Nonlinear Vibrations of Aerospace Structures

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Why Do We Need High-Fidelity Models?

For better **decision-making** capability!

Using models, we can access non measurable information (e.g., stress).

Particular operational conditions (e.g., explosions, earthquakes) that are difficult/impossible/dangerous to reproduce experimentally can be simulated.
Why Do We Need High-Fidelity Models?

But also to:

- Reduce dependence on testing (cost and time issues)
- Test design (e.g., sensor and actuator placement)
- Perform **virtual prototyping**: A model can predict the behavior of a structure before its construction. The parameters of a model can easily be modified to improve the design (optimization).
Different Approaches to Model Nonlinear Structures

1. Large displacements and rotations

Displacement: \( u = x - X \)

Cauchy strain tensor: \( \varepsilon_{ij}^c = \frac{1}{2} \left( \frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right) \)

Small displacements and rotations.
1. Large displacements and rotations

Displacement: \( \mathbf{u} = \mathbf{x} - \mathbf{X} \)

Cauchy strain tensor: \( \varepsilon_{ij}^c = \frac{1}{2} \left( \frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right) \)

- Small displacements and rotations.

- Not invariant under rigid-body motion. Cauchy strains cannot be used if rotation amplitudes are finite.
1. Large displacements and rotations

Displacement: \[ \mathbf{u} = \mathbf{x} - \mathbf{X} \]

Green strain tensor: \[ \epsilon_{ij}^G = \frac{1}{2} \left( \frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \sum_{k=1}^{3} \frac{\partial u_k}{\partial X_i} \frac{\partial u_k}{\partial X_j} \right) \]

- Large displacements and rotations.
- Nonlinear measure of deformation. Geometrical nonlinearities can be considered in the elastic force model.
Different Approaches to Model Nonlinear Structures

1. Large displacements and rotations

Landing gear mechanism
*Prof. O. Brüls, ULiège*

Deployable space structure
*Prof. O. Brüls, ULiège*
Different Approaches to Model Nonlinear Structures

2. Large deformations

Nonlinear constitutive laws

Plasticity/Irreversibility

Contacts/Impacts
Different Approaches to Model Nonlinear Structures

2. Large deformations

Fan Blade containment test
Prof. J.-P. Ponthot, ULiège

Buckling of blade in LP compressor
Prof. J.-P. Ponthot, ULiège
3. Linear structure with localized nonlinearities

FOCUS OF THIS COURSE
High-fidelity and fast-running modeling of structures with localized nonlinearities
Integration of Data-Driven and Computer-Aided Models

Accurate modeling of localized nonlinearities identified from experimental data (see next lectures).
Development of Fast-Running Models

Finite element models may involve thousands (even millions) of degrees of freedom (DOFs).

For structures with localized nonlinearities, only a few DOFs are generally involved in nonlinear connections.

Model reduction and substructuring can be applied to speed up simulations.
Model Reduction and Substructuring

Reminders from “Mechanical vibrations: Theory and Applications to Structural Dynamics” (Géradin and Rixen):

**Reduction**: In most cases, engineers are interested in a smaller system capturing only lower frequency dynamics. In this case, a genuine reduction is performed, the reduction method being seen as a DOF economizer.

**Substructuring**: In the context of large projects, the analysis is frequently subdivided into several parts. A separate model is constructed for each part of the system and reduced (**super-element**). The different parts and super-elements are finally combined to simulate the dynamics of the whole system.
Model Reduction and Substructuring

Most methods for reducing the size $n$ of a system consist in partitioning the degrees of freedom into $n_R$ dynamic retained coordinates ($n_R << n$) and $n_C$ condensed coordinates.

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_R \\ \mathbf{x}_C \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} \mathbf{K}_{RR} & \mathbf{K}_{RC} \\ \mathbf{K}_{CR} & \mathbf{K}_{CC} \end{bmatrix} \quad \mathbf{M} = \begin{bmatrix} \mathbf{M}_{RR} & \mathbf{M}_{RC} \\ \mathbf{M}_{CR} & \mathbf{M}_{CC} \end{bmatrix}$$

The dynamical behavior of the structure is usually described by the retained coordinates only.

In this course, the DOFs retained are those connected to nonlinearities.
Craig-Bampton Method

Let us consider a substructure which is connected to the rest of the system by a set of boundary degrees of freedom $x_R$.

The originality of the method is to consider in the condensation, in addition to the boundary DOFs $x_R$, the contribution of the internal vibration modes to the reduced model.
Craig-Bampton Method

The dynamical behavior of a substructure is fully described by:

- the static boundary modes resulting from the static condensation,
- the subsystem eigenmodes in clamped boundary configuration.

