Harmonic 3
2. Frequency-amplitude dependence
Variation of Nonlinear FRFs
Measured (Unscaled) FRFs

[Image: A photograph of a test setup with labeled points and lines indicating frequency and displacement.]

[Graph: A graph showing the relationship between frequency (Hz) and displacement (m). The y-axis is labeled as Displ. (m) and ranges from 0 to 2.5 x 10^{-3}. The x-axis is labeled as Frequency (Hz) and ranges from 30 to 45.]
3. Bifurcations
Bifurcations Generate Multi-Valued Response
These Complex Phenomena Exist in Practice

E.g., flutter in aircraft structures

Diagram:
- Amplitude vs. Airspeed
- Bifurcation point
4. Stability
Bifurcations Change Stability
What Is Stability/Instability?
Starting from a Stable/Unstable Solutions

32 Hz

6.555e-4
5.677e-4
9.098e-5

x (m)

Frequency (Hz)
Starting from a Stable/Unstable Solutions

\[ x_1 = 6.500 \times 10^{-4} \]

\[ -0.06 \cos(2\pi - 32t) \]
Starting from a Perturbed Stable Solution

6.6e-4 (stable sol.)
Starting from a Perturbed Unstable Solution

\[ \times 10^{-4} \]

\[ 6.6e-4 \] (stable sol.)
The Competition Between the Two Stable Solutions

Response amplitude

dx/dt (m/s)

x (m)
Evolution of the Basins of Attraction

Frequency (Hz)

$2 \times 10^{-3}$

x (m)

32 Hz

34 Hz

36 Hz

37
Is that all?
What’s Going On (Far From Resonance)?
What’s Going On (5N)?
Superposition principle

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Linear

**NO ! By definition ...**

Nonlinear

Simple frequency content

**NO ! Harmonics**

Uniqueness of the solutions

**NO ! Bifurcations**

Invariance of FRFs

**NO ! Freq.-energy dependence**
4 New Key Concepts

- **Harmonics**
  - Rich frequency content
  - New resonances

- **Frequency-amplitude dependence**
  - Don’t extrapolate!

- **Bifurcations**
  - Nonuniqueness.
  - New resonances

- **Stability**
  - Stable/unstable solutions
Usefulness of Nonlinear FRFs

![Graph showing the stability of nonlinear FRFs over a range of frequencies.]