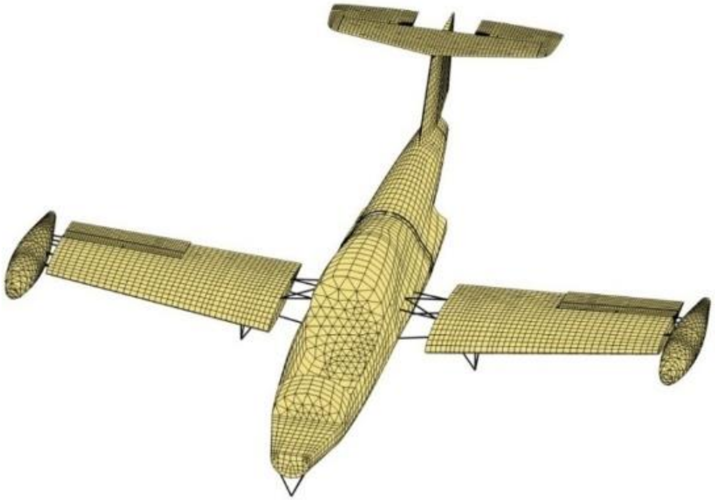
Nonlinear Vibrations of Aerospace Structures

University of Liège, Belgium

L05 Nonlinear Simulations

Modeling and Reduction
Time Integration
Periodic Solution
Continuation



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2017-2021:

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Topics of interest:

- Nonlinear modal analysis
- Vibration mitigation

Why Do We Need High-Fidelity Models?

For better decision-making capability!

AIRBUS A350XWB
Govers et al., ISMA 2014.

Using models, we can access non measurable information (e.g., stress).

Particular operational conditions (e.g., explosions, earthquakes) that are difficult/impossible/dangerous to reproduce experimentally can be simulated.

Why Do We Need High-Fidelity Models?

But also to:

- Reduce dependence on testing (cost and time issues)
- Test design (e.g., sensor and actuator placement)
- Perform virtual prototyping:

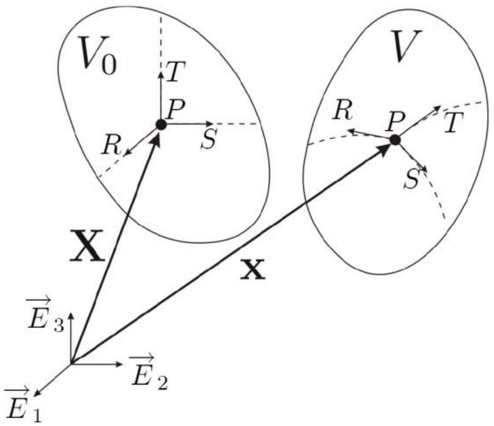
  A model can predict the behavior of a structure before its construction.

  The parameters of a model can easily be modified to improve the design (optimization).

Different Approaches to Model Nonlinear Structures

1. Large displacements and rotations

Displacement: $\mathbf{u} = \mathbf{x} - \mathbf{X}$

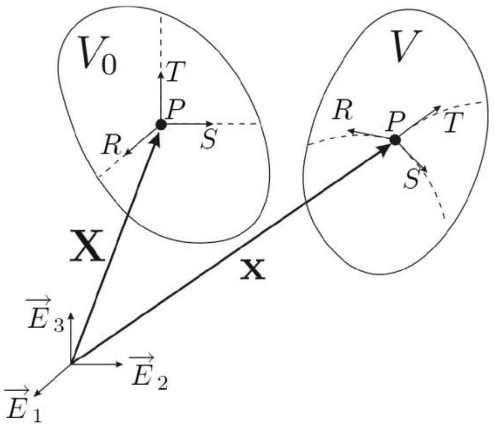


Cauchy strain tensor: $\epsilon_{ij}^{C} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)$

✓ Small displacements and rotations.

Different Approaches to Model Nonlinear Structures

1. Large displacements and rotations

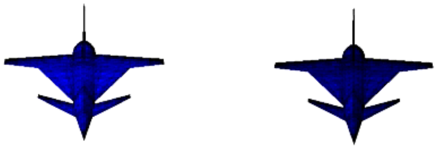


Displacement: $\mathbf{u} = \mathbf{x} - \mathbf{X}$

Cauchy strain tensor: $\epsilon_{ij}^{C} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)$

✓ Small displacements and rotations.

✗ Not invariant under rigid-body motion. Cauchy strains cannot be used if rotation amplitudes are finite.



Different Approaches to Model Nonlinear Structures

1. Large displacements and rotations

Displacement: $\mathbf{u} = \mathbf{x} - \mathbf{X}$

Green strain tensor: $$\epsilon_{ij}^{G} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \sum_{k=1}^{3} \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right)$$

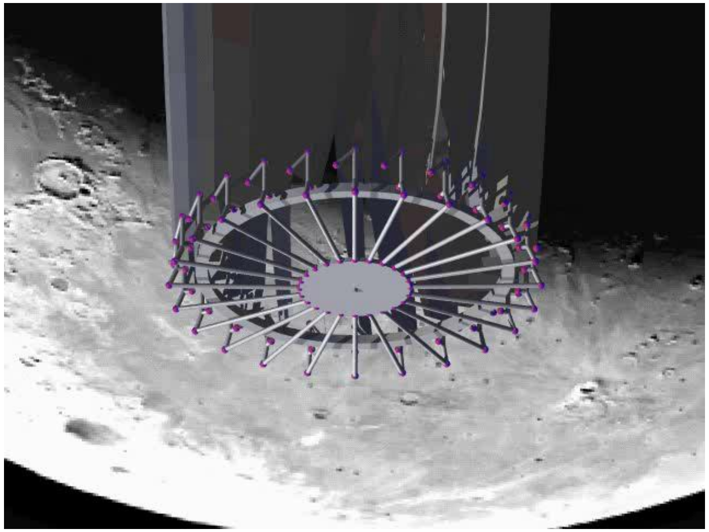
✓ Large displacements and rotations.

✓ Nonlinear measure of deformation. Geometrical nonlinearities can be considered in the elastic force model.

Different Approaches to Model Nonlinear Structures

1. Large displacements and rotations

Landing gear mechanism
Prof. O. Brüls, ULiège

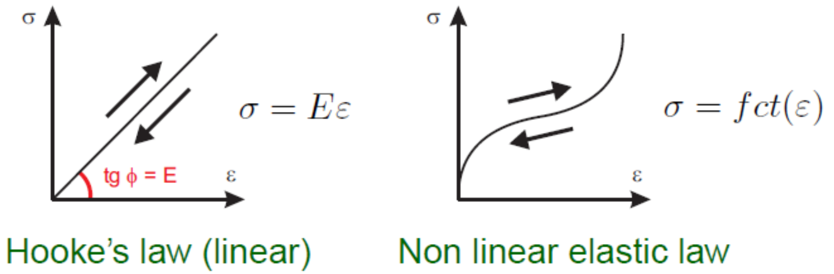


Deployable space structure
Prof. O. Brüls, ULiège

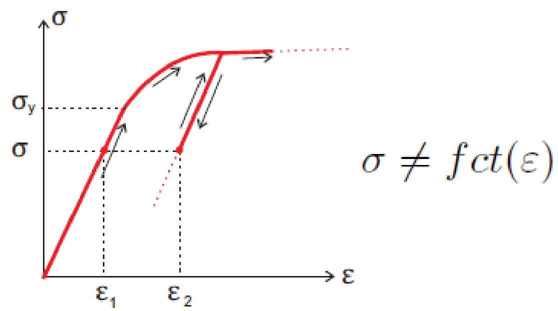
Different Approaches to Model Nonlinear Structures

2. Large deformations

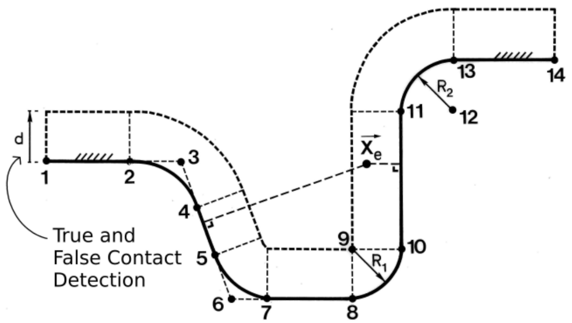
Nonlinear constitutive laws



Plasticity/Irreversibility

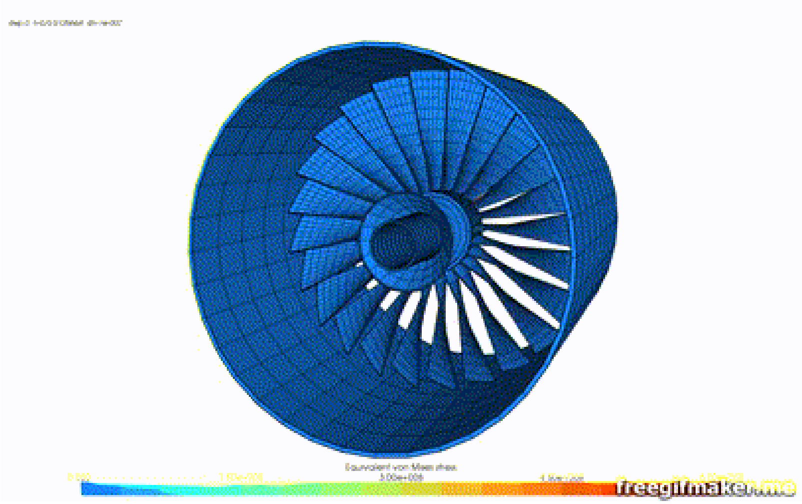


Contacts/Impacts

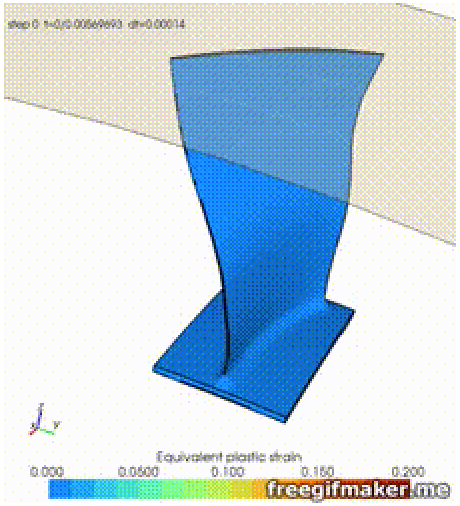


Different Approaches to Model Nonlinear Structures

2. Large deformations



Fan Blade containment test
Prof. J.-P. Ponthot, ULiège



Buckling of blade in LP compressor
Prof. J.-P. Ponthot, ULiège

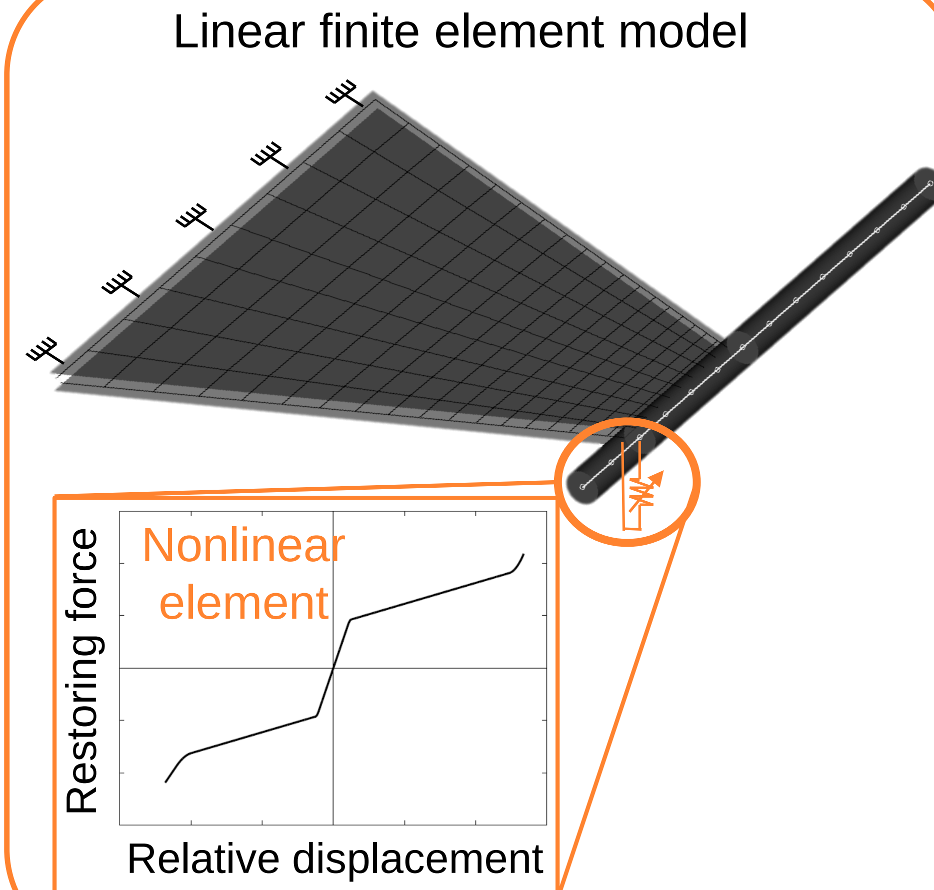
Different Approaches to Model Nonlinear Structures

3. Linear structure with localized nonlinearities

FOCUS OF THIS COURSE

High-fidelity and fast-running modeling of structures with localized nonlinearities

Integration of Data-Driven and Computer-Aided Models



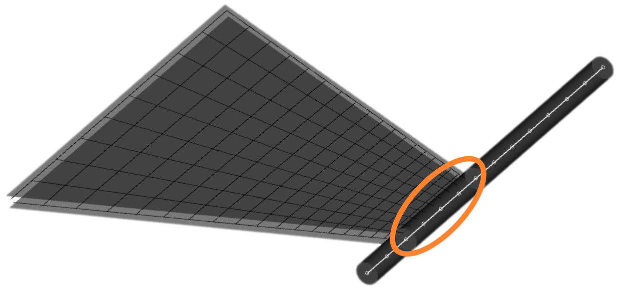
Accurate modeling of localized nonlinearities identified from experimental data (see next lectures).

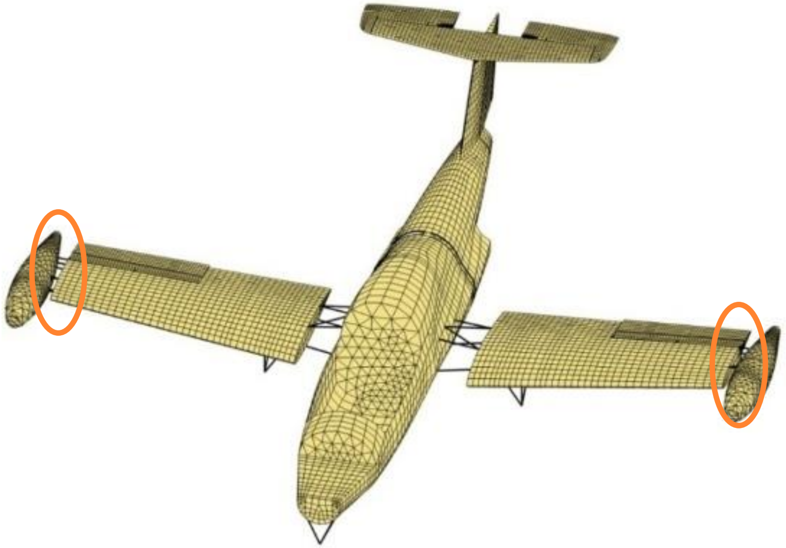
Development of Fast-Running Models

Finite element models may involve thousands (even millions) of degrees of freedom (DOFs).

For structures with localized nonlinearities, only a few DOFs are generally involved in nonlinear connections.

- Model reduction and substructuring can be applied to speed up simulations.





Model Reduction and Substructuring

Reminders from “*Mechanical vibrations: Theory and Applications to Structural Dynamics*” (Géradin and Rixen):

Reduction: In most cases, engineers are interested in a smaller system capturing only lower frequency dynamics. In this case, a genuine reduction is performed, the reduction method being seen as a DOF economizer.

Substructuring: In the context of large projects, the analysis is frequently subdivided into several parts. A separate model is constructed for each part of the system and reduced (*super-element*). The different parts and super-elements are finally combined to simulate the dynamics of the whole system.

Model Reduction and Substructuring

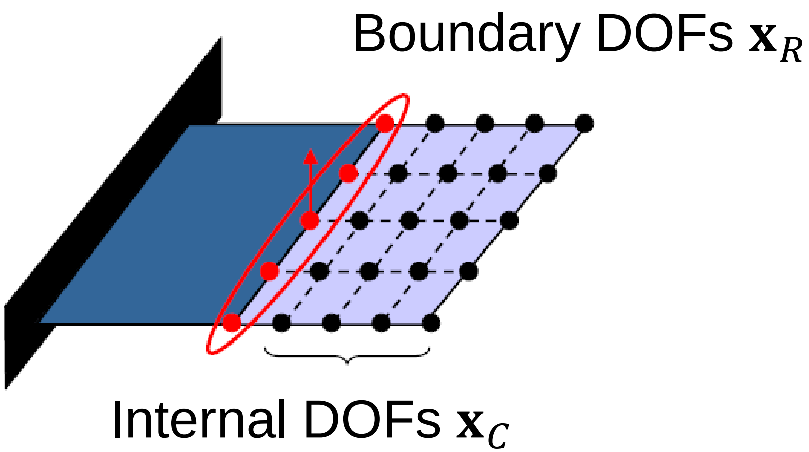
Most methods for reducing the size n of a system consist in partitioning the degrees of freedom into n_R dynamic retained coordinates ($n_R << n$) and n_C condensed coordinates.

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_R \\ \mathbf{x}_C \end{bmatrix} \qquad \mathbf{K} = \begin{bmatrix} \mathbf{K}_{RR} & \mathbf{K}_{RC} \\ \mathbf{K}_{CR} & \mathbf{K}_{CC} \end{bmatrix} \qquad \mathbf{M} = \begin{bmatrix} \mathbf{M}_{RR} & \mathbf{M}_{RC} \\ \mathbf{M}_{CR} & \mathbf{M}_{CC} \end{bmatrix}$$

The dynamical behavior of the structure is usually described by the retained coordinates only.

Craig-Bampton Method

Let us consider a substructure which is connected to the rest of the system by a set of boundary degrees of freedom $\mathbf{x}_R$.

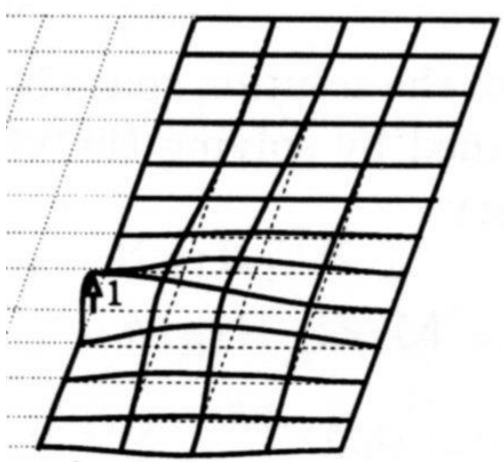


The originality of the method is to consider in the condensation, in addition to the boundary DOFs $\mathbf{x}_R$, the contribution of the internal vibration modes to the reduced model.

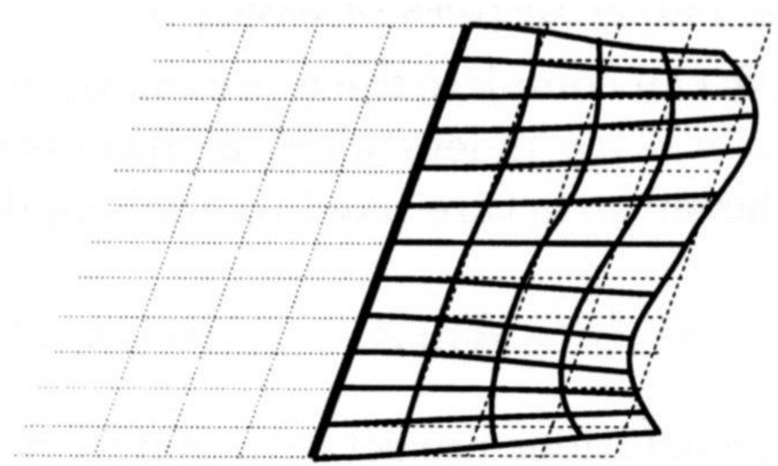
Craig-Bampton Method

The dynamical behavior of a substructure is fully described by:

- the static boundary modes resulting from the static condensation,
- the subsystem eigenmodes in clamped boundary configuration.