Static mode

Vibration mode
Accordingly, it means that the following transformation may be applied to the initial degrees of freedom:

\[
x = \begin{bmatrix}
I & 0 \\
-K_{CC}^{-1}K_{CR} & \Phi_C
\end{bmatrix}
\begin{bmatrix}
x_R \\
y_C
\end{bmatrix}
\]

where the Guyan’s reduction matrix has been complemented by the set of \( n_C \) internal vibration modes \( \tilde{x} \) obtained by solving:

\[
(K_{CC} - \tilde{\omega}^2 M_{CC}) \tilde{x} = 0
\]

\[
\Phi_C = [\tilde{x}(1) \ldots \tilde{x}(n_C)]
\]
Craig-Bampton Method

In practice, only a certain number $m < n_C$ of internal vibration modes are kept:

$$\Phi_C \rightarrow \Phi_m = [\tilde{x}(1) \ldots \tilde{x}(m)]$$

$$y_C \rightarrow y_m$$

This subset of internal vibration modes should be selected in order to cover a frequency range that is large enough to approximate the dynamics in play. Convergence of the reduced-order model should be carefully assessed!
Craig-Bampton Method

Final reduction matrix of dimension $n \times (n_R + m)$:

$$R = \begin{bmatrix} I & 0 \\ -K_{CC}^{-1}K_{CR} & \Phi_m \end{bmatrix}$$

Reduced stiffness and mass matrices:

$$\bar{K} = R^T KR \quad \bar{M} = R^T MR$$

Under the assumption of proportional damping, reduced damping matrix can be defined as

$$\bar{C} = \alpha \bar{K} + \beta \bar{M}$$
What types of simulation can be performed using a reduced-order model with localized nonlinearities?
Standard Nonlinear Simulations: Nonlinear Time Integration
Time Integration Is a Simulation Standard

Simulate the time response of a nonlinear system by solving its governing equations of motion using numerical algorithms:

\[ M\ddot{\mathbf{x}}(t) + C\dot{\mathbf{x}}(t) + K\mathbf{x}(t) + f_{nl}(\mathbf{x}, \dot{\mathbf{x}}) = f_{ext}(t) \]

\( q_1 \) (m)  \hspace{2cm} \( q_n \) (m)
Time Integration Is a Simulation Standard

Given

\[
\begin{align*}
\text{EOMs: } & \quad M\ddot{x}(t) + C\dot{x}(t) + Kx(t) + f_{nl}(x, \dot{x}) \\
& = f_{ext}(t) \\
\text{Initial cond.: } & \quad x_0 = x(t_0), \dot{x}_0 = \dot{x}(t_0)
\end{align*}
\]

Compute

\[
x_{n+1} = x(t_{n+1})
\]

Such that

\[
M\ddot{x}_{n+1} + C\dot{x}_{n+1} + Kx_{n+1} + f_{nl,n+1} = f_{ext,n+1}
\]
Newmark’s Iterative Scheme for Nonlinear Systems

\[ M, f, f_{ext}, S \quad x_0, \dot{x}_0 \]

Compute \( \ddot{x}_0 \)
\[ \ddot{x}_0 = M^{-1} \left( f_{ext,0} - f(\dot{x}_0, x_0) \right) \]

Time incrementation
\[ t_{n+1} = t_n + h \]

Prediction
\[ \dot{x}_{n+1} = \dot{x}_n + \left( 1 - \gamma \right) h \ddot{x}_n \]
\[ x_{n+1} = x_n + h \dot{x}_n + \left( 0.5 - \beta \right) h^2 \ddot{x}_n \]
\[ \dddot{x}_{n+1} = 0 \]

Residual vector evaluation
\[ r_{n+1} = M \dddot{x}_{n+1} + f_{n+1} - f_{ext,n+1} \]

Convergence?
\[ \left\| r_{n+1} \right\| < \varepsilon \left\| f_{n+1} \right\| \]

Yes

Calculation of the correction
\[ S(x_{n+1}) \Delta x = -r_{n+1} \]

Correction
\[ x_{n+1} = x_{n+1} + \Delta x \]
\[ \dot{x}_{n+1} = \dot{x}_{n+1} + \frac{\gamma}{\beta h} \Delta x \]
\[ \dddot{x}_{n+1} = \dddot{x}_{n+1} + \frac{1}{\beta h^2} \Delta x \]

No

(See Géradin and Rixen’s book for more details)
Time Step $h$, $\beta$ and $\gamma$ Are Key Parameters

Compute $\ddot{x}_0$

\[
\ddot{x}_0 = M^{-1} \left( f_{ext,0} - f(\dot{x}_0, x_0) \right)
\]

Time incrementation

\[
t_{n+1} = t_n + h
\]

Prediction

\[
\dot{x}_{n+1} = \dot{x}_n + (1 - \gamma) h \ddot{x}_n
\]

\[
x_{n+1} = x_n + h \dot{x}_n + (0.5 - \beta) h^2 \ddot{x}_n
\]

\[
\ddot{x}_{n+1} = 0
\]

Residual vector evaluation

\[
r_{n+1} = M \ddot{x}_{n+1} + f_{n+1} - f_{ext,n+1}
\]

Convergence?

\[
|r_{n+1}| < \varepsilon |f_{n+1}|
\]

Yes

No

Calculation of the correction

\[
S(x_{n+1}) \Delta x = -r_{n+1}
\]

Correction

\[
x_{n+1} = x_{n+1} + \Delta x
\]

\[
\dot{x}_{n+1} = \dot{x}_{n+1} + \frac{\gamma}{\beta h} \Delta x
\]

\[
\ddot{x}_{n+1} = \ddot{x}_{n+1} + \frac{1}{\beta h^2} \Delta x
\]

(See Gérardin and Rixen's book for more details)
### Stability of Newmark’s Scheme for Linear Systems

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>$\gamma$</th>
<th>$\beta$</th>
<th>$\omega h$</th>
<th>$\rho - 1$</th>
<th>$\frac{\Delta T}{T}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Purely explicit</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{\omega^2 h^2}{4}$</td>
<td>–</td>
</tr>
<tr>
<td>Central difference</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>$-\frac{\omega^2 h^2}{24}$</td>
</tr>
<tr>
<td>Fox &amp; Goodwin</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{12}$</td>
<td>2.45</td>
<td>0</td>
<td>$O(h^3)$</td>
</tr>
<tr>
<td>Linear acceleration</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{6}$</td>
<td>3.46</td>
<td>0</td>
<td>$\frac{\omega^2 h^2}{24}$</td>
</tr>
<tr>
<td>Average constant acceleration</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{4}$</td>
<td>$\infty$</td>
<td>0</td>
<td>$\frac{\omega^2 h^2}{12}$</td>
</tr>
<tr>
<td>Average constant acceleration (modified)</td>
<td>$\frac{1}{2} + \alpha$</td>
<td>$\frac{(1 + \alpha)^2}{4}$</td>
<td>$\infty$</td>
<td>$-\alpha \frac{\omega^2 h^2}{2}$</td>
<td>$\frac{\omega^2 h^2}{12}$</td>
</tr>
</tbody>
</table>

Implemented in NI2D
Why Newmark and Not Runge-Kutta (ode45)?

- **Fixed time step**
  Convenient for FE models with high eigenfrequencies.

- **Control on stability and accuracy**
  Demonstrated for linear systems with $\beta, \gamma$ and time step $h$.

- **Possibility to add numerical damping**
  Use of the $\alpha$ parameter, or HHT scheme (more accurate).

Newmark’s scheme is **implemented in most commercial FE software**.
Influence of the Time Step / Sampling Frequency

Rule of thumb: For a periodicity error of 1%, taking higher harmonics into account, consider at least

\[ f_s > 200f \]

Sampling frequency = 1/time step

Frequency of interest in the signal
Influence of the Time Step / Sampling Frequency

Mode 1: 31.2801 Hz / 1.2%
Advanced Nonlinear Simulations:
Nonlinear Frequency Responses and Modes
Limitations of Time Integration

Time simulations provide useful information about structural dynamics but they can be time consuming.

EXCITATION: sine, swept-sine, etc.  →  NL SYSTEM  →  Disp.

Time
Limitations of Time Integration

Time simulations may reveal nonlinear phenomena but cannot explain their origin.

SINE EXCITATION → NL SYSTEM → Disp. → BISTABILITY

- Initial conditions A
- Initial conditions B

Time
Limitations of Time Integration

Time simulations may reveal nonlinear phenomena but cannot explain their origin.
Limitations of Time Integration

Time simulations may reveal nonlinear phenomena but cannot explain their origin.
Limitations of Time Integration

Time simulations may reveal nonlinear phenomena but cannot explain their origin.

SWEPT-SINE EXCITATION → NL SYSTEM → Disp.