Static mode



Vibration mode

Craig-Bampton Method

Accordingly, it means that the following transformation may be applied to the initial degrees of freedom:

$$\mathbf{x} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{K}_{CC}^{-1}\mathbf{K}_{CR} & \mathbf{\Phi}_C \end{bmatrix} \begin{bmatrix} \mathbf{x}_R \\ \mathbf{y}_C \end{bmatrix}$$

n_R boundary DOFs

n_C intensity parameters of the internal modes

where the Guyan's reduction matrix has been complemented by the set of n_C internal vibration modes $\tilde{\mathbf{x}}$ obtained by solving:

$$(\mathbf{K}_{CC} - \tilde{\omega}^2 \mathbf{M}_{CC})\tilde{\mathbf{x}} = \mathbf{0}$$

$$\mathbf{\Phi}_C = [\tilde{\mathbf{x}}_{(1)} \quad \cdots \quad \tilde{\mathbf{x}}_{(n_C)}]$$

Craig-Bampton Method

In practice, only a certain number $m < n_C$ of internal vibration modes are kept:

$$\boldsymbol{\Phi}_C \to \boldsymbol{\Phi}_m = [\tilde{\mathbf{x}}_{(1)} \quad \cdots \quad \tilde{\mathbf{x}}_{(m)}]$$

$$\mathbf{y}_C \to \mathbf{y}_m$$

This subset of internal vibration modes should be selected in order to cover a frequency range that is large enough to approximate the dynamics in play. Convergence of the reduced-order model should be carefully assessed!

Craig-Bampton Method

Final reduction matrix of dimension $n \times (n_R + m)$:

$$\mathbf{R} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{K}_{CC}^{-1}\mathbf{K}_{CR} & \mathbf{\Phi}_m \end{bmatrix}$$

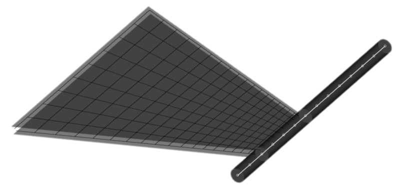
Reduced stiffness and mass matrices:

$$\overline{\mathbf{K}} = \mathbf{R}^T\mathbf{K}\mathbf{R} \qquad \overline{\mathbf{M}} = \mathbf{R}^T\mathbf{M}\mathbf{R}$$

Under the assumption of proportional damping, reduced damping matrix can be defined as

$$\overline{\mathbf{C}} = \alpha\overline{\mathbf{K}} + \beta\overline{\boldsymbol{M}}$$

Design Cycle of a Nonlinear Structure



Computer-aided
modeling (FEM, …)

MEASURE → IDENTIFY → MODEL → UNDERSTD UNCOVER → DESIGN

What types of simulation can be performed using a reduced-order model with localized nonlinearities?

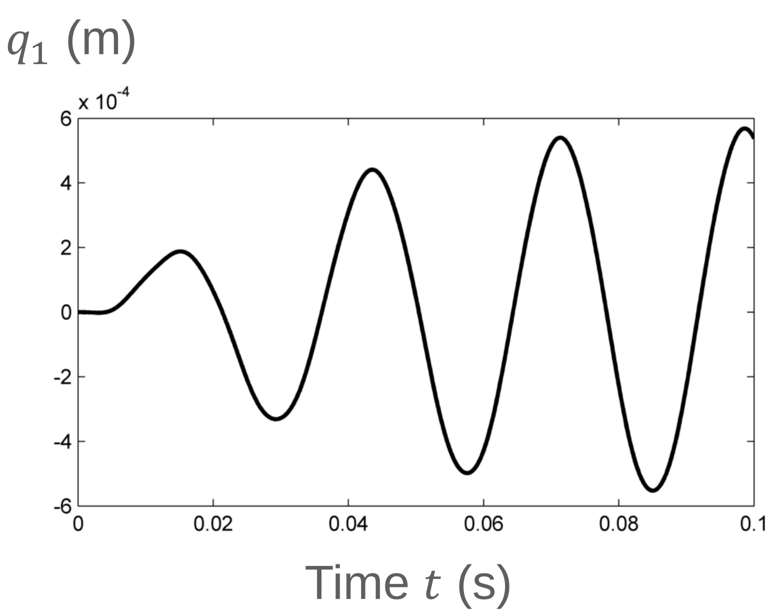
Standard Nonlinear Simulations:

Nonlinear Time Integration

Time Integration Is a Simulation Standard

Simulate the time response of a nonlinear system by solving its governing equations of motion using numerical algorithms

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) + \mathbf{f}_{nl}(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{f}_{ext}(t)$$



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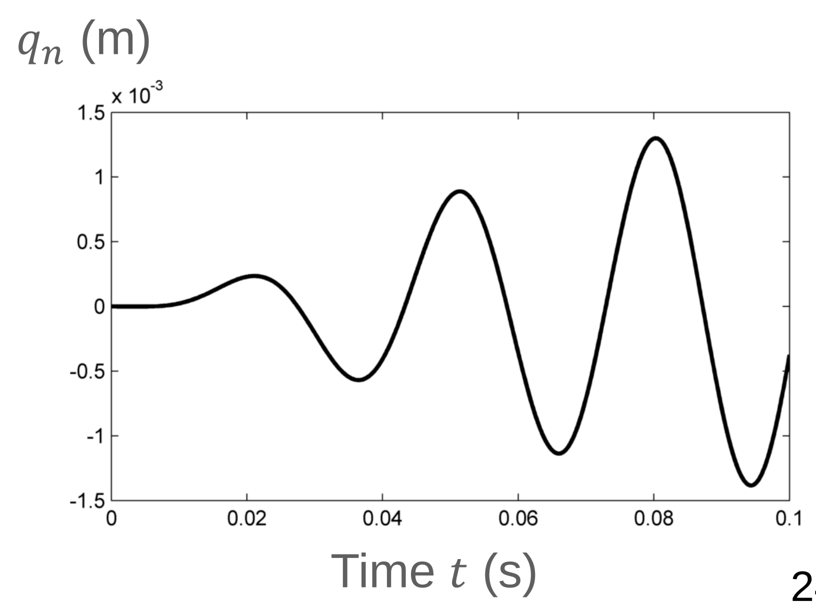


Time Integration Is a Simulation Standard

Given

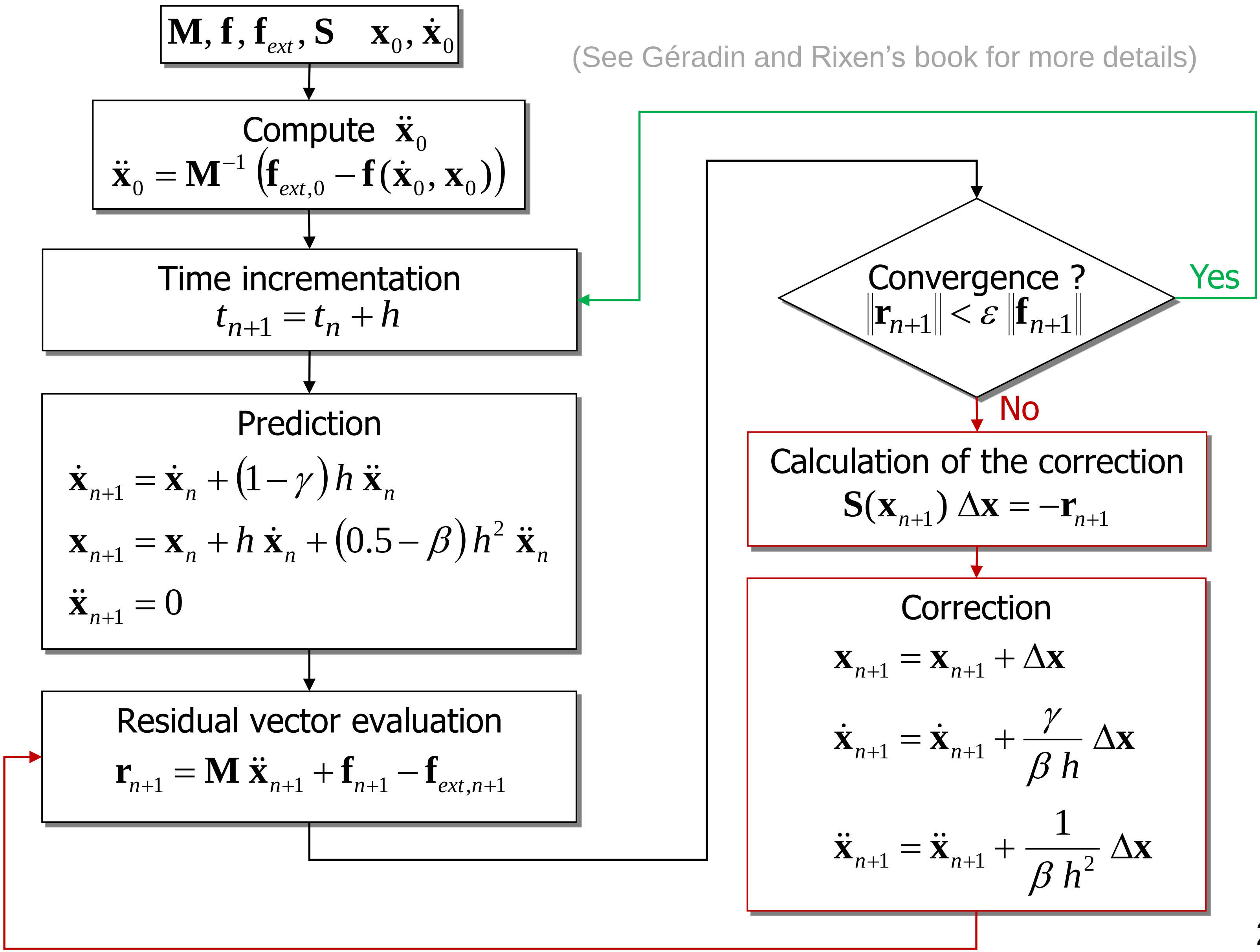
- EOMs: $\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) + \mathbf{f}_{nl}(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{f}_{ext}(t)$
- Initial cond.: $\mathbf{x}_0 = \mathbf{x}(t_0), \dot{\mathbf{x}}_0 = \dot{\mathbf{x}}(t_0)$

Compute $\mathbf{x}_{n+1} = \mathbf{x}(t_{n+1})$

Such that $\mathbf{M}\ddot{\mathbf{x}}_{n+1} + \mathbf{C}\dot{\mathbf{x}}_{n+1} + \mathbf{K}\mathbf{x}_{n+1} + \mathbf{f}_{nl,n+1} = \mathbf{f}_{ext,n+1}$

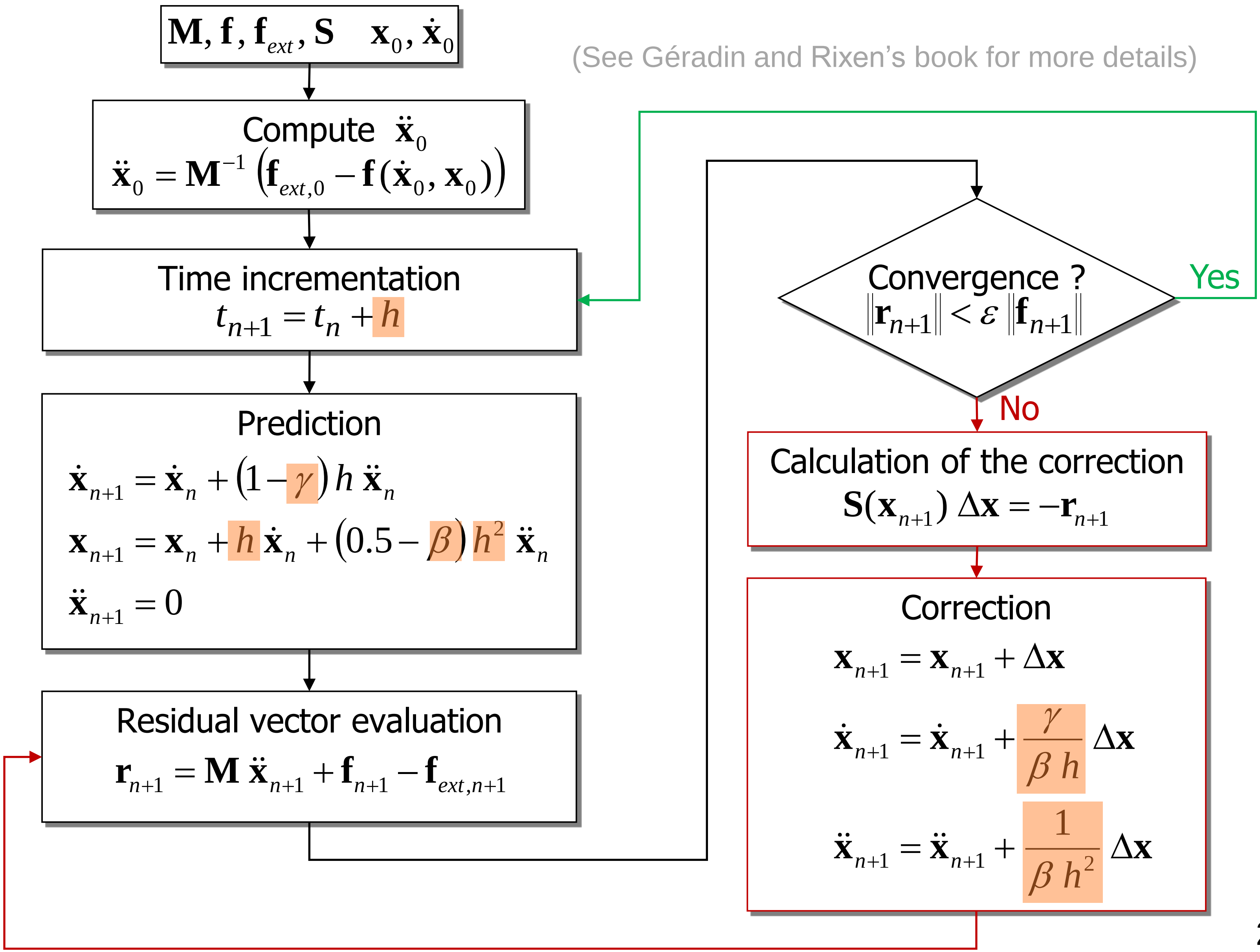
Newmark's Iterative Scheme for Nonlinear Systems

(See Géradin and Rixen's book for more details)



Time Step h, β and γ Are Key Parameters

(See Géradin and Rixen's book for more details)



Stability of Newmark's Scheme for Linear Systems

Algorithm	γ	β	ωh (Stability limit)	$\rho - 1$ (Accuracy: Amplitude error)	$\frac{\Delta T}{T}$ (Accuracy: Periodicity error)
Purely explicit	0	0	0	$\frac{\omega^2 h^2}{4}$	$-$
Central difference	$\frac{1}{2}$	0	2	0	$-\frac{\omega^2 h^2}{24}$
Fox & Goodwin	$\frac{1}{2}$	$\frac{1}{12}$	2.45	0	$O(h^3)$
Linear acceleration	$\frac{1}{2}$	$\frac{1}{6}$	3.46	0	$\frac{\omega^2 h^2}{24}$
Average constant acceleration	$\frac{1}{2}$	$\frac{1}{4}$	∞	0	$\frac{\omega^2 h^2}{12}$
Average constant acceleration (modified)	$\frac{1}{2}+\alpha$	$\frac{(1+\alpha)^2}{4}$	∞	$-\alpha \frac{\omega^2 h^2}{2}$	$\frac{\omega^2 h^2}{12}$

Implemented in NI2D (Average constant acceleration)

Why Newmark and Not Runge-Kutta (ode45)?

- Fixed time step

  Convenient for FE models with high eigenfrequencies.

- Control on stability and accuracy

  Demonstrated for linear systems with β,γ and time step h.

- Possibility to add numerical damping

  Use of the α parameter, or HHT scheme (more accurate).

Newmark's scheme is implemented in most commercial FE software.

Influence of the Time Step / Sampling Frequency

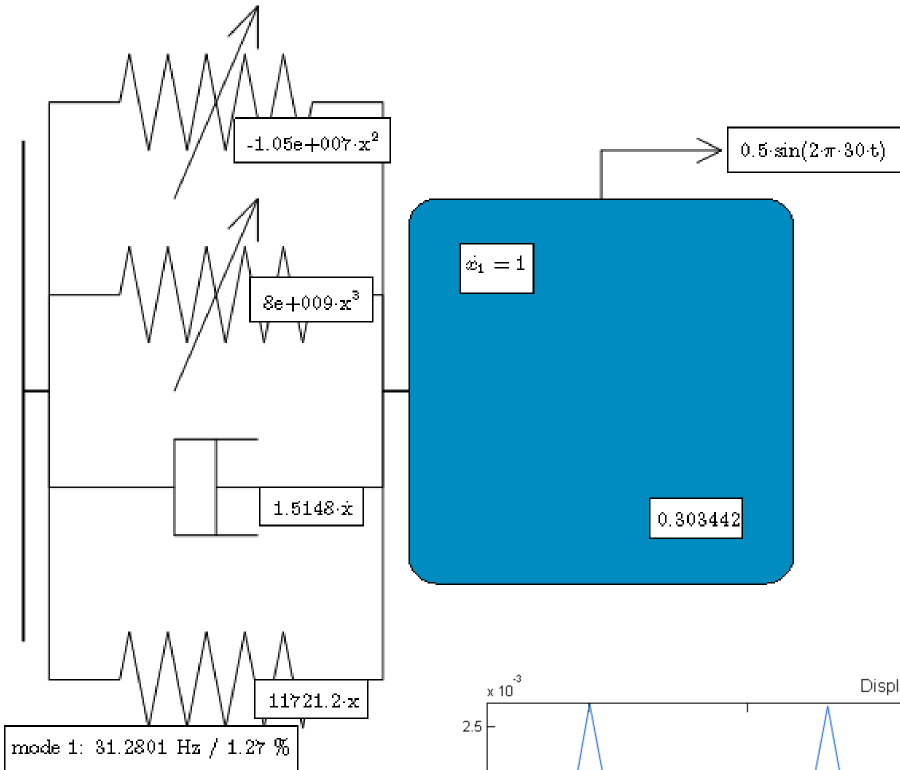
Rule of thumb: For a periodicity error of 1%, taking higher harmonics into account, consider at least