NONLINEAR RESONANCE

Time / sweep frequency
Nonlinear normal modes (NNMs) – See Next Lecture

NNMs are obtained by computing branches of periodic solutions of the underlying undamped and unforced model:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) + \mathbf{f}_{nl}(\mathbf{x}) = 0$$

NNMs are useful because:

1. They describe the deformations at resonance of the structure.
2. They describe how modal parameters evolve with motion amplitude.
Nonlinear normal modes (NNMs) – See Next Lecture

NNMs also help to uncover complex phenomena such as modal interactions / internal resonances.
Nonlinear Frequency Response Curves (NFRCs)

NFRCs are obtained by computing branches of periodic solutions of the damped model when submitted to a harmonic excitation:

\[
M\ddot{x}(t) + C\dot{x}(t) + Kx(t) + f_{nl}(x, \dot{x}) = f_{ext}(\omega, t)
\]
Nonlinear Frequency Response Curves (NFRCs)

NFRCs are useful because they describe the evolution of amplitude of the steady-state responses of the structure, i.e., after the transients.
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Nonlinear Frequency Response Curves (NFRCs)

The representative variable is usually chosen as the vibration amplitude of one of the DOFs, and is represented with respect to the frequency $\omega$. 
NFRCs can be seen as the nonlinear extension of linear frequency response curves (LFRCs), or FRFs.
Nonlinear Frequency Response Curves (NFRCs)

NFRCs can be seen as the nonlinear extension of linear frequency response curves (LFRCs), or FRFs.

… But

<table>
<thead>
<tr>
<th></th>
<th>LFRCs</th>
<th>NFRCs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Superposition</td>
<td>Yes (✓)</td>
<td>No (✗)</td>
</tr>
<tr>
<td>Uniqueness</td>
<td>Yes (✓)</td>
<td>No (✗)</td>
</tr>
<tr>
<td>Frequency</td>
<td>Energy independent</td>
<td>Energy dependent</td>
</tr>
<tr>
<td>Stability</td>
<td>Always stable</td>
<td>Stable or unstable</td>
</tr>
</tbody>
</table>
Nonlinear Frequency Response Curves (NFRCs)

Locus of LNMs for various energies

Locus of NNMs for various energies

Linear: resonances occur in neighborhoods of LNMs.

Nonlinear: resonances occur in neighborhoods of NNMs.
Nonlinear Frequency Response Curves (NFRCs)

NNMs also help to uncover complex phenomena such as amplitude jumps.
Nonlinear Frequency Response Curves (NFRCs)

NNMs also help to uncover complex phenomena such as quasiperiodic regime.

Frequency $\omega$ vs. Amplitude
Nonlinear Frequency Response Curves (NFRCs)

NNMs also help to uncover complex phenomena such as bistability.
Towards the Continuation of NNMs and NFRCs

1. Computation of Periodic Solutions

Displacement

Time (s)

TOPIC OF THIS LECTURE
Towards the Continuation of NNMs and NFRCs

TOPIC OF THIS LECTURE

2. Continuation procedure

Frequency $\omega$

Amplitude
Towards the Continuation of NNMs and NFRCs

3. Stability analysis

SEE NEXT LECTURES
Towards the Continuation of NNMs and NFRCs

4. Bifurcation analysis

SEE NEXT LECTURES
Computation of Periodic Solutions
Mathematical Representation of a Periodic Solution

There are at least 3 approaches to describe a periodic solution.
There are at least 3 approaches to describe a periodic solution.

Initial conditions $[x_0 \quad \dot{x}_0]^T$ and the period $T$. 

Time integration over $T$
There are at least 3 approaches to describe a periodic solution.

Piecewise polynomial functions and the period $T$. 

\[
\text{Diagram showing periodic functions and their sum.}
\]
Mathematical Representation of a Periodic Solution

There are at least 3 approaches to describe a periodic solution.

Fourier series and the period $T$. 
Computing the periodic solution of a nonlinear system means searching for a solution $\mathbf{x}$ that satisfies

\[
M\ddot{\mathbf{x}}(t) + C\dot{\mathbf{x}}(t) + K\mathbf{x}(t) + f_{nl}(\mathbf{x}, \dot{\mathbf{x}}) = f_{ext}(\omega, t)
\]

with a periodicity condition

\[
\mathbf{x}(t + T) = \mathbf{x}(t)
\]

This represents a boundary-value problem (BVP).
Computation of a Periodic Solution

There are three approaches to solve this BVP.

- Based on initial conditions \([x_0, \dot{x}_0]^T\).
  
  **Shooting technique**

- Based on piecewise polynomial functions.
  
  **Orthogonal collocation (not discussed here)**

- Based on Fourier series.
  
  **Harmonic balance method**
Optimization of the initial state of a system $[x_0 \hspace{0.5cm} \dot{x}_0]^T$ to obtain a periodic solution after time integration over a period $T$. 