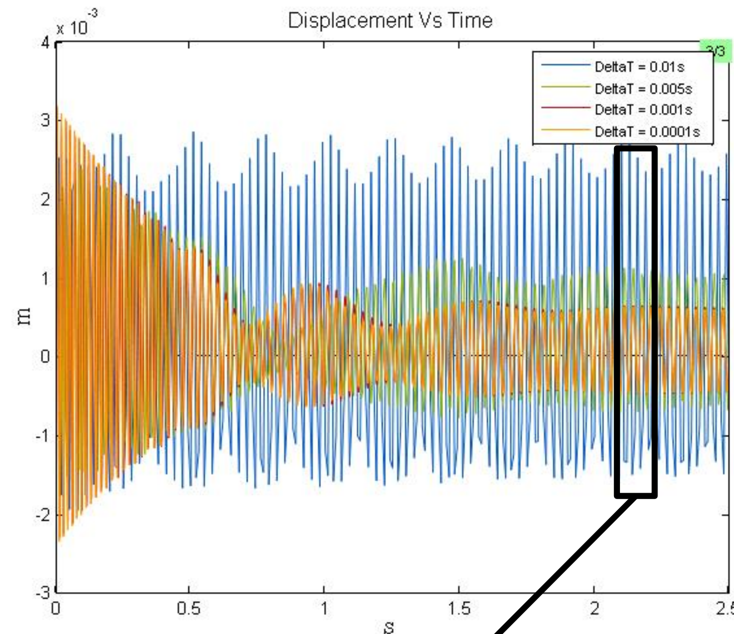
Sampling frequency = 1/time step

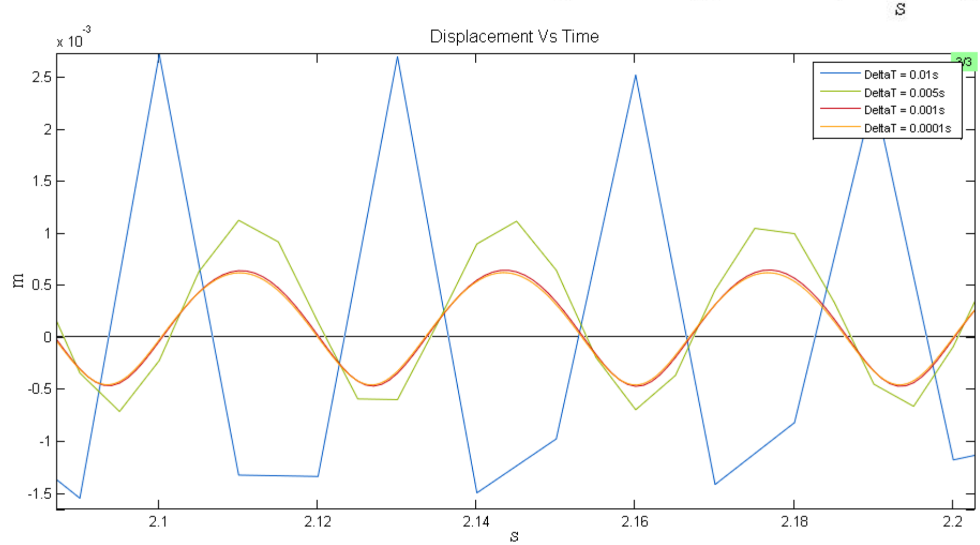
$$f_s > 200f$$

Frequency of interest in the signal

Influence of the Time Step / Sampling Frequency





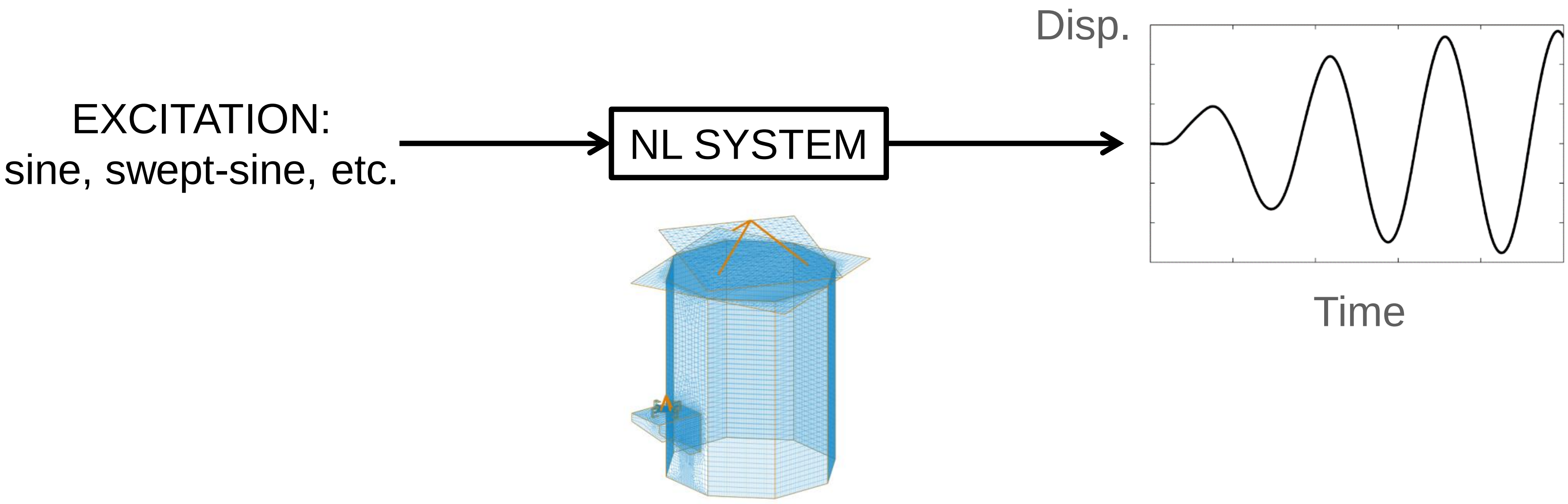


Advanced Nonlinear Simulations:

Nonlinear Frequency Responses and Modes

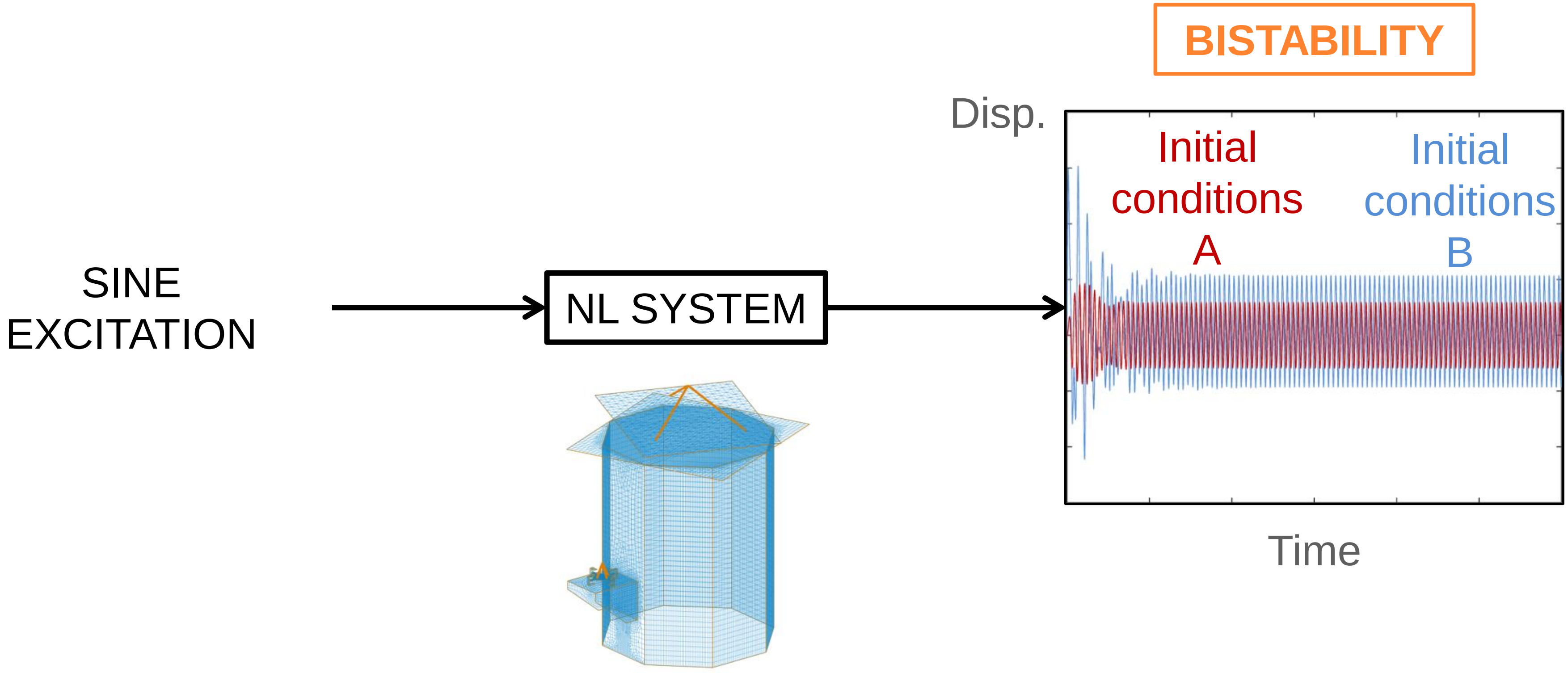
Limitations of Time Integration

Time simulations provide useful information about structural dynamics but they can be time consuming.



Limitations of Time Integration

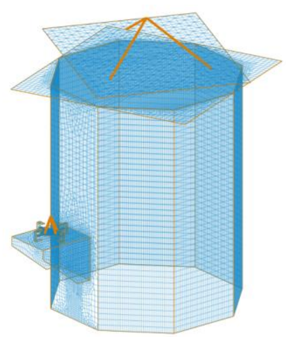
Time simulations may reveal nonlinear phenomena but cannot explain their origin.



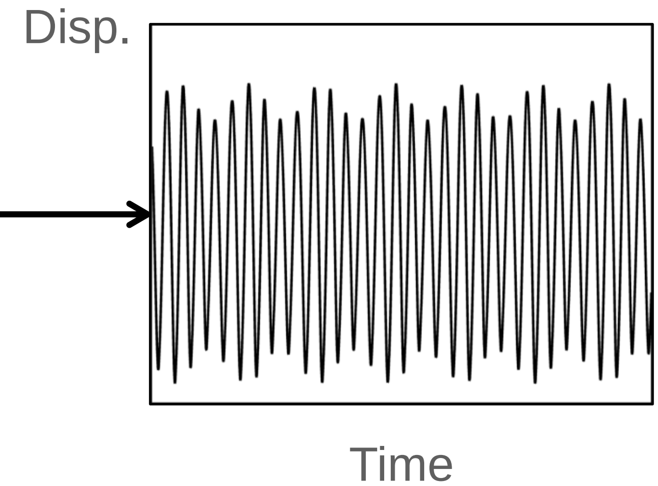
Limitations of Time Integration

Time simulations may reveal nonlinear phenomena but cannot explain their origin.

SINE EXCITATION → NL SYSTEM →



QUASIPERIODIC REGIME



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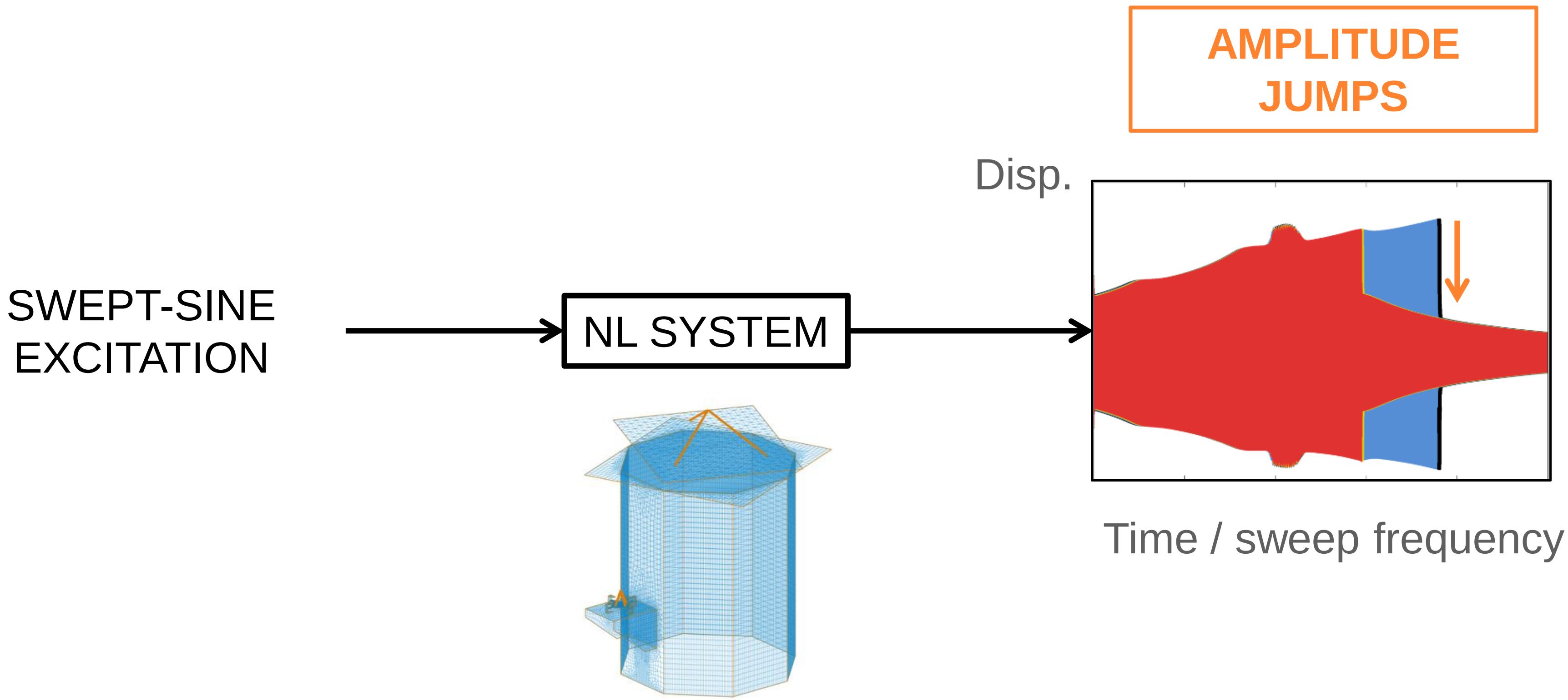

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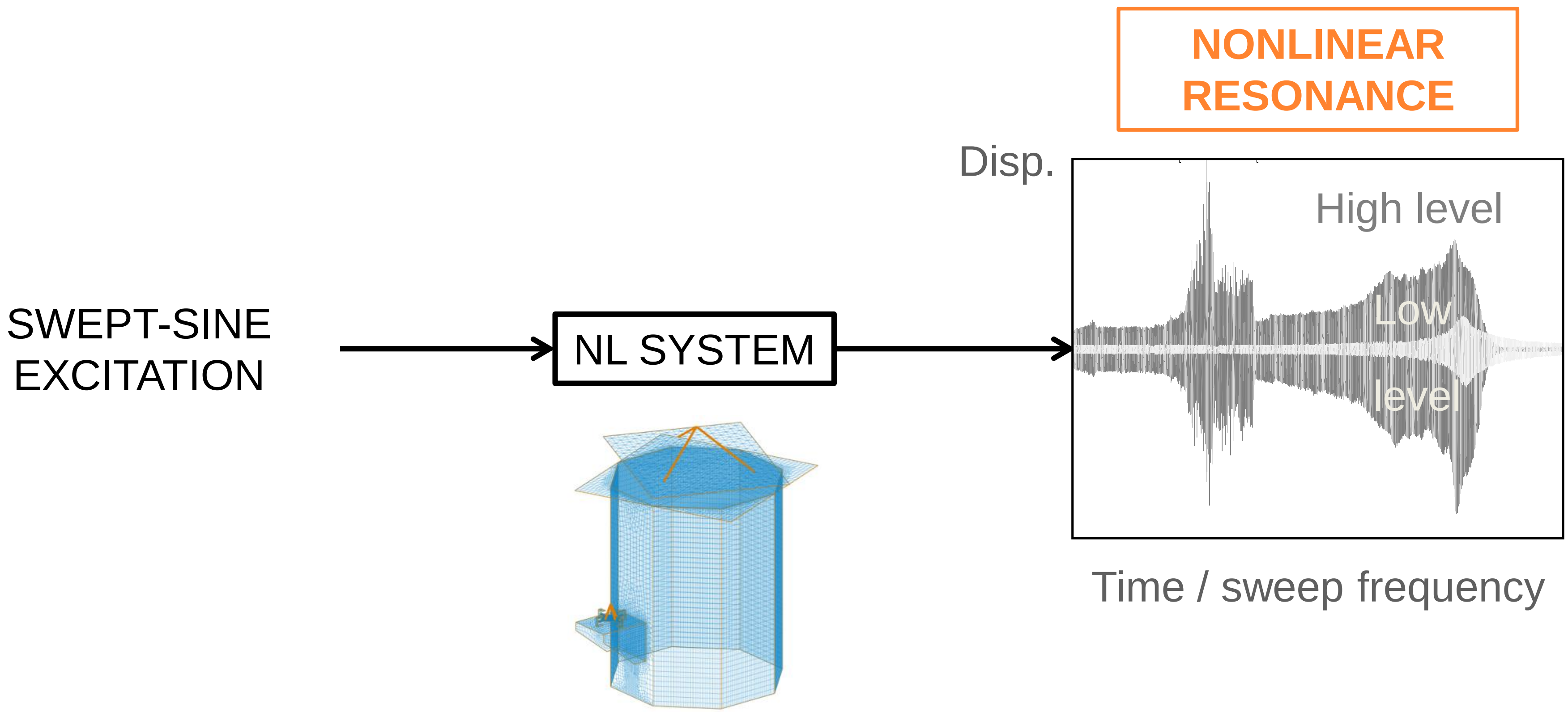


Nonlinear normal modes (NNMs) – See Lecture 3

NNMs are obtained by computing branches of periodic solutions of the underlying undamped and unforced model:

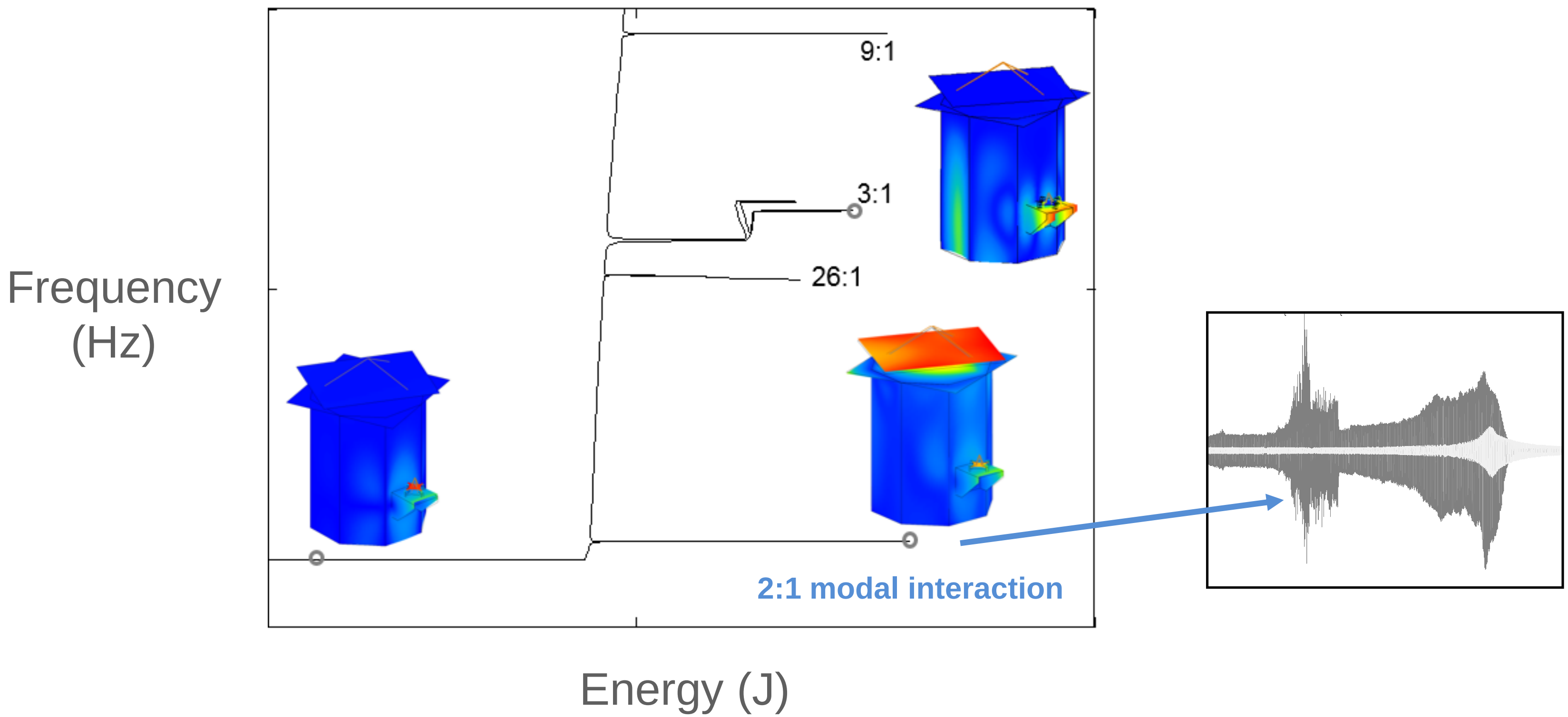
$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) + \mathbf{f}_{nl}(\mathbf{x}) = 0$$

NNMs are useful because:

- They describe the deformations at resonance of the structure.
- They describe how modal parameters evolve with motion amplitude.