« Angle » = $x_0$
« Power » = $\dot{x}_0$
Shooting Technique

The equations of motion are first recast in state-space form:

\[ \dot{y}(t) = Ly(t) - g_{nl}(y) + g_{ext}(\omega, t) \]

with

\[ y = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \quad L = \begin{bmatrix} 0 & I_n \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \]

\[ g_{nl} = \begin{bmatrix} 0 \\ M^{-1}f_{nl}(x, \dot{x}) \end{bmatrix} \quad g_{ext} = \begin{bmatrix} 0 \\ M^{-1}f_{ext}(\omega, t) \end{bmatrix} \]

The state of this system at time \( t \) and given initial condition \( y_0 \) is denoted as \( y = y(t, y_0) \).
Shooting Technique

An initial state $y_{0,p}$ leads to a periodic solution if

$$h_{shooting} \equiv y(T, y_{0,p}) - y_{0,p} = 0$$

where $y(T, y_{0,p})$ is computed from time integration of the EOMs.

The shooting technique consists in computing $y_{0,p}$ that satisfies $h_{shooting} = 0$ for $T$ known a priori (NFRC) or not (NNM).

In the case of a harmonic excitation with frequency $\omega$, $T$ can be approximated as $T = 2\pi/\omega$. 
Shooting Technique Scheme (for NFRCs)

Initial guess for initial state $y_{0,p}^0$

Initial state at iteration $i$ $y_{0,p}^i$

Evaluation of $h_{\text{shooting}} = y(T, y_{0,p}^i) - y_{0,p}^i$

$T$

$< \epsilon$? NO: Correction (e.g., Newton-Raphson)

YES

Time integration

END
Periodic Solutions of Large Structures

The shooting technique is efficient and accurate for small nonlinear systems (1-30 DOFs).

For larger systems however, demand in CPU time (multiple time integrations) and memory space can be problematic.

For such cases, one usually relies on the harmonic balance method.
Fourier Series Approximation

\[ M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = f(x, \dot{x}, \omega, t) \]

where \( f(x, \dot{x}, \omega, t) \) gathers both nonlinear and external forces.

The harmonic balance (HB) method consists in approximating the displacements \( x(t) \) with Fourier series truncated to the order \( N_H \).
Fourier Series Approximation

\[
\mathbf{M}\dddot{x}(t) + \mathbf{C}\ddot{x}(t) + \mathbf{K}x(t) = f(x, \dot{x}, \omega, t)
\]

\[
x(t) = \sum_{k=1}^{N_H} \left( \mathbf{s}_k^x \sin(k\omega t) + \mathbf{c}_k^x \cos(k\omega t) \right) + \frac{\mathbf{c}_0^x}{\sqrt{2}}
\]

The new unknowns are the Fourier coefficients \( z \), with

\[
z = \begin{bmatrix}
\mathbf{c}_0^x & \mathbf{s}_1^x & \mathbf{c}_1^x & \cdots & \mathbf{s}_{N_H}^x & \mathbf{c}_{N_H}^x
\end{bmatrix}^T
\]

\( n_Z = n(2N_H + 1) \) unknowns
Fourier Series Approximation

\[ M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = f(x, \dot{x}, \omega, t) \]

\[ f(x, \dot{x}, \omega, t) = \frac{c_0^f}{\sqrt{2}} + \sum_{k=1}^{NH} (s_k^f \sin(k\omega t) + c_k^f \cos(k\omega t)) \]

The Fourier coefficients of \( f \) are denoted by \( b \), with

\[ b = \begin{bmatrix} c_0^f & s_1^f & c_1^f & ... & s_{NH}^f & c_{NH}^f \end{bmatrix}^T \]

\[ = b(z) \text{ since } f \text{ depends on } x. \]
Fourier Series Approximation

Displacements and forces can be recast into a more compact form

\[ x(t) = (Q(t) \otimes I_n)z \]
\[ f(t) = (Q(t) \otimes I_n)b \]

where \( \otimes \) denotes the Kronecker tensor product, \( I_n \) represents the identity matrix and where \( Q(t) \) is the orthogonal trigonometric basis:

\[ Q(t) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \sin(\omega t) & \cos(\omega t) & \ldots & \sin(N_H \omega t) & \cos(N_H \omega t) \end{bmatrix} \]
Fourier Series Approximation

With this formulation, velocities can also be defined using Fourier series:

$$\dot{x}(t) = (\dot{Q}(t) \otimes I_n)z = ((Q(t)\nabla) \otimes I_n)z$$

where

$$\nabla = \begin{bmatrix} 0 & \cdots & \nabla_k & \cdots & \nabla_{NH} \end{bmatrix}$$

$$\nabla_k = \begin{bmatrix} 0 & -k\omega \\ k\omega & 0 \end{bmatrix}$$
Fourier Series Approximation