Nonlinear normal modes (NNMs) – See Lecture 4

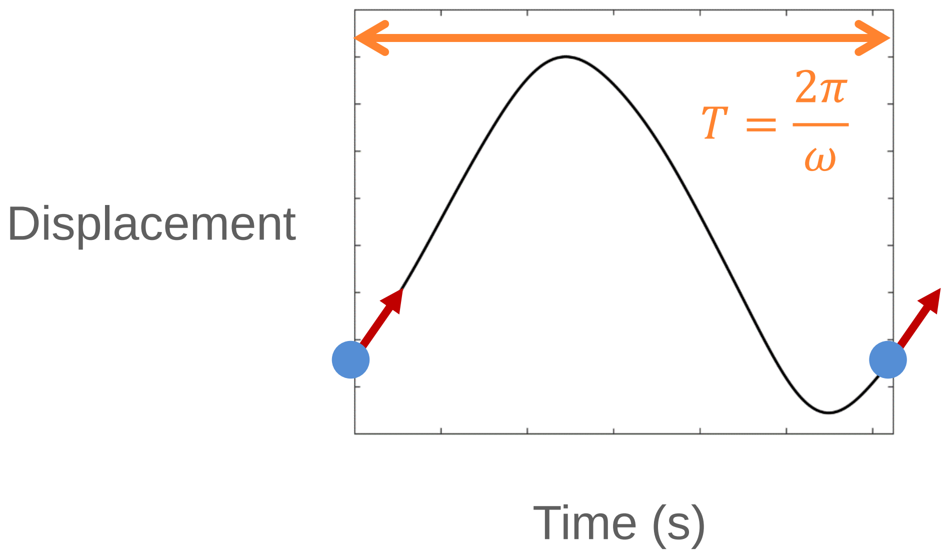
NNMs also help to uncover complex phenomena such as modal interactions / internal resonances.



Nonlinear Frequency Response Curves (NFRCs)

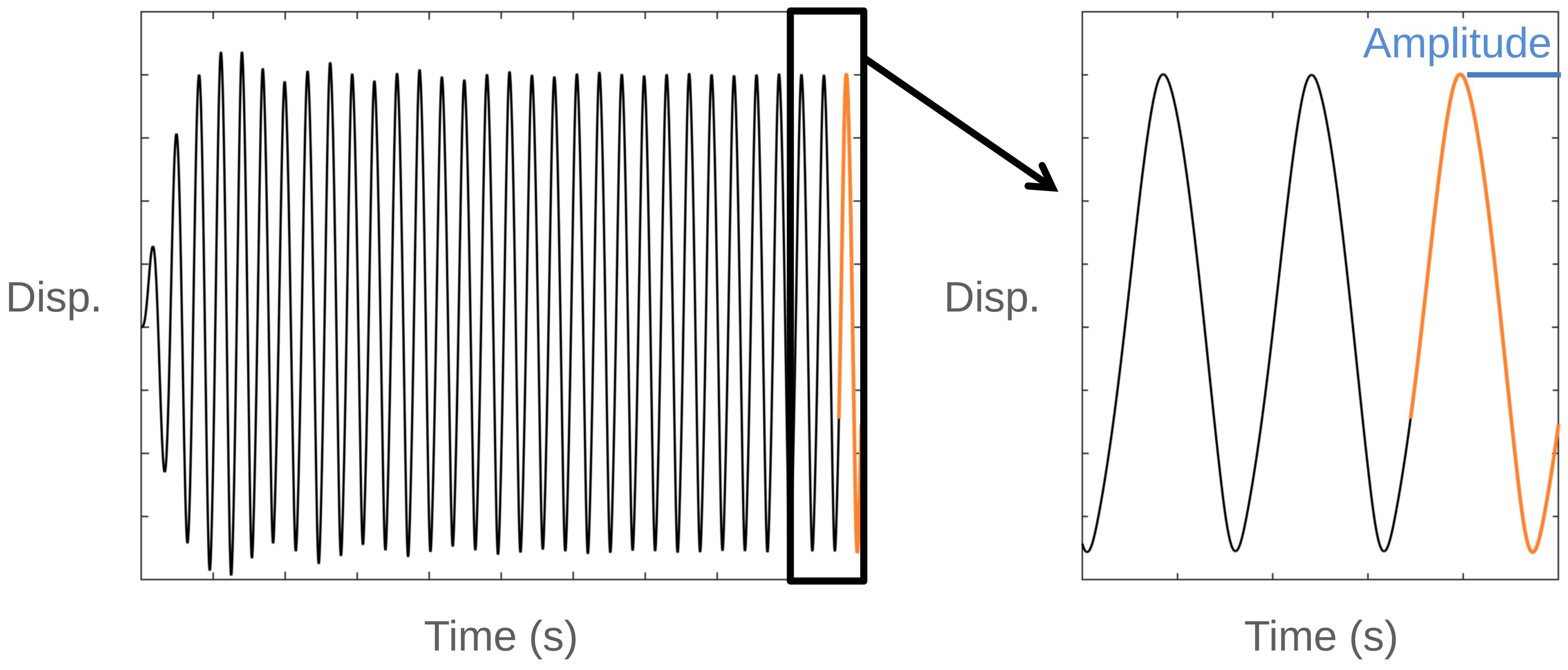
NFRCs are obtained by computing branches of periodic solutions of the damped model when submitted to a harmonic excitation:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) + \mathbf{f}_{nl}(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{f}_{ext}(\omega, t)$$



Nonlinear Frequency Response Curves (NFRCs)

NFRCs are useful because they describe the evolution of amplitude of the steady-state responses of the structure, *i.e.*, after the transients.



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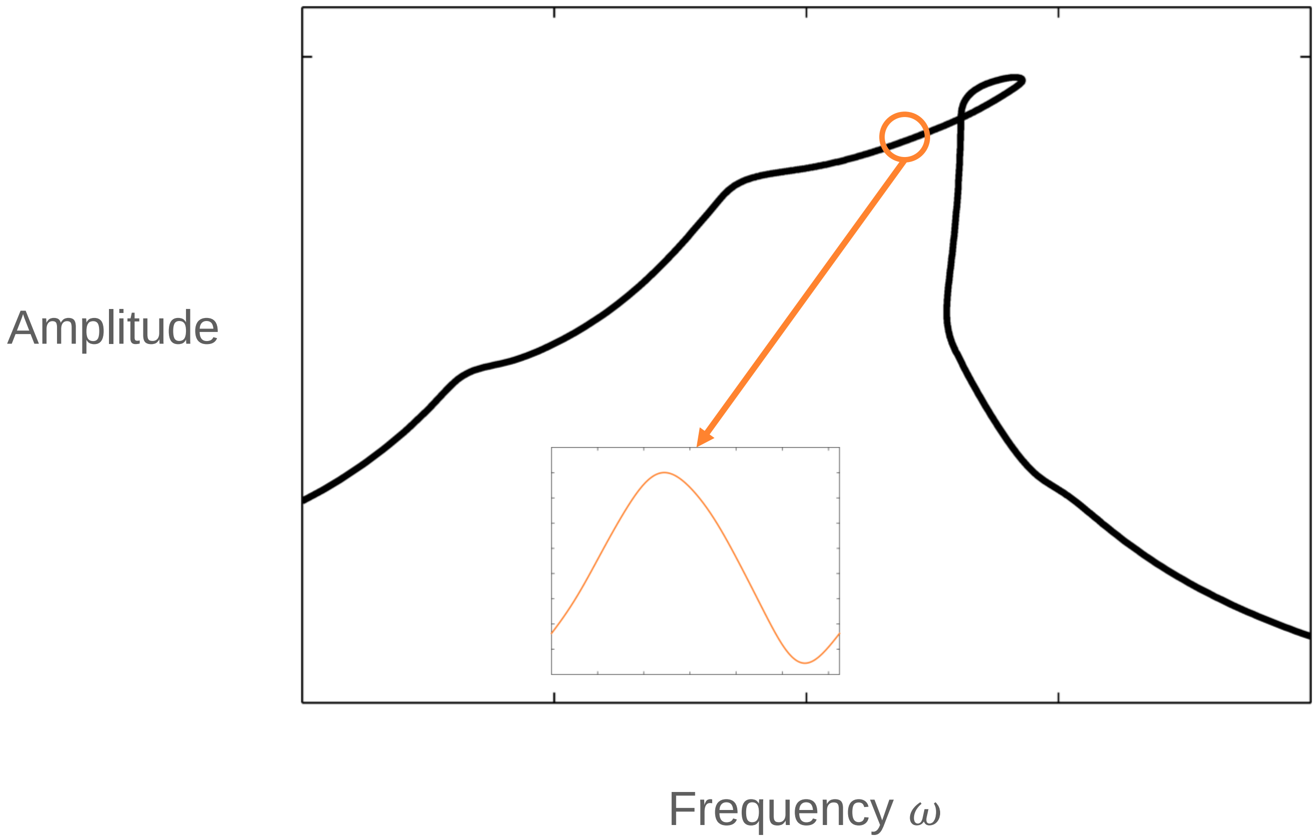
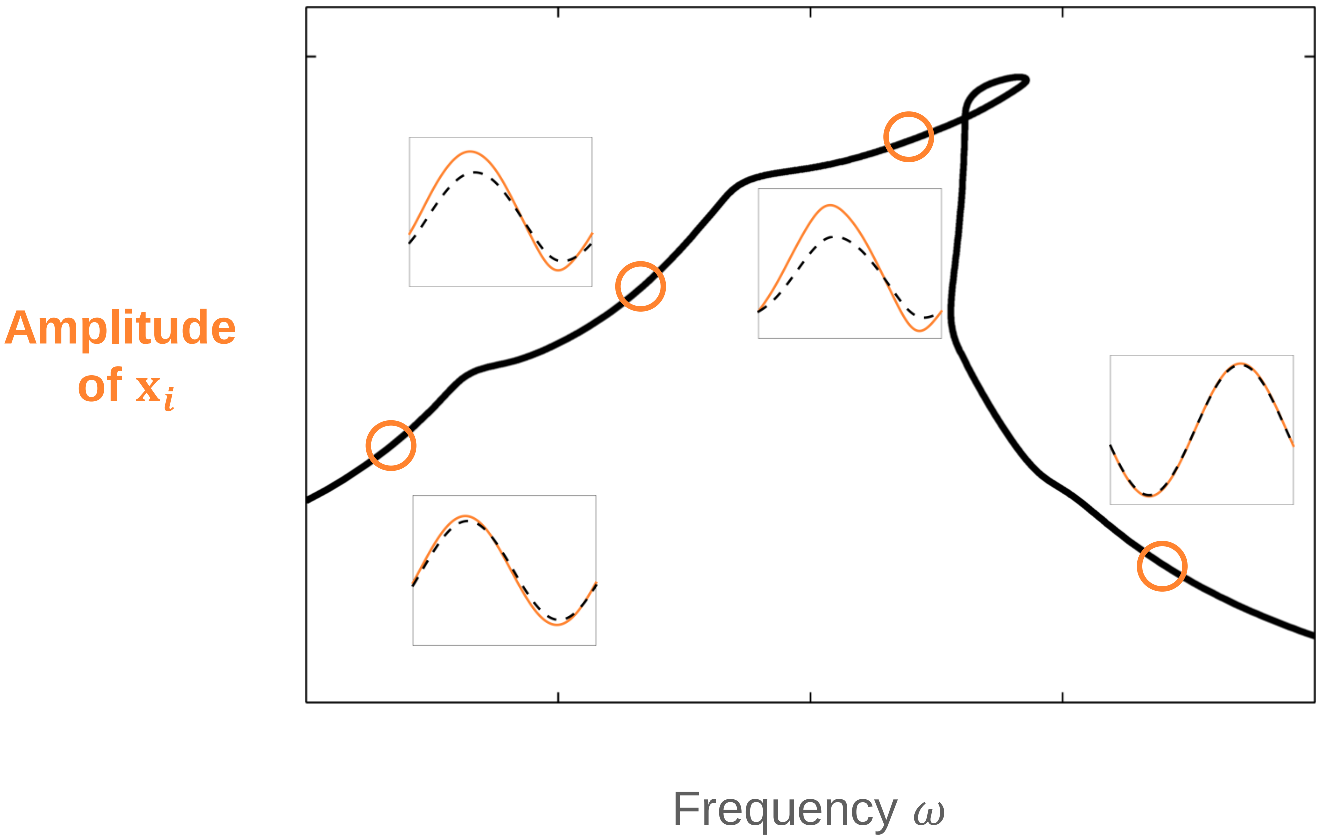


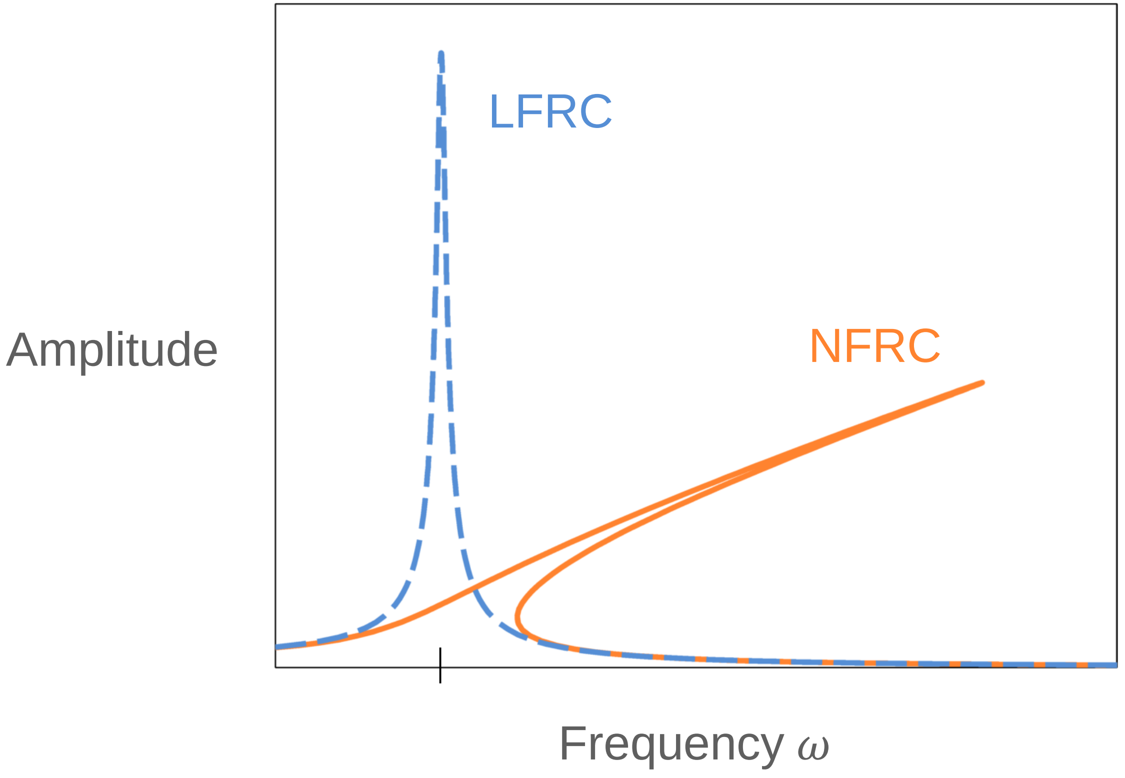
Nonlinear Frequency Response Curves (NFRCs)

The representative variable is usually chosen as the vibration amplitude of one of the DOFs, and is represented with respect to the frequency ω.



Nonlinear Frequency Response Curves (NFRCs)

NFRCs can be seen as the nonlinear extension of linear frequency response curves (LFRCs), or FRFs.



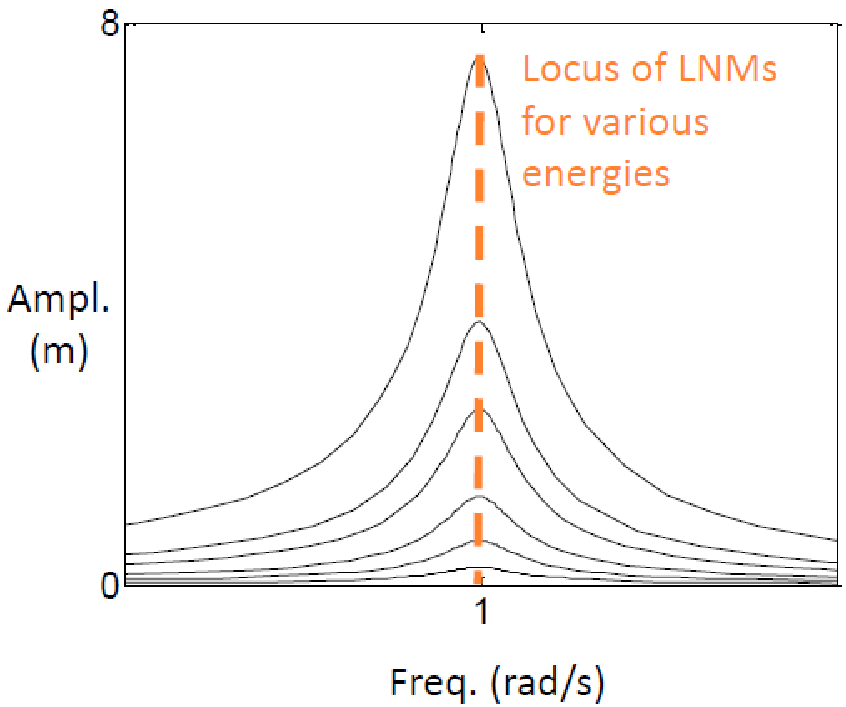
Nonlinear Frequency Response Curves (NFRCs)

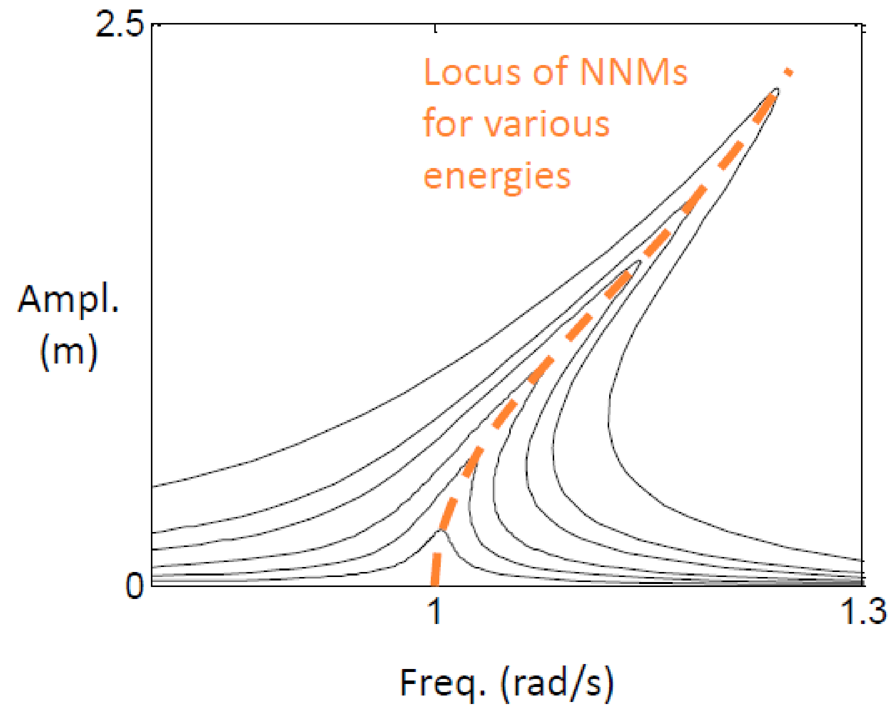
NFRCs can be seen as the nonlinear extension of linear frequency response curves (LFRCs), or FRFs.

… But

	LFRCs	NFRCs
Superposition	✓	✗
Uniqueness	✓	✗
Frequency	Energy independent	Energy dependent
Stability	Always stable	Stable or unstable

Nonlinear Frequency Response Curves (NFRCs)



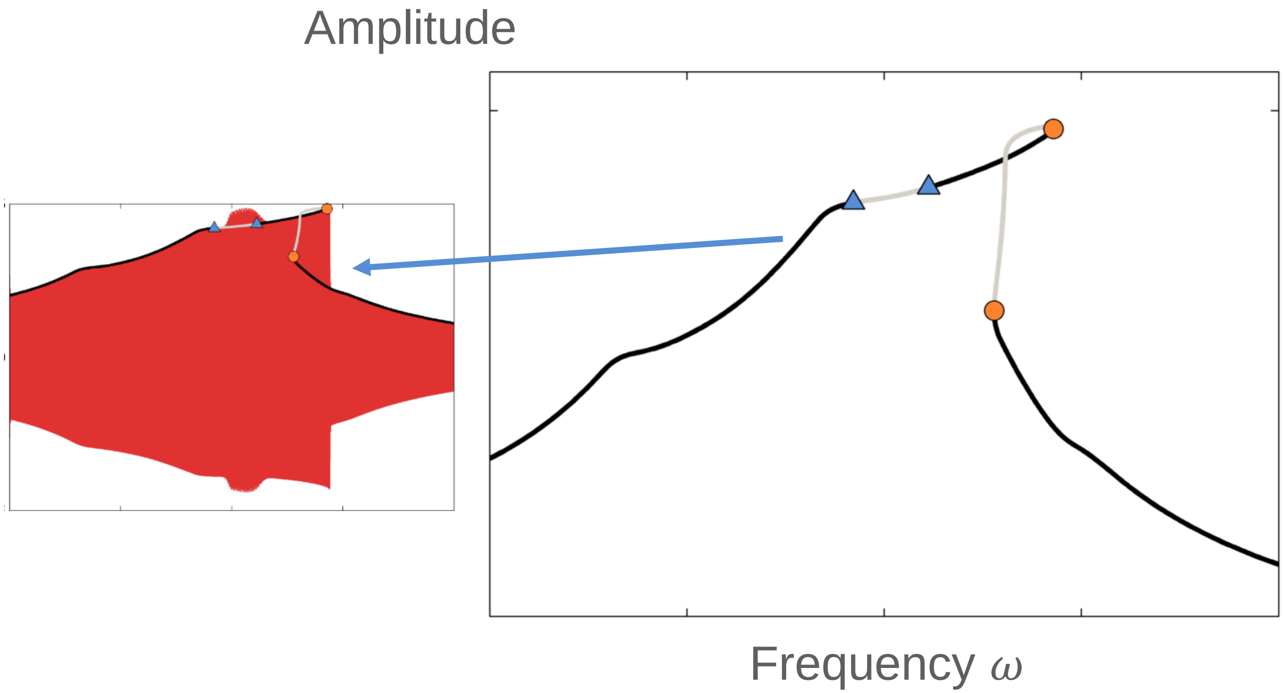


Linear: resonances occur in neighborhoods of LNMs.

Nonlinear: resonances occur in neighborhoods of NNMs.

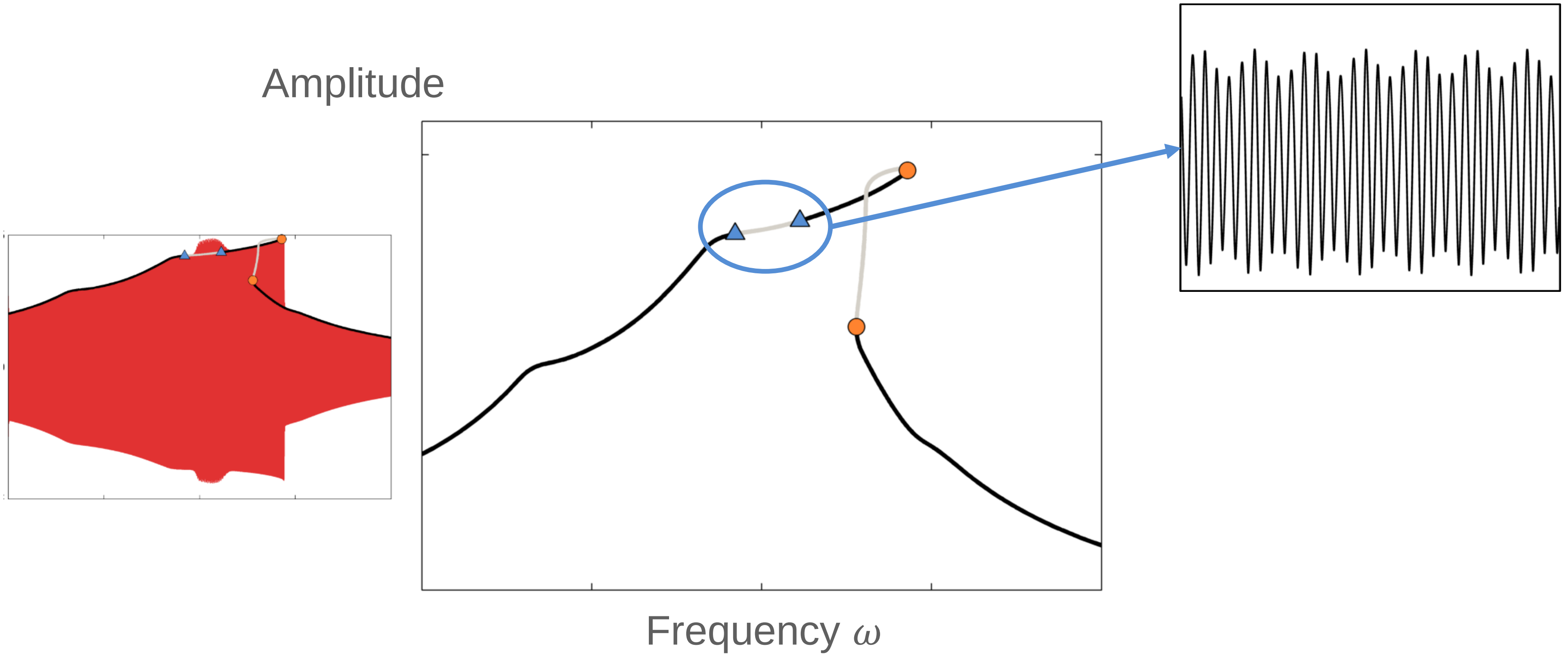
Nonlinear Frequency Response Curves (NFRCs)

NFRCs also help to uncover complex phenomena such as amplitude jumps.



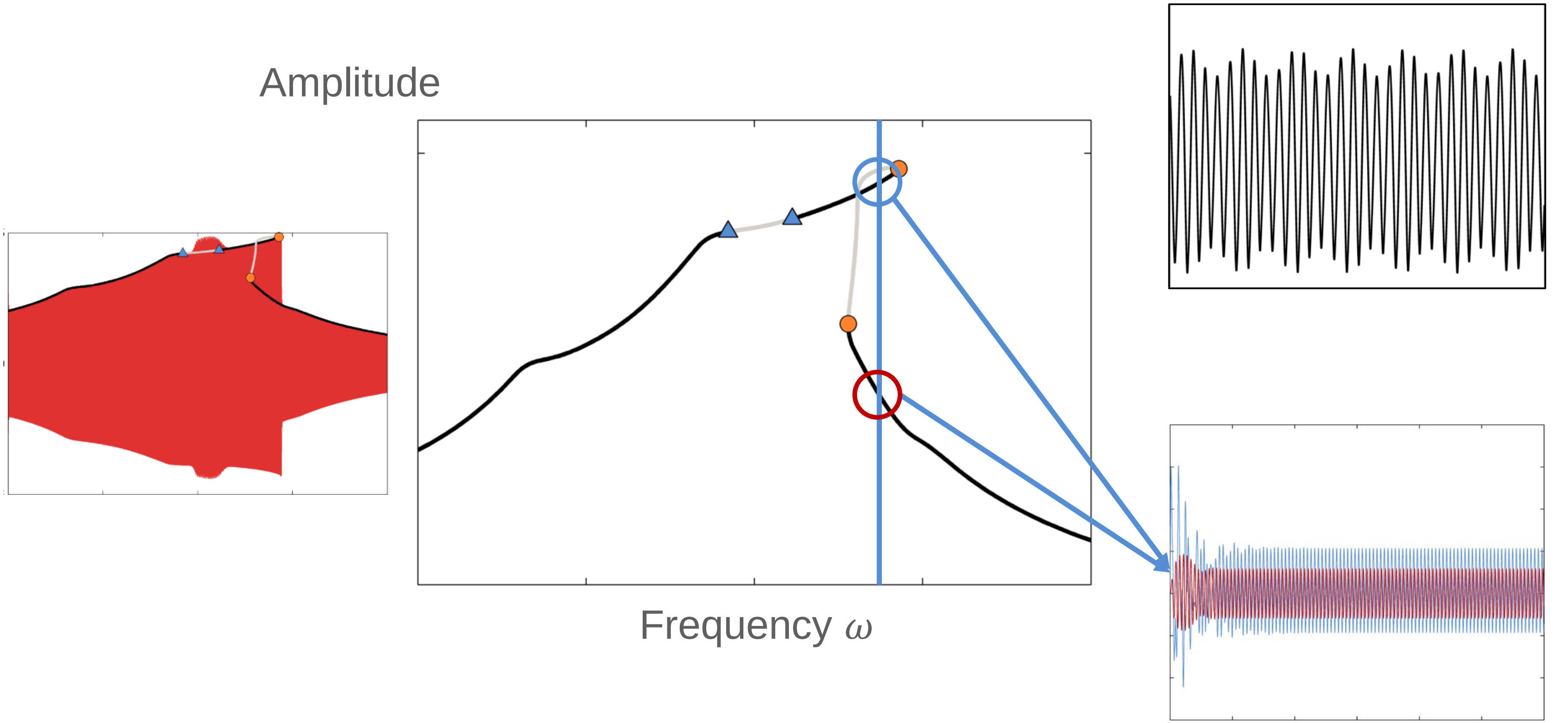
Nonlinear Frequency Response Curves (NFRCs)

NFRCs also help to uncover complex phenomena such as quasiperiodic regime.



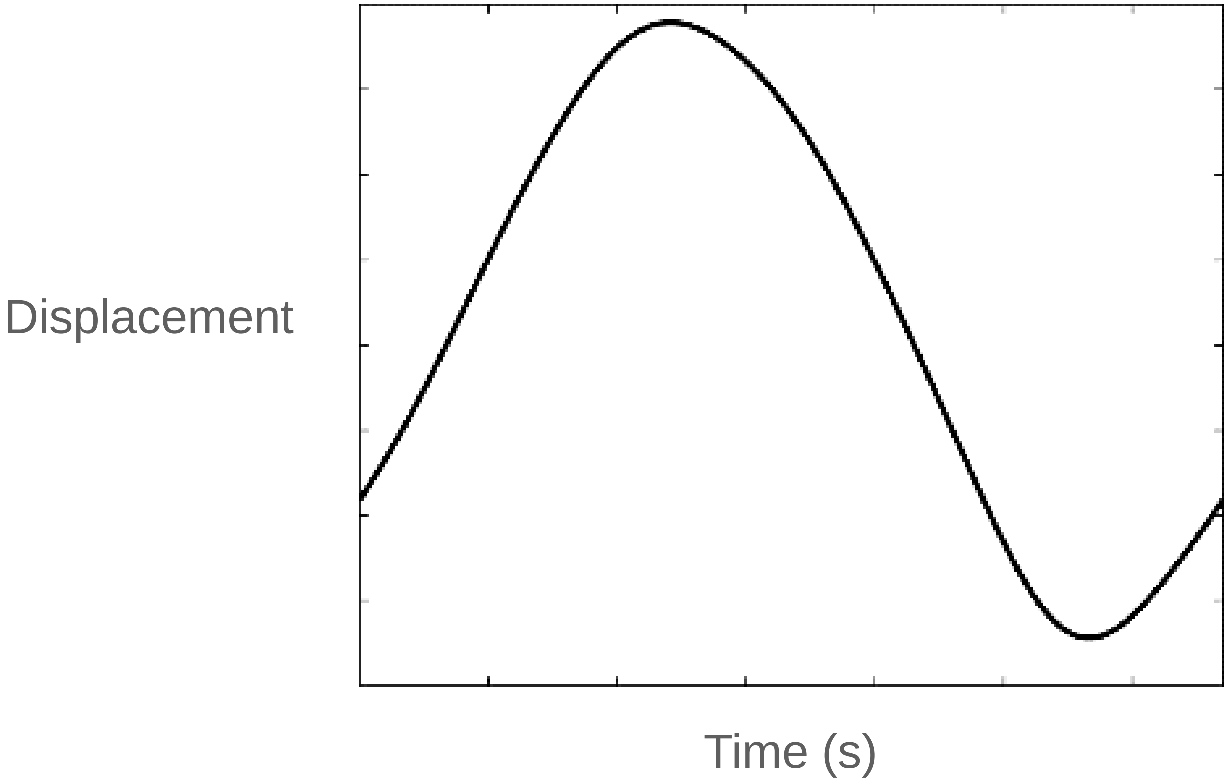
Nonlinear Frequency Response Curves (NFRCs)

NFRCs also help to uncover complex phenomena such as bistability.