With this formulation, accelerations can also be defined using Fourier series:

$$\ddot{x}(t) = (\ddot{Q}(t) \otimes I_n)z = \left((Q(t)\nabla^2) \otimes I_n\right)z$$

where

$$\nabla^2 = \nabla \nabla = \begin{bmatrix} 0 & & \nabla^2_k & & \nabla^2_{NH} \\ & \ddots & & \ddots & \\ & & 0 & & \nabla^2_{NH} \end{bmatrix}$$

$$\nabla^2_k = \begin{bmatrix} -(k\omega)^2 & 0 \\ 0 & -(k\omega)^2 \end{bmatrix}$$
Equations of Motion in the Frequency Domain

\[ \mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{Kx}(t) = \mathbf{f}(\mathbf{x}, \dot{\mathbf{x}}, \omega, t) \]

Fourier series approximation

\[ \mathbf{M}\left(\nabla^2 (\mathbf{Q}(t) \mathbf{z}) \otimes \mathbf{I}_n\right) + \mathbf{C}\left(\nabla (\mathbf{Q}(t) \mathbf{z}) \otimes \mathbf{I}_n\right) + \mathbf{K}(\mathbf{Q}(t) \mathbf{z}) \otimes \mathbf{I}_n = (\mathbf{Q}(t) \mathbf{z}) \otimes \mathbf{I}_n \mathbf{b} \]

This expression can be further simplified using:
- Galerkin procedure (to remove time dependency).
- Kronecker product properties.
Equations of Motion in the Frequency Domain

In a more compact form:

\[ h(z, \omega) \equiv A(\omega)z - b(z) = 0 \]

where \( A \) describes the linear dynamics

\[
A = \nabla^2 \otimes M + \nabla \otimes C + I_{2N_H+1} \otimes K
\]

\[
= \begin{bmatrix}
K \\
K - \omega^2 M & -\omega C \\
\omega C & K - \omega^2 M \\
\vdots \\
K - (N_H \omega)^2 M & -N_H \omega C \\
N_H \omega C & K - (N_H \omega)^2 M
\end{bmatrix}
\]
Equations of Motion in the Frequency Domain

In a more compact form:

$$\mathbf{h}(\mathbf{z}, \omega) \equiv \mathbf{A}(\omega)\mathbf{z} - \mathbf{b}(\mathbf{z}) = 0$$

where \( \mathbf{b} \) is the Fourier coefficients vector of nonlinear and external forces

$$f(x, \dot{x}, \omega, t) = f_{\text{ext}}(\omega, t) - f_{\text{nl}}(x, \dot{x})$$

$$= \frac{c_0^f}{\sqrt{2}} + \sum_{k=1}^{N_H} \left( s_k^f \sin(k\omega t) + c_k^f \cos(k\omega t) \right)$$

$$\mathbf{b} = \begin{bmatrix} c_0^f & s_1^f & c_1^f & \ldots & s_{N_H}^f & c_{N_H}^f \end{bmatrix}^T$$
Equations of Motion in the Frequency Domain

In a more compact form:

\[ \mathbf{h}(\mathbf{z}, \omega) \equiv \mathbf{A}(\omega)\mathbf{z} - \mathbf{b}(\mathbf{z}) = 0 \]

If for a given forcing frequency \( \omega \), one finds a vector \( \mathbf{z}^* \) such that

\[ \mathbf{h}(\mathbf{z}^*, \omega) = 0 \]

Then the time series \( \mathbf{x}^*(t) \) reconstructed from \( \mathbf{z}^* \)

\[ \begin{align*} &\text{verify the EOMs of the system.} \\ &\text{are periodic.} \end{align*} \]
Equations of Motion in the Frequency Domain

\[ h(z, \omega) \equiv A(\omega)z - b(z) = 0 \]

- \( h(z, \omega) = 0 \) is a nonlinear algebraic equation (easier to solve than time integrations as in shooting technique).

- \( z \) are the Fourier coefficients of the displacements and the new unknowns of the problem (usually less than for orthogonal collocation).

- For NFRCs, \( \omega \) is the forcing frequency and is a system parameter.
Harmonic Balance Parameters

Number of harmonics $N_H$ retained in the Fourier series.
Harmonic Balance Parameters

Number of time samples $N$ in the Fourier transform.

- **Number of harmonics**: 5
- **Number of points**: 512
- **Amplitude of 1st guess**: 0.001 m
- **Maximum number of iterations**: 15
- **Relative precision**: $1e-06$
- **Scaling factor for displacements**: $5e-06$
- **Scaling factor for time**: 3000
Harmonic Balance Parameters

Stability parameters (see next lectures)
Amplitude of the sine series used as initial guess for all DOFs.
Harmonic Balance Parameters

The Newton-Raphson procedure fails if this number of iterations is exceeded.
Harmonic Balance Parameters

The Newton-Raphson procedure stops if the relative error is smaller than this precision.
Because the frequency (e.g., 30Hz = 188rad/s) and the amplitude (e.g., 0.001m) have different orders of magnitude, time and displacements have to be rescaled to avoid ill conditioning.
Harmonic Balance Method: In Summary

<table>
<thead>
<tr>
<th>PROS</th>
<th>CONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Efficient</td>
<td>Less accurate</td>
</tr>
<tr>
<td>Harmonic coefficients available</td>
<td>Many harmonics are sometimes required</td>
</tr>
<tr>
<td>Filtering</td>
<td></td>
</tr>
</tbody>
</table>

Adaptations of the method improve its performance (alternating time-frequency method, chain rule, ...) – not discussed here.
Computation of Periodic Solutions: In Summary

Periodic solutions of nonlinear structures can be computed with time-domain (shooting, orthogonal collocation) or frequency-domain method (harmonic balance).

The differences between these methods lie in their accuracy and execution time.

Without adaptation, however, the harmonic balance:

- Fails at computing periodic responses in severe nonlinear regimes (need for continuation procedure).
- Does not indicate if the solutions can be observed experimentally or not (need for a stability analysis).
Computation of Branches of Periodic Solutions
Computation of Branches of Periodic Solutions

Numerical methods to go from single periodic solutions...

... to a branch of periodic solutions
Mathematical Definition of a Branch of Periodic Solutions

Let us consider a function $F: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$. A branch is a set of solutions $F(x, \lambda) = 0$, where $x$ are the state variables and $\lambda$ is a system parameter.

The branch can be represented in a 2D plane through the evolution of a representative variable $y = y(x)$ w.r.t. $\lambda$. 
In this course, the branch is composed by solutions of the harmonic balance equation for a nonlinear system:

\[ h(z, \omega): \mathbb{R}^{n_z+1} \rightarrow \mathbb{R}^{n_z} \]

- **Nonlinear Frequency Response Curves**
  - Forced and damped system

- **Nonlinear Normal Modes**
  - Unforced and undamped system
Sequential Continuation – A Straightforward Approach

Increase the period and use the previously computed periodic solution as an initial guess for the next computation.

\[
\Delta \omega \quad \omega
\]

- Previous solution as prediction
- Optimization with fixed frequency
- Solutions of the branch
If HB method is already implemented, sequential continuation is programmed in a few lines.

Initial solution
\[ z_i = z_0 \]

Next frequency
\[ \omega_i = \omega_{i-1} + \Delta \omega \]

New iteration
\[ z_i = z_{i-1} \]

Convergence
\[ h(z_i, \omega_i) = 0? \]

Correction
(Newton-Raphson, fminunc, etc.)

If \textbf{Yes}, proceed with new iteration.

If \textbf{No}, apply correction and check convergence again.
Sequential Continuation Fails at Turning Points

Amplitude

Frequency $\omega$

$0.1 \sin(\omega t)$

$\Delta \omega < 0$

$\Delta \omega > 0$
A New Continuation Scheme

In order to pass through turning points, both the state $z$ and the parameter $\omega$ should vary. This is done through a 2-step procedure:
In order to pass through turning points, both the state $z$ and the parameter $\omega$ should vary. This is done through a 2-step procedure:
Different predictors can be considered:

\[ \mathbf{X}^i_{\text{pred}} = \mathbf{X}^{i-1} + s^i \mathbf{t}^i \]

where \( \mathbf{X} = [\mathbf{z} \quad \omega]^T \) denotes the unknown vector.

**Secant predictor**

\[ \mathbf{t}^i = \frac{\mathbf{X}^{i-1} - \mathbf{X}^{i-2}}{\|\mathbf{X}^{i-1} - \mathbf{X}^{i-2}\|} \]

Diagram showing the secant predictor with arrows indicating the unit vector and stepsizes.
Different predictors can be considered:

\[ X^i_{\text{pred}} = X^{i-1} + s^i t^i \]

- **Tangent predictor**

\[
\begin{bmatrix}
  h_z & h_\omega \\
  t_{i-1}^T
\end{bmatrix} t^i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

More accurate but requires the computation of the Jacobian matrices.
We are looking for a solution of $h(z, \omega) = 0$, with

$$h(z, \omega): \mathbb{R}^{nz+1} \rightarrow \mathbb{R}^{nz}$$

Two possibilities:

- Fix the parameter $\omega$ and only optimize $z$.
  