Towards the Continuation of NNMs and NFRCs

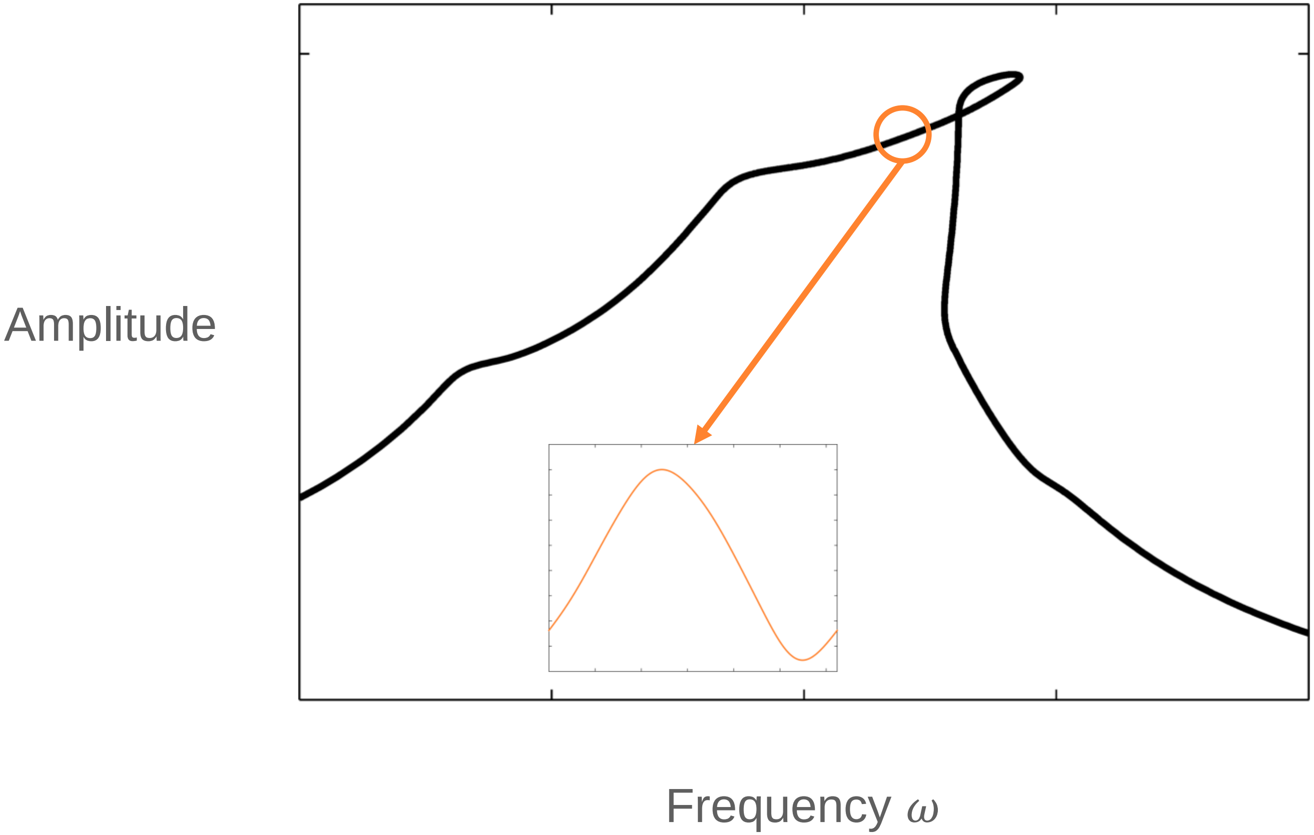
1. Computation of Periodic Solutions



TOPIC OF THIS LECTURE

Towards the Continuation of NNMs and NFRCs

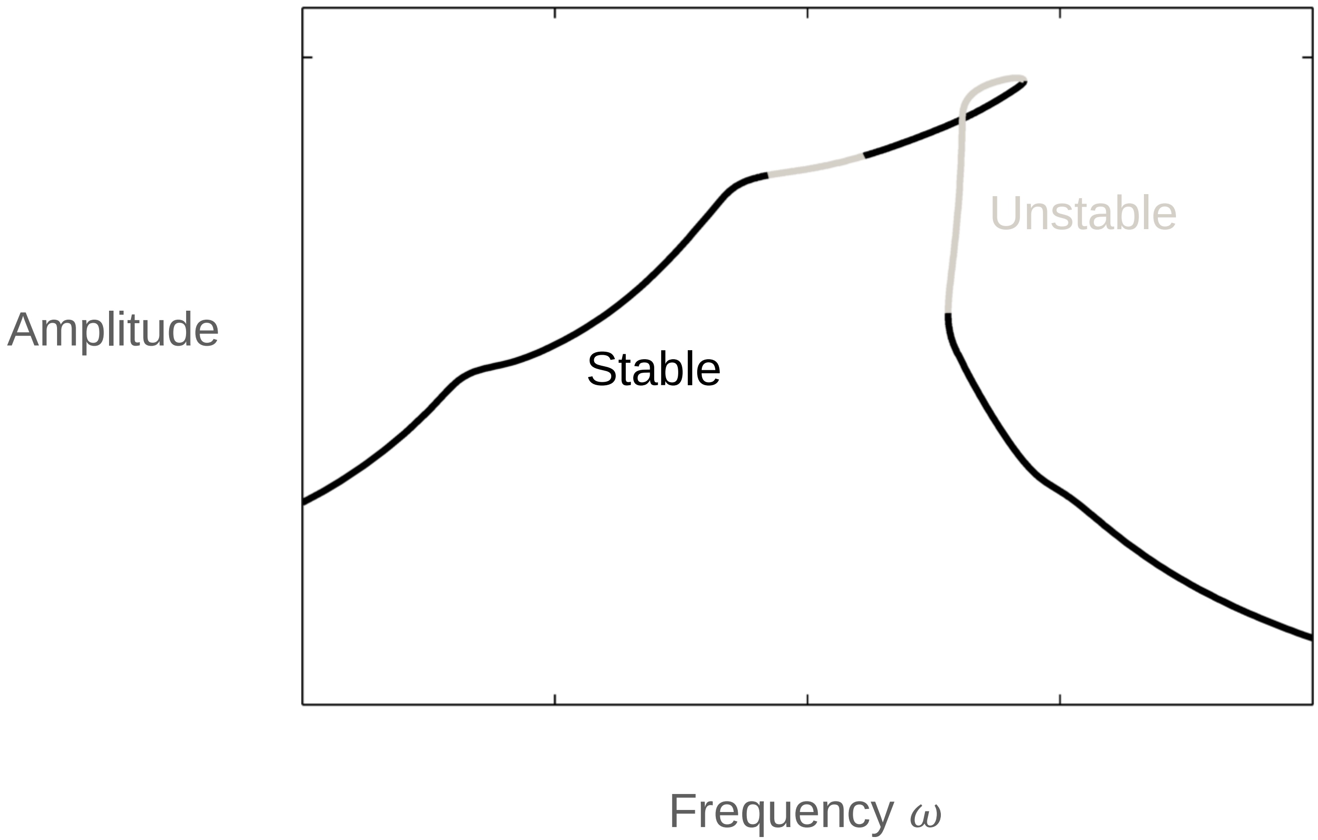
2. Continuation procedure



TOPIC OF THIS LECTURE

Towards the Continuation of NNMs and NFRCs

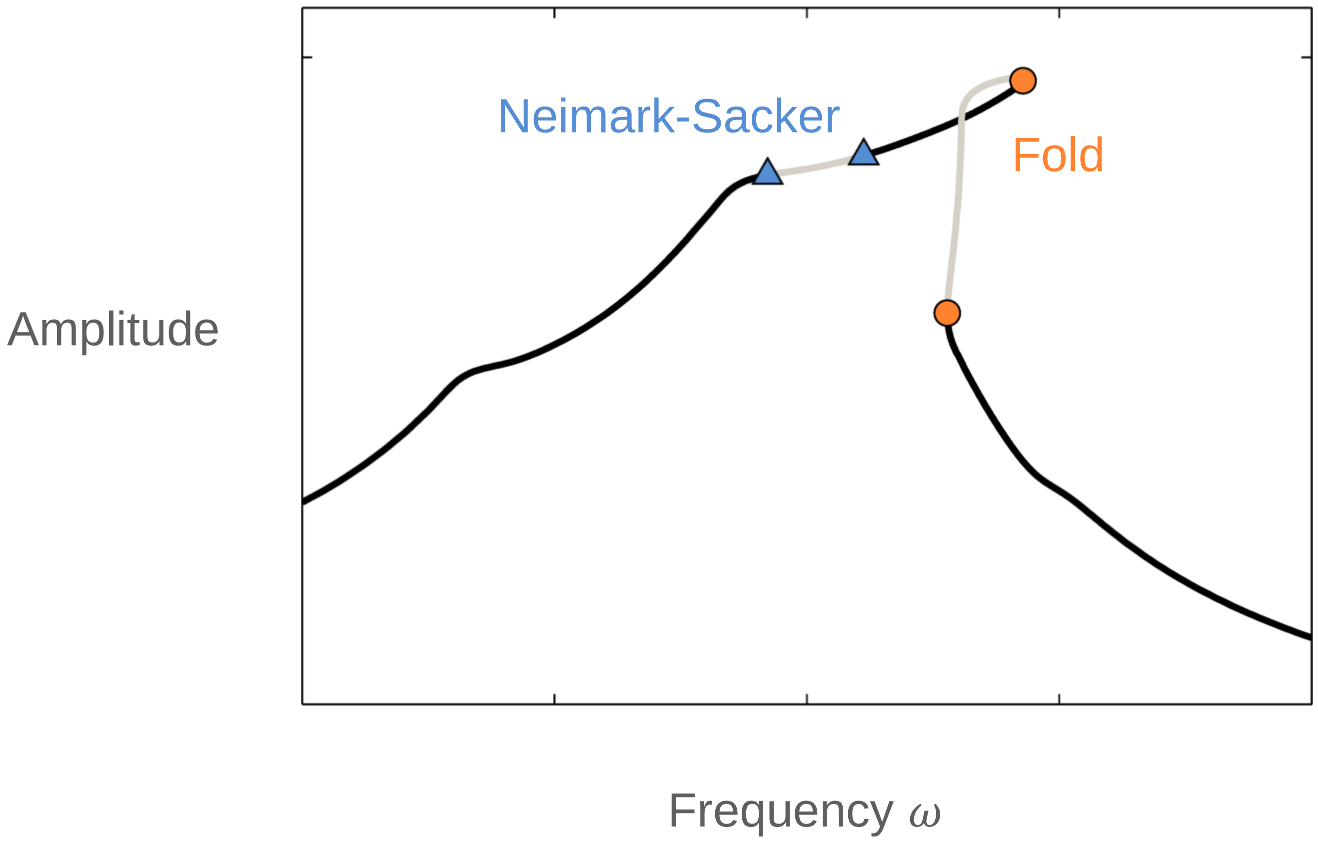
3. Stability analysis



SEE NEXT LECTURES

Towards the Continuation of NNMs and NFRCs

4. Bifurcation analysis

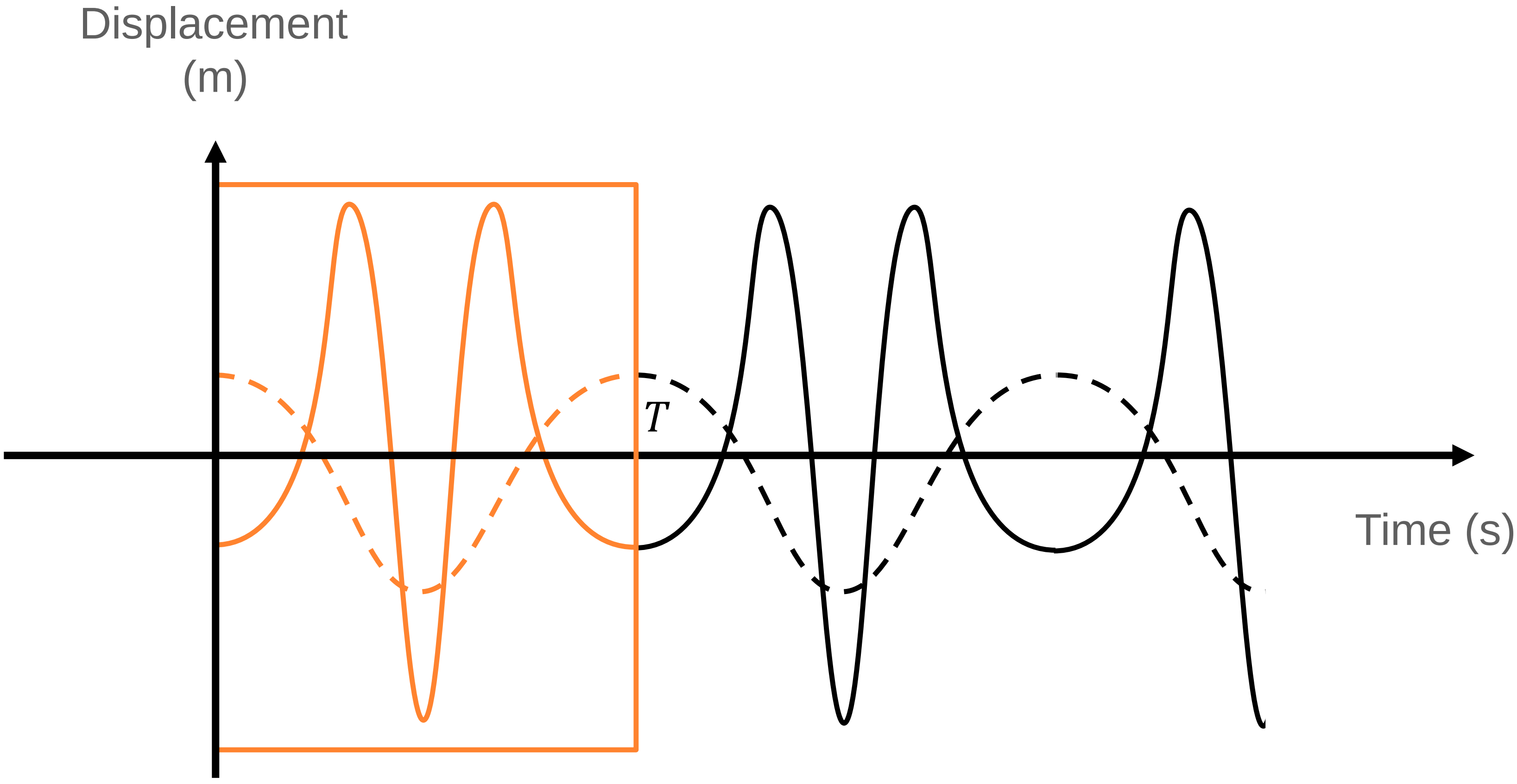


SEE NEXT LECTURES

Computation of Periodic Solutions

Mathematical Representation of a Periodic Solution

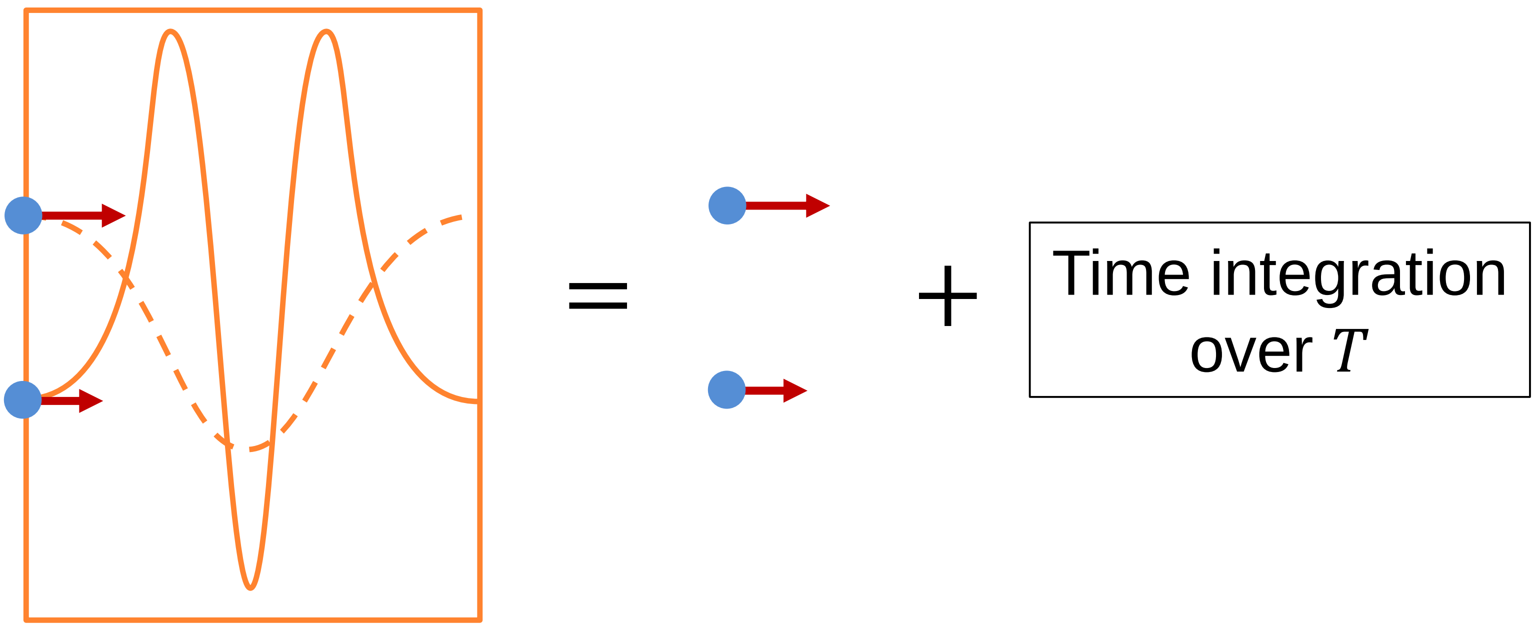
There are at least 3 approaches to describe a periodic solution.



Mathematical Representation of a Periodic Solution

There are at least 3 approaches to describe a periodic solution.

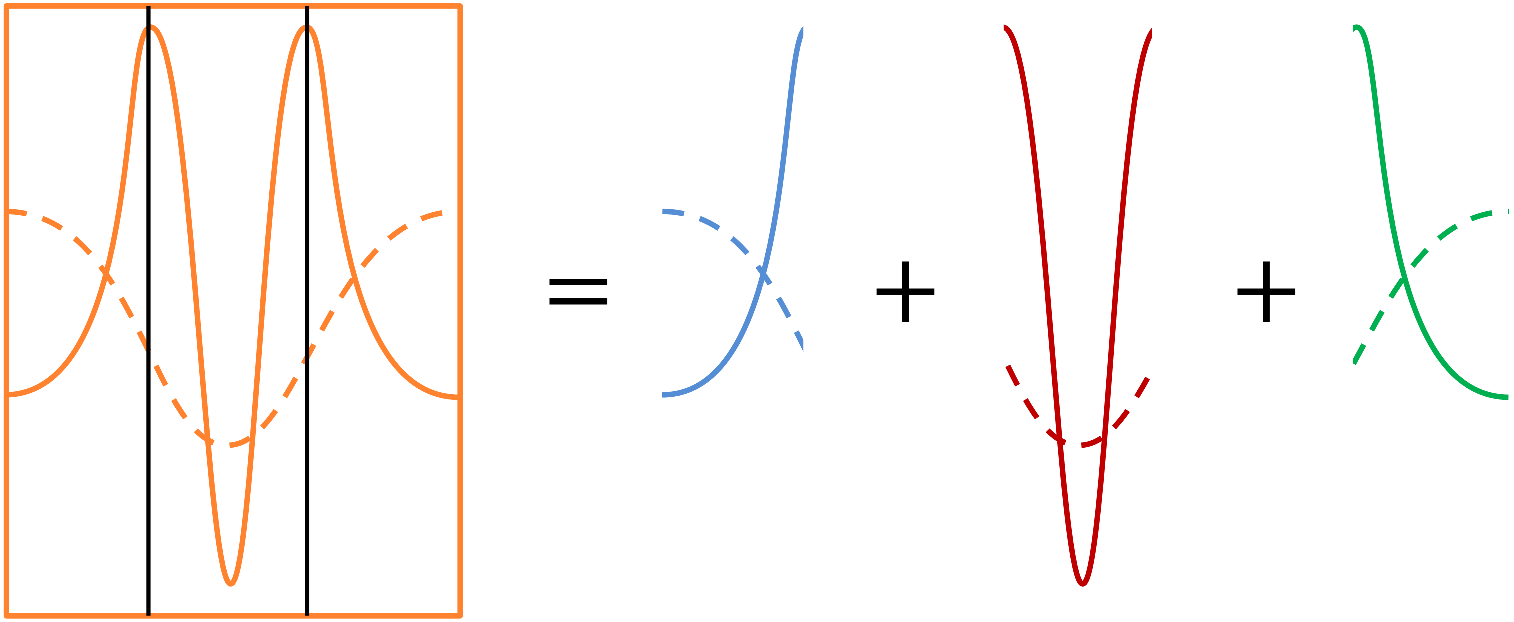
- Initial conditions $[\mathbf{x}_0 \quad \dot{\mathbf{x}}_0]^T$ and the period T.



Mathematical Representation of a Periodic Solution

There are at least 3 approaches to describe a periodic solution.

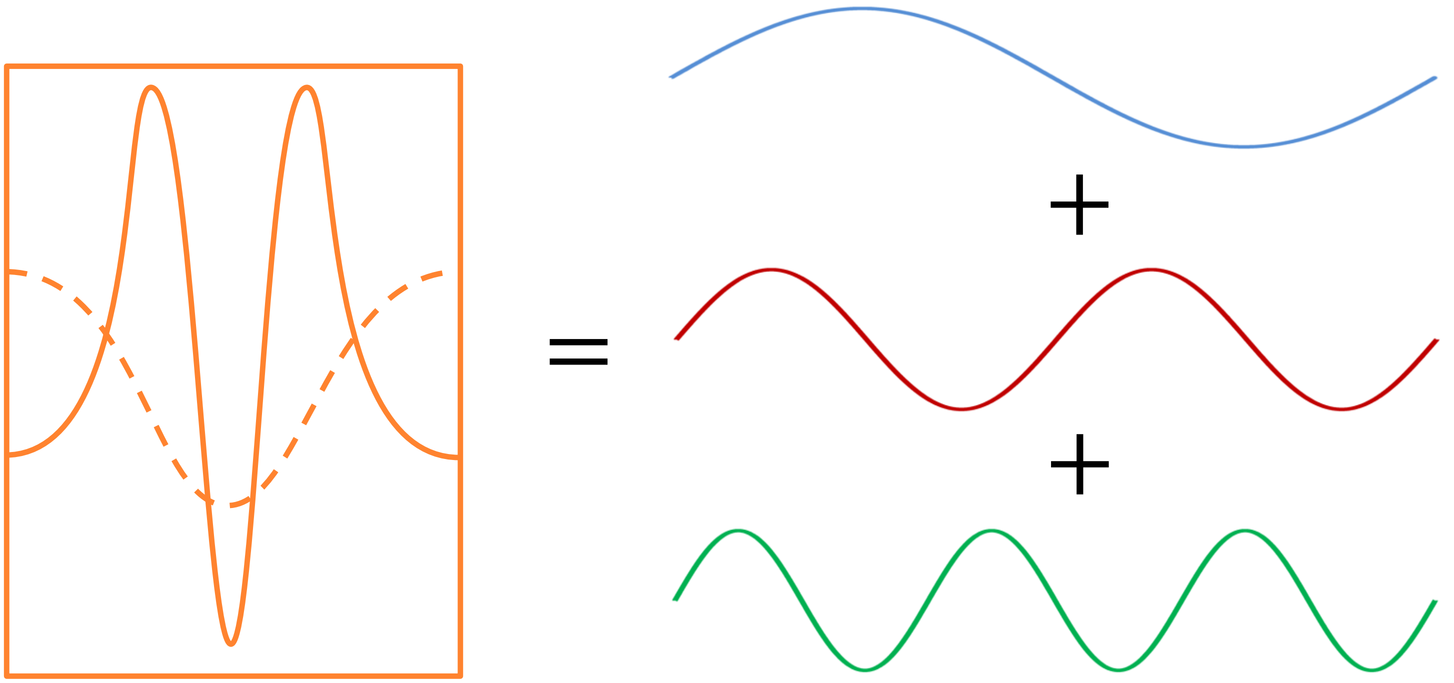
- Piecewise polynomial functions and the period T.



Mathematical Representation of a Periodic Solution

There are at least 3 approaches to describe a periodic solution.

- Fourier series and the period T.



Computation of a Periodic Solution

Computing the periodic solution of a nonlinear system means searching for a solution $\mathbf{x}$ that satisfies

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) + \mathbf{f}_{nl}(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{f}_{ext}(\omega, t)$$

with a periodicity condition

$$\mathbf{x}(t + T) = \mathbf{x}(t)$$

This represents a boundary-value problem (BVP).

Computation of a Periodic Solution

There are three approaches to solve this BVP.

- Based on initial conditions $[\mathbf{x}_0 \quad \dot{\mathbf{x}}_0]^T$.

  Shooting technique

- Based on piecewise polynomial functions.

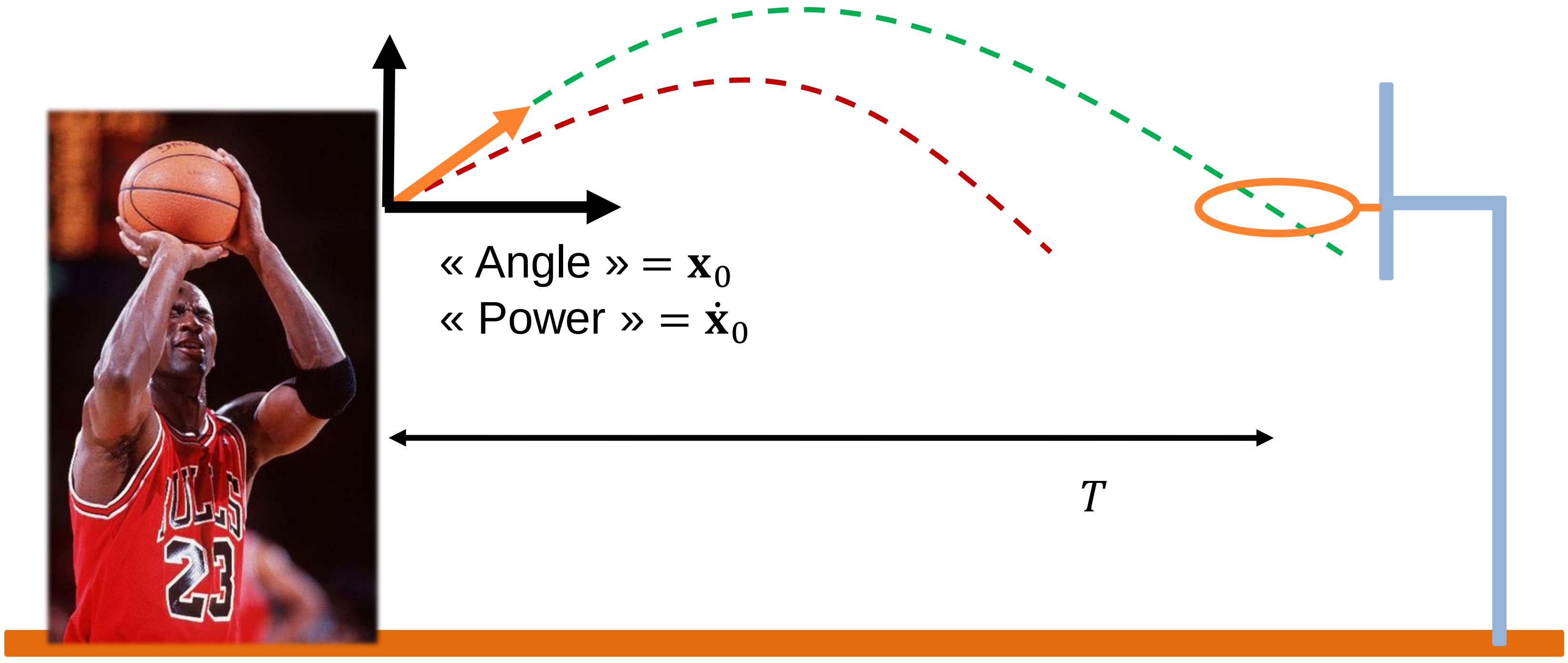
  Orthogonal collocation (not discussed here)

- Based on Fourier series.

  Harmonic balance method

Shooting Technique

Optimization of the initial state of a system $[\mathbf{x}_0 \quad \dot{\mathbf{x}}_0]^T$ to obtain a periodic solution after time integration over a period T.



Shooting Technique

The equations of motion are first recast in state-space form:

$$\dot{\mathbf{y}}(t) = \mathbf{L}\mathbf{y}(t) - \mathbf{g}_{nl}(\mathbf{y}) + \mathbf{g}_{ext}(\omega, t)$$

with

$$\mathbf{y} = \begin{bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{bmatrix} \qquad \mathbf{L} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_n \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}$$

$$\mathbf{g}_{nl} = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{f}_{nl}(\mathbf{x}, \dot{\mathbf{x}}) \end{bmatrix} \qquad \mathbf{g}_{ext} = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{f}_{ext}(\omega, t) \end{bmatrix}$$

The state of this system at time t and given initial condition $\mathbf{y}_0$ is denoted as $\mathbf{y} = \mathbf{y}(t; \mathbf{y}_0)$.

Shooting Technique

An initial state $\mathbf{y}_{0,p}$ leads to a periodic solution if

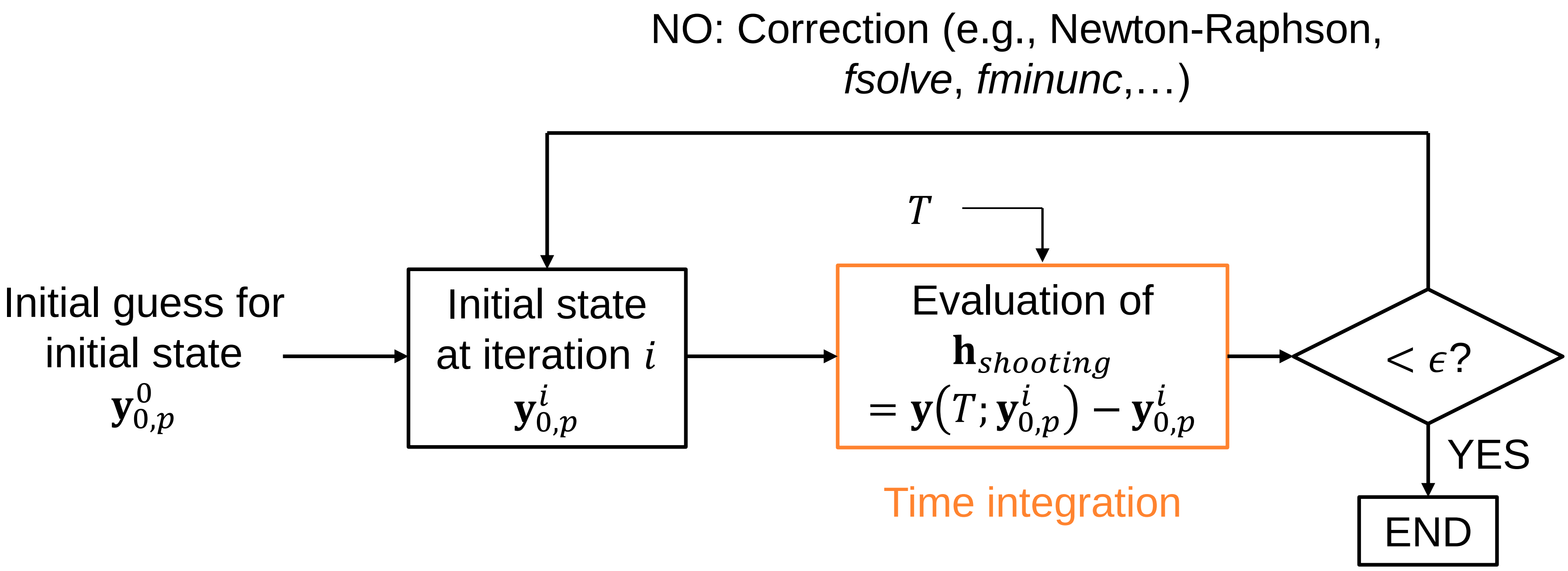
$$\mathbf{h}_{shooting} \equiv \mathbf{y}(T; \mathbf{y}_{0,p}) - \mathbf{y}_{0,p} = \mathbf{0}$$

where $\mathbf{y}(T; \mathbf{y}_{0,p})$ is computed from time integration of the EOMs.

The shooting technique consists in computing $\mathbf{y}_{0,p}$ that satisfies $\mathbf{h}_{shooting} = \mathbf{0}$ for T known a priori (NFRC) or not (NNM).

In the case of a harmonic excitation with frequency ω, T can be approximated as $T = 2\pi/\omega$.

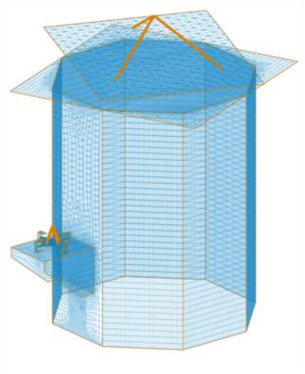
Shooting Technique Scheme (for NFRCs)

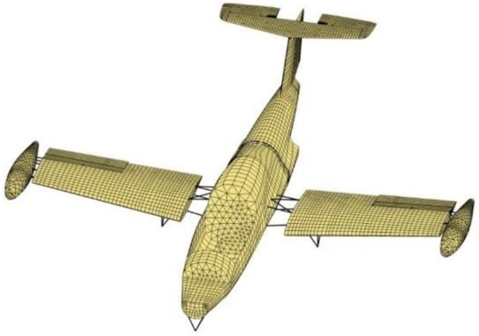


Periodic Solutions of Large Structures

The shooting technique is efficient and accurate for small nonlinear systems (1-30 DOFs).

For larger systems however, demand in CPU time (multiple time integrations) and memory space can be problematic.





For such cases, one usually relies on the harmonic balance method.

Fourier Series Approximation

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{f}(\mathbf{x}, \dot{\mathbf{x}}, \omega, t)$$
$$= \mathbf{f}_{ext}(\omega, t) - \mathbf{f}_{nl}(\mathbf{x}, \dot{\mathbf{x}})$$

where $\mathbf{f}(\mathbf{x}, \dot{\mathbf{x}}, \omega, t)$ gathers both nonlinear and external forces.

The harmonic balance (HB) method consists in approximating the displacements $\mathbf{x}(t)$ with Fourier series truncated to the order N_H.

Fourier Series Approximation

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{f}(\mathbf{x}, \dot{\mathbf{x}}, \omega, t)$$

$$\mathbf{x}(t) = \frac{\mathbf{c}_0^{\mathbf{x}}}{\sqrt{2}} + \sum_{k=1}^{N_H} \left(\mathbf{s}_k^{\mathbf{x}} \sin(k\omega t) + \mathbf{c}_k^{\mathbf{x}} \cos(k\omega t)\right)$$

The new unknowns are the Fourier coefficients $\mathbf{z}$, with

$$\mathbf{z} = \underbrace{\left[\mathbf{c}_0^{\mathbf{x}T} \quad \mathbf{s}_1^{\mathbf{x}T} \quad \mathbf{c}_1^{\mathbf{x}T} \quad \dots \quad \mathbf{s}_{N_H}^{\mathbf{x}}{}^T \quad \mathbf{c}_{N_H}^{\mathbf{x}}{}^T\right]^T}_{n_z = n(2N_H + 1) \text{ unknowns}}$$

Fourier Series Approximation

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{f}(\mathbf{x}, \dot{\mathbf{x}}, \omega, t)$$

$$\mathbf{f}(\mathbf{x}, \dot{\mathbf{x}}, \omega, t) = \frac{\mathbf{c}_0^{\mathbf{f}}}{\sqrt{2}} + \sum_{k=1}^{N_H} \left(\mathbf{s}_k^{\mathbf{f}} \sin(k\omega t) + \mathbf{c}_k^{\mathbf{f}} \cos(k\omega t) \right)$$

The Fourier coefficients of $\mathbf{f}$ are denoted by $\mathbf{b}$, with

$$\mathbf{b} = \left[\mathbf{c}_0^{\mathbf{f}\,T} \quad \mathbf{s}_1^{\mathbf{f}\,T} \quad \mathbf{c}_1^{\mathbf{f}\,T} \quad \ldots \quad \mathbf{s}_{N_H}^{\mathbf{f}\,T} \quad \mathbf{c}_{N_H}^{\mathbf{f}\,T} \right]^T$$

$= \mathbf{b}(\mathbf{z})$ since $\mathbf{f}$ depends on $\mathbf{x}$.

Fourier Series Approximation

Displacements and forces can be recast into a more compact form

$$\mathbf{x}(t) = (\mathbf{Q}(t) \otimes \mathbf{I}_n)\mathbf{z}$$

$$\mathbf{f}(t) = (\mathbf{Q}(t) \otimes \mathbf{I}_n)\mathbf{b}$$

where $\otimes$ denotes the Kronecker tensor product, $\mathbf{I}_n$ represents the identity matrix and where $\mathbf{Q}(t)$ is the orthogonal trigonometric basis:

$$\mathbf{Q}(t) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \sin(\omega t) & \cos(\omega t) & \dots & \sin(N_H \omega t) & \cos(N_H \omega t) \end{bmatrix}$$

Fourier Series Approximation

With this formulation, velocities can also be defined using Fourier series:

$$\dot{\mathbf{x}}(t) = \left(\dot{\mathbf{Q}}(t) \otimes \mathbf{I}_n\right)\mathbf{z} = \left((\mathbf{Q}(t)\boldsymbol{\nabla}) \otimes \mathbf{I}_n\right)\mathbf{z}$$

where

$$\boldsymbol{\nabla} = \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & \boldsymbol{\nabla}_k & & \\ & & & \ddots & \\ & & & & \boldsymbol{\nabla}_{N_H} \end{bmatrix} \qquad \boldsymbol{\nabla}_k = \begin{bmatrix} 0 & -k\omega \\ k\omega & 0 \end{bmatrix}$$

Fourier Series Approximation

With this formulation, accelerations can also be defined using Fourier series:

$$\ddot{\mathbf{x}}(t) = \left(\ddot{\mathbf{Q}}(t) \otimes \mathbf{I}_n\right)\mathbf{z} = \left(\left(\mathbf{Q}(t)\boldsymbol{\nabla}^2\right) \otimes \mathbf{I}_n\right)\mathbf{z}$$

where

$$\boldsymbol{\nabla}^{\mathbf{2}} = \boldsymbol{\nabla}\boldsymbol{\nabla} = \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & \boldsymbol{\nabla}_k^2 & & \\ & & & \ddots & \\ & & & & \boldsymbol{\nabla}_{N_H}^2 \end{bmatrix} \qquad \boldsymbol{\nabla}_k^2 = \begin{bmatrix} -(k\omega)^2 & 0 \\ 0 & -(k\omega)^2 \end{bmatrix}$$

Equations of Motion in the Frequency Domain

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{f}(\mathbf{x}, \dot{\mathbf{x}}, \omega, t)$$

Fourier series approximation

$$\mathbf{M}\big((\mathbf{Q}(t)\boldsymbol{\nabla}^2) \otimes \mathbf{I}_n\big)\mathbf{z} + \mathbf{C}\big((\mathbf{Q}(t)\boldsymbol{\nabla}) \otimes \mathbf{I}_n\big)\mathbf{z}$$
$$+\mathbf{K}(\mathbf{Q}(t) \otimes \mathbf{I}_n)\mathbf{z} = (\mathbf{Q}(t) \otimes \mathbf{I}_n)\mathbf{b}$$

This expression can be further simplified using:
- Galerkin procedure (to remove time dependency).
- Kronecker product properties.

Equations of Motion in the Frequency Domain

In a more compact form:

$$\mathbf{h}(\mathbf{z}, \omega) \equiv \mathbf{A}(\omega)\mathbf{z} - \mathbf{b}(\mathbf{z}) = \mathbf{0}$$

where $\mathbf{A}$ describes the linear dynamics

$$\mathbf{A} = \nabla^2 \otimes \mathbf{M} + \nabla \otimes \mathbf{C} + \mathbf{I}_{2N_H+1} \otimes \mathbf{K}$$

$$= \begin{bmatrix} \mathbf{K} & & & & & \\ & \mathbf{K} - \omega^2\mathbf{M} & -\omega\mathbf{C} & & & \\ & \omega\mathbf{C} & \mathbf{K} - \omega^2\mathbf{M} & & & \\ & & & \ddots & & \\ & & & & \mathbf{K} - (N_H\omega)^2\mathbf{M} & -N_H\omega\mathbf{C} \\ & & & & N_H\omega\mathbf{C} & \mathbf{K} - (N_H\omega)^2\mathbf{M} \end{bmatrix}$$

Equations of Motion in the Frequency Domain

In a more compact form:

$$\mathbf{h}(\mathbf{z},\omega) \equiv \mathbf{A}(\omega)\mathbf{z} - \mathbf{b}(\mathbf{z}) = \mathbf{0}$$

where **b** is the Fourier coefficients vector of nonlinear and external forces

$$\mathbf{f}(\mathbf{x},\dot{\mathbf{x}},\omega,t) = \mathbf{f}_{ext}(\omega,t) - \mathbf{f}_{nl}(\mathbf{x},\dot{\mathbf{x}})$$

$$= \frac{\mathbf{c}_0^{\mathbf{f}}}{\sqrt{2}} + \sum_{k=1}^{N_H}\left(\mathbf{s}_k^{\mathbf{f}}\sin(k\omega t) + \mathbf{c}_k^{\mathbf{f}}\cos(k\omega t)\right)$$

$$\mathbf{b} = \left[\mathbf{c}_0^{\mathbf{f}\,T} \quad \boldsymbol{s}_1^{\mathbf{f}\,T} \quad \mathbf{c}_1^{\mathbf{f}\,T} \quad \ldots \quad \boldsymbol{s}_{N_H}^{\mathbf{f}\;\;T} \quad \mathbf{c}_{N_H}^{\mathbf{f}\;\;T}\right]^T$$

Equations of Motion in the Frequency Domain

In a more compact form:

$$\mathbf{h}(\mathbf{z}, \omega) \equiv \mathbf{A}(\omega)\mathbf{z} - \mathbf{b}(\mathbf{z}) = \mathbf{0}$$

If for a given forcing frequency ω, one finds a vector $\mathbf{z}^*$ such that

$$\mathbf{h}(\mathbf{z}^*, \omega) = \mathbf{0}$$

Then the time series $\mathbf{x}^*(t)$ reconstructed from $\mathbf{z}^*$

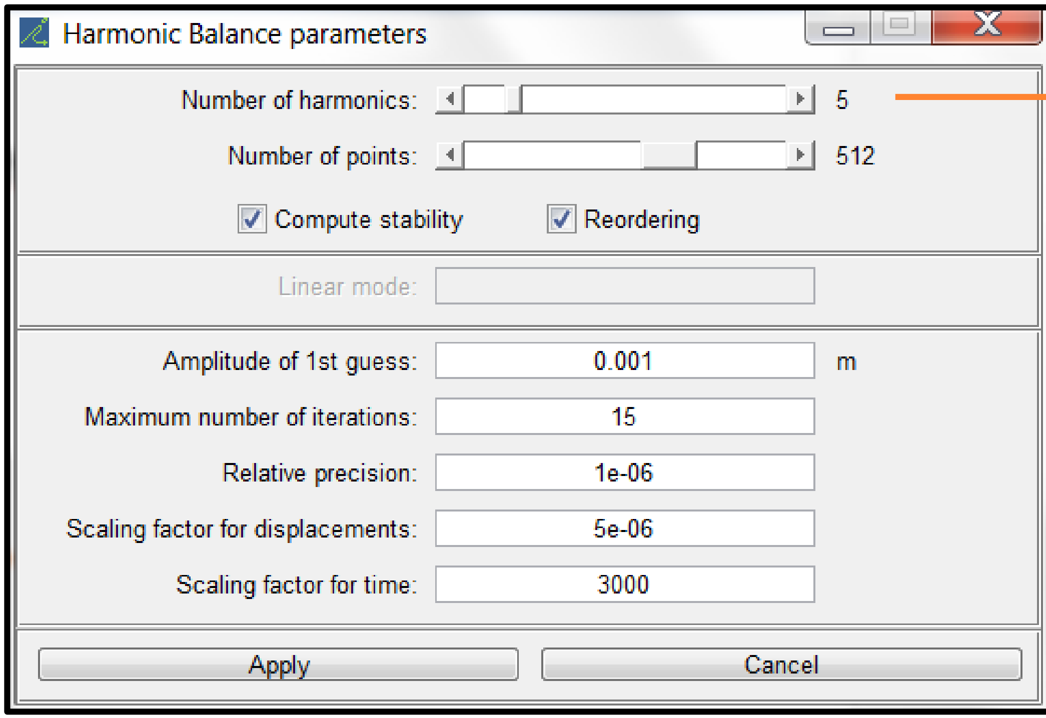
- verify the EOMs of the system.
- are periodic.

Equations of Motion in the Frequency Domain

$$\mathbf{h}(\mathbf{z}, \omega) \equiv \mathbf{A}(\omega)\mathbf{z} - \mathbf{b}(\mathbf{z}) = \mathbf{0}$$

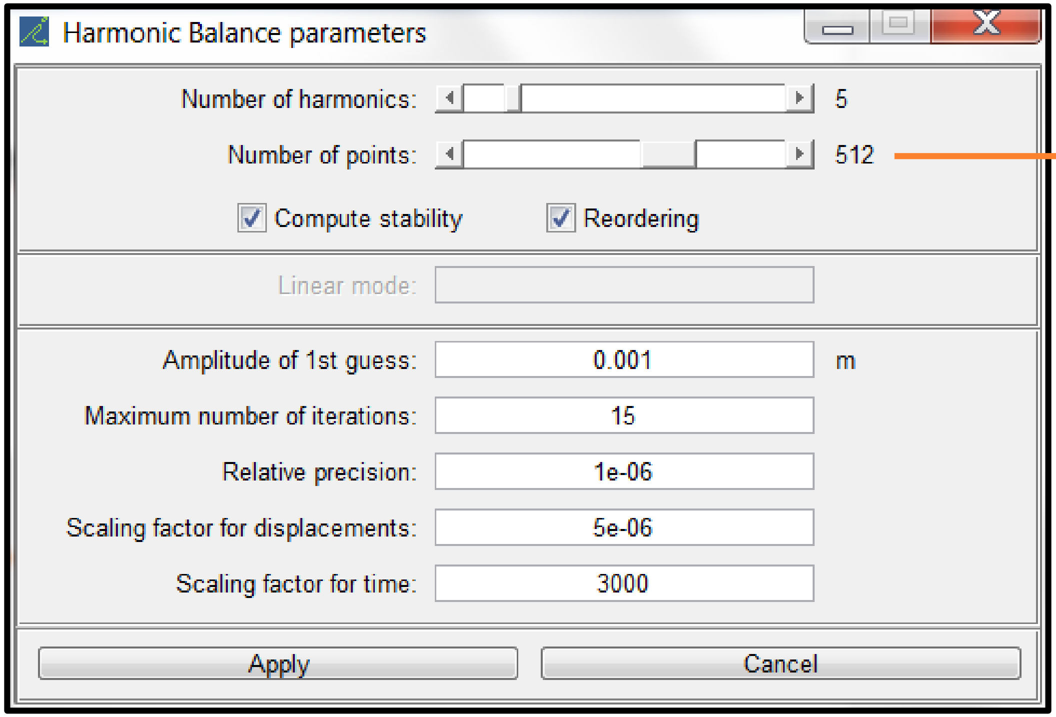
- $\mathbf{h}(\mathbf{z}, \omega) = 0$ is a nonlinear algebraic equation (easier to solve than time integrations as in shooting technique).
- $\mathbf{z}$ are the Fourier coefficients of the displacements and the new unknowns of the problem (usually less than for orthogonal collocation).
- For NFRCs, ω is the forcing frequency and is a system parameter.