  Cf. sequential continuation

- Add another equation to the system.
  
  Pseudo-arclength and Moore-Penrose schemes
Pseudo-arclength Corrector Step

With the **pseudo-arclength** scheme, a solution is sought in the perpendicular direction w.r.t. the prediction.
Pseudo-arclength Corrector Step

With the **pseudo-arclength** scheme, a solution is sought in the perpendicular direction w.r.t. the prediction.

\[
\begin{align*}
\mathbf{z}^i_{(j+1)} &= \mathbf{z}^i_{(j)} + \Delta \mathbf{z}(j) \\
\omega^i_{(j+1)} &= \omega^i_{(j)} + \Delta \omega(j)
\end{align*}
\]

\(i = \) continuation iteration \( (j) = \) corrector iteration

with

\[
\begin{bmatrix}
\mathbf{h}_z(\mathbf{z}^i_{(j)}, \omega_{(j)}) & \mathbf{h}_\omega(\mathbf{z}^i_{(j)}, \omega_{(j)}) \\
\mathbf{t}_z^i & \mathbf{t}_\omega^i
\end{bmatrix}
\begin{bmatrix}
\Delta \mathbf{z}(j) \\
\Delta \omega(j)
\end{bmatrix}
= 
\begin{bmatrix}
-\mathbf{h}(\mathbf{z}^i_{(j)}, \omega_{(j)}) \\
0
\end{bmatrix}
\]

⇒ **Taylor series expansion**

⇒ **Orthogonality condition**
Other Correctors

Other corrector definitions can also be used.

With the Moore-Penrose scheme for instance, the correction direction is updated at each corrector step.
Stability of Periodic Solutions Varies Along the Branch

Amplitude

Frequency $\omega$

Stable

Unstable

$0.1 \sin(\omega t)$

1

0.02

1

1
Periodic Solutions Can be Stable or Unstable

Floquet exponents (see next lectures)
Periodic Solutions Can be Stable or Unstable

Amplitude vs. Frequency

Floquet exponents
(see next lectures)
Periodic Solutions Can be Stable or Unstable

Floquet exponents (see next lectures)
Periodic Solutions Can be Stable or Unstable

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Floquet exponents
(see next lectures)
Periodic Solutions Can be Stable or Unstable

More generally, these changes are governed by bifurcations (see next lectures).

Stability changes occur in the neighborhood of turning points.
Influence of the Step size

Stepsize is a key parameter for the continuation procedure.

\[ X_{pred}^i = X^{i-1} + s^i t^i \]

- Unit vector
- Stepsize

Amp.

Frequency $\omega$
Small Stepsize

- Small number of corrections
- Good resolution for the branch
- Slow continuation procedure

Amplitude vs. Frequency $\omega$
Large Stepsize

- Fast continuation procedure
- Large number of corrections
- Poor resolution for the branch

\[ \text{Amplitude} \]

\[ \text{Frequency } \omega \]
Stepsize Strategy

**Fixed stepsize**

\[ s^i = \text{constant} \]

**Adaptative stepsize**

\[ s^i = \frac{M^*}{M} s^{i-1} \]

where \( M \) is the iteration number for the current correction, and \( M^* \) is the optimal iteration number.
Influence of Harmonic Balance Parameters

With the harmonic balance method, the displacements are approximated with Fourier series.

\[ x(t) = c_0^x + \sum_{k=1}^{N_H} (s_k^x \sin(k\omega t) + c_k^x \cos(k\omega t)) \]

Fourier coefficients \( z \) are computed with the discrete Fourier transform:

\[ z = \Gamma^+([N])\tilde{x} \]

Number of time samples (power of 2)
Influence of the Number of Harmonics $N_H$

$N_H$ has a direct influence on the accuracy of the harmonic balance solution, and hence on the accuracy of the branch.

Always make sure to check for convergence!
Influence of the Number of Time Samples $N$

$N$ has a direct influence on the discrete Fourier transform, and the accuracy of the alternating frequency/time-domain method.
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$N$ has a direct influence on the discrete Fourier transform, and the accuracy of the alternating frequency/time-domain method.

Increasing $N$

Always make sure to check for convergence!
Continuation: In Summary

Sequential continuation can be easily implemented to represent the evolution of the periodic solutions w.r.t. to the frequency \( \omega \) but it fails at turning points.

Continuation schemes based on predictor/corrector steps give the evolution of the periodic solutions in both stable and unstable regions.

HB and continuation parameters have to be carefully selected to ensure accuracy and good resolution of the branches.