Harmonic Balance Parameters



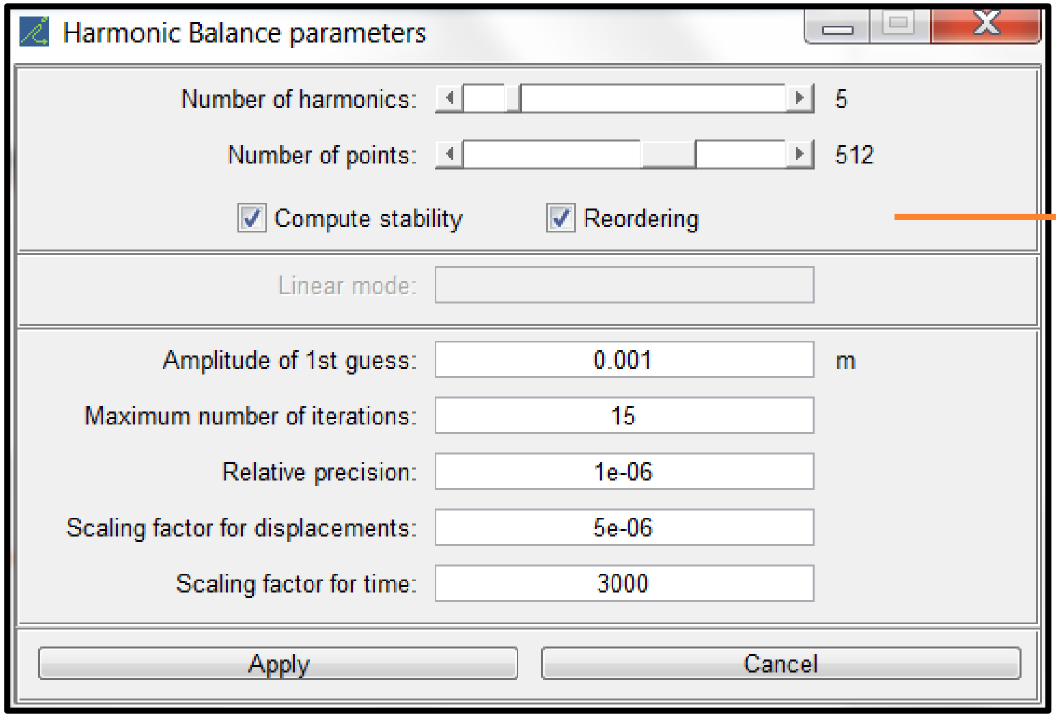
Number of harmonics N_H retained in the Fourier series.

Harmonic Balance Parameters



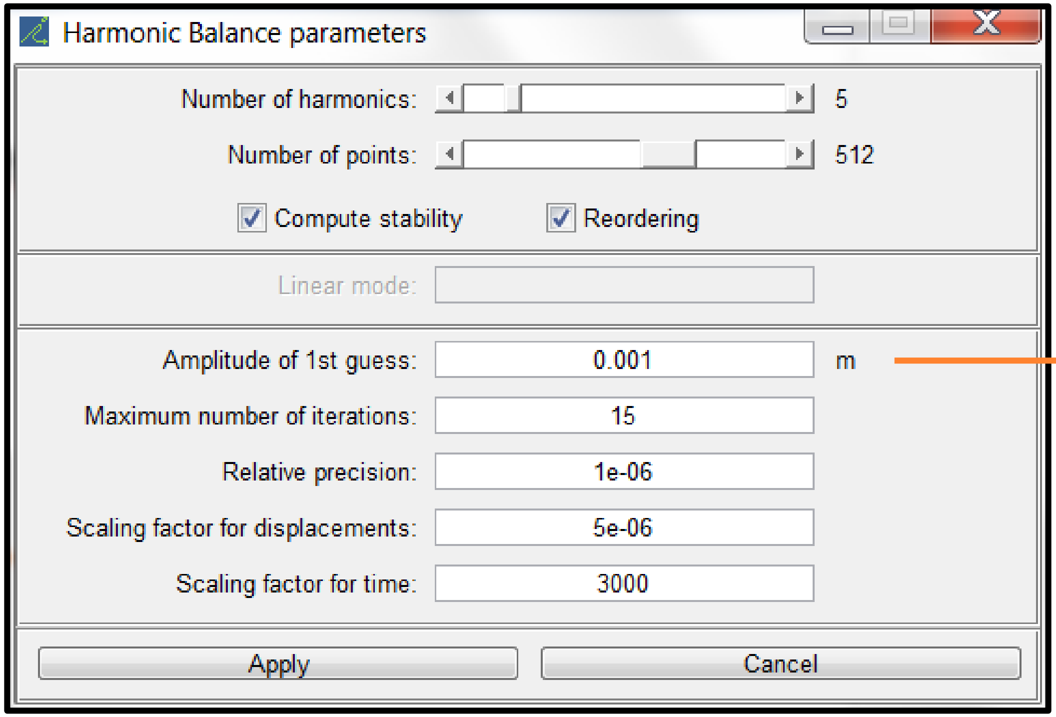
Number of time samples N in the Fourier transform.

Harmonic Balance Parameters



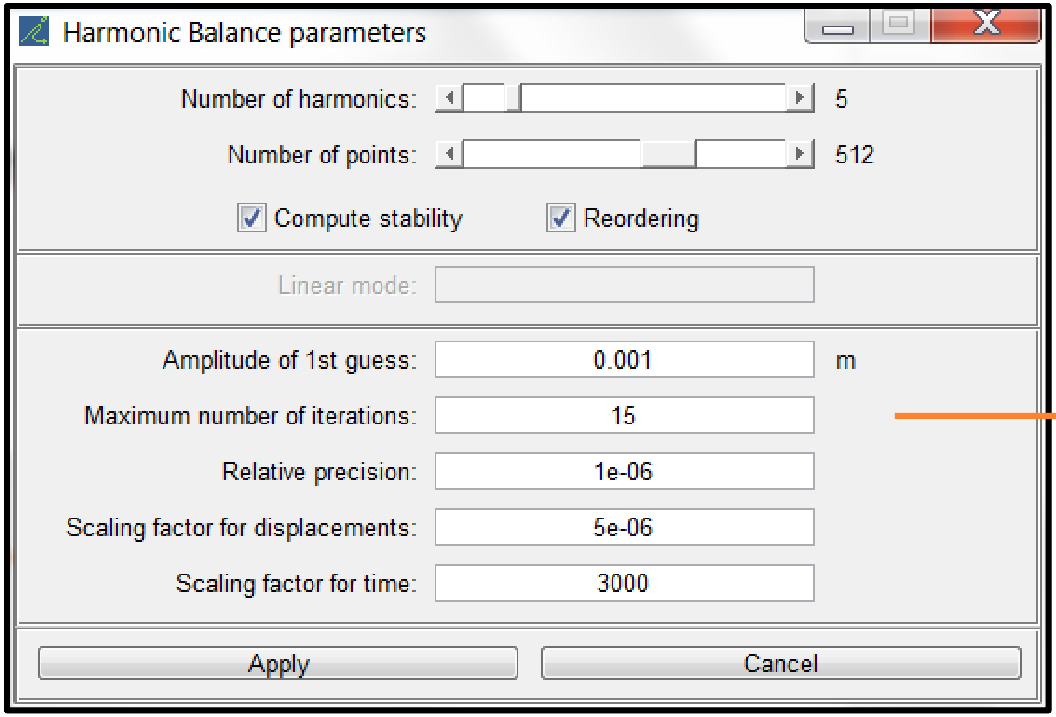
Stability parameters (see next lectures)

Harmonic Balance Parameters



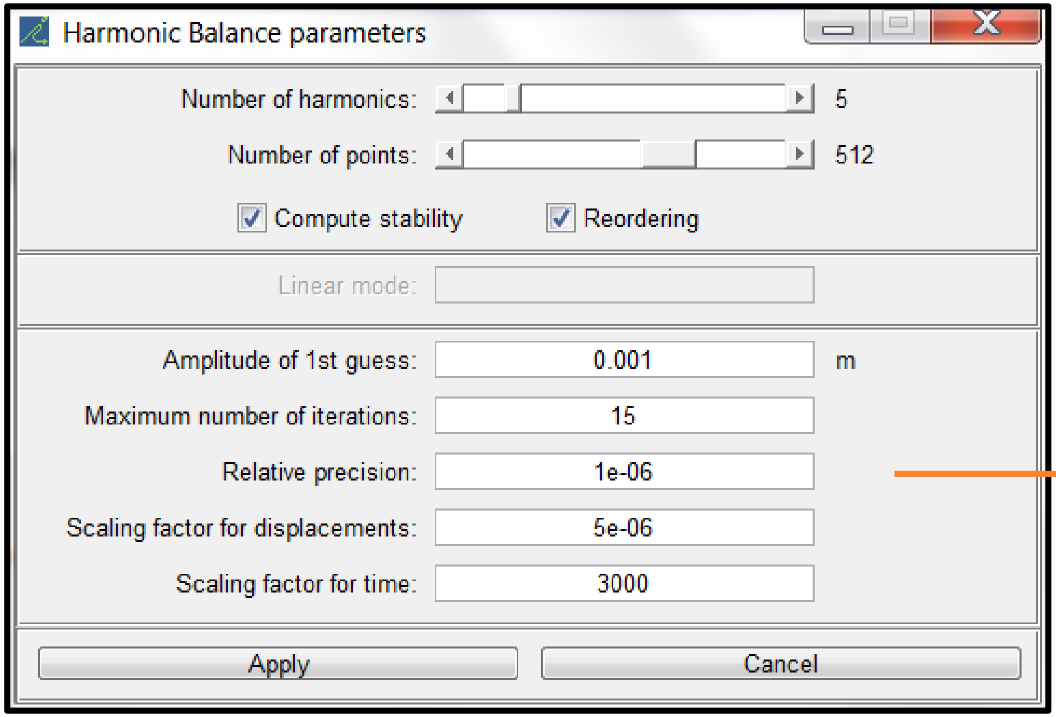
Amplitude of the sine series used as initial guess for all DOFs.

Harmonic Balance Parameters



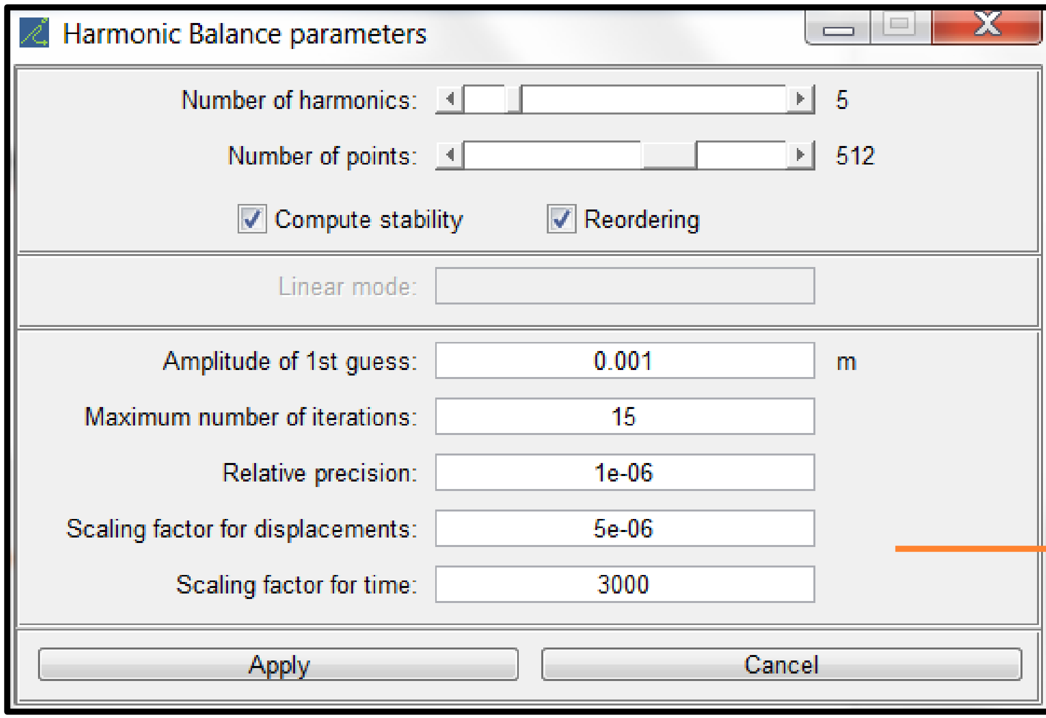
The Newton-Raphson procedure fails if this number of iterations is exceeded.

Harmonic Balance Parameters



The Newton-Raphson procedure stops if the relative error is smaller than this precision.

Harmonic Balance Parameters



Because the frequency (e.g., 30Hz = 188rad/s) and the amplitude (e.g., 0.001m) have different orders of magnitude, time and displacements have to be rescaled to avoid ill conditioning.

Harmonic Balance Method: In Summary

PROS	CONS
Efficient	Less accurate
Harmonic coefficients available	Many harmonics are sometimes required
Filtering	

Adaptations of the method improve its performance (alternating time-frequency method, chain rule, …) – not discussed here.

Computation of Periodic Solutions: In Summary

Periodic solutions of nonlinear structures can be computed with time-domain (shooting, orthogonal collocation) or frequency-domain method (harmonic balance).

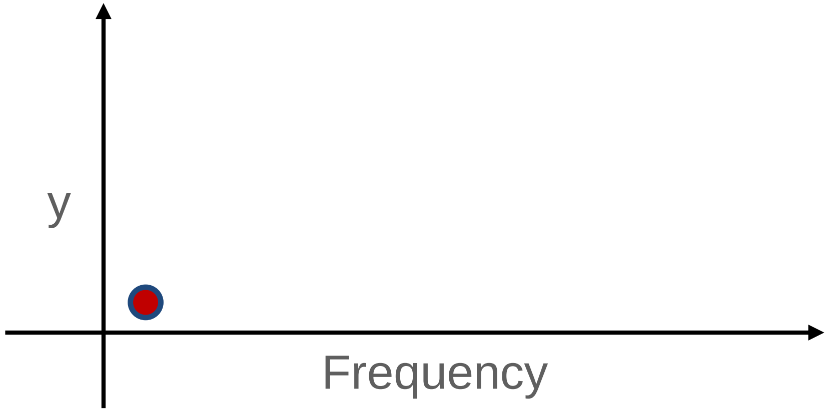
The differences between these methods lie in their accuracy and execution time.

Without adaptation, however, the harmonic balance:

- Fails at computing periodic reponses in severe nonlinear regimes (need for continuation procedure).
- Does not indicate if the solutions can be observed experimentally or not (need for a stability analysis).

Computation of Branches of Periodic Solutions

Computation of Branches of Periodic Solutions

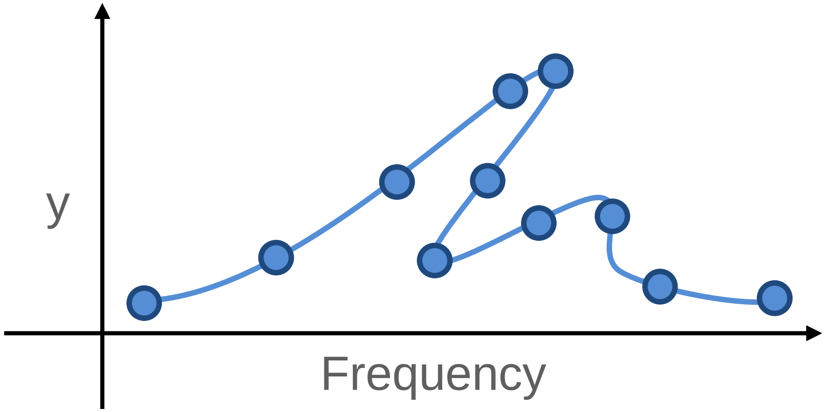


Numerical methods to go from

single periodic solutions…

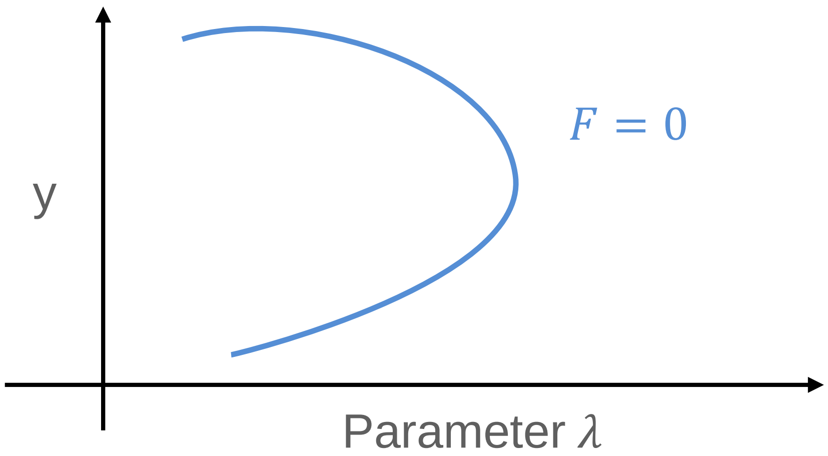
… to a

branch of periodic solutions



Mathematical Definition of a Branch of Periodic Solutions

Let us consider a function $\boldsymbol{F}: \boldsymbol{R}^{n+1} \rightarrow \boldsymbol{R}^{n}$. A branch is a set of solutions $\boldsymbol{F}(\boldsymbol{x}, \lambda) = \boldsymbol{0}$, where $\boldsymbol{x}$ are the state variables and λ is a system parameter.



The branch can be represented in a 2D plane through the evolution of a representative variable $y = y(\boldsymbol{x})$ w.r.t. λ.

(For a more formal definition, see the implicit function theorem.)

Types of Branch

In this course, the branch is composed by solutions of the harmonic balance equation for a nonlinear system:

$$\mathbf{h}(\mathbf{z}, \omega): \mathbf{R}^{n_z+1} \rightarrow \boldsymbol{R}^{n_z}$$

ω → Frequency (= system parameter)

$\mathbf{z}$ → Fourier coefficients (= state variables)

- Nonlinear Frequency Response Curves

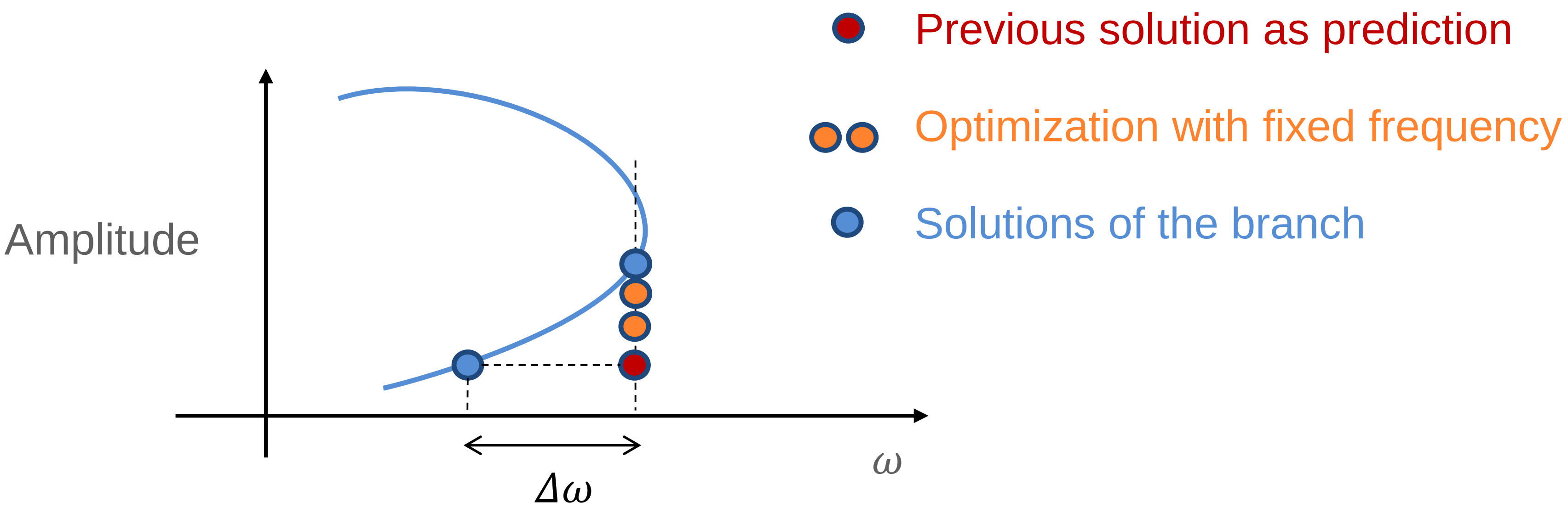
  Forced and damped system

- Nonlinear Normal Modes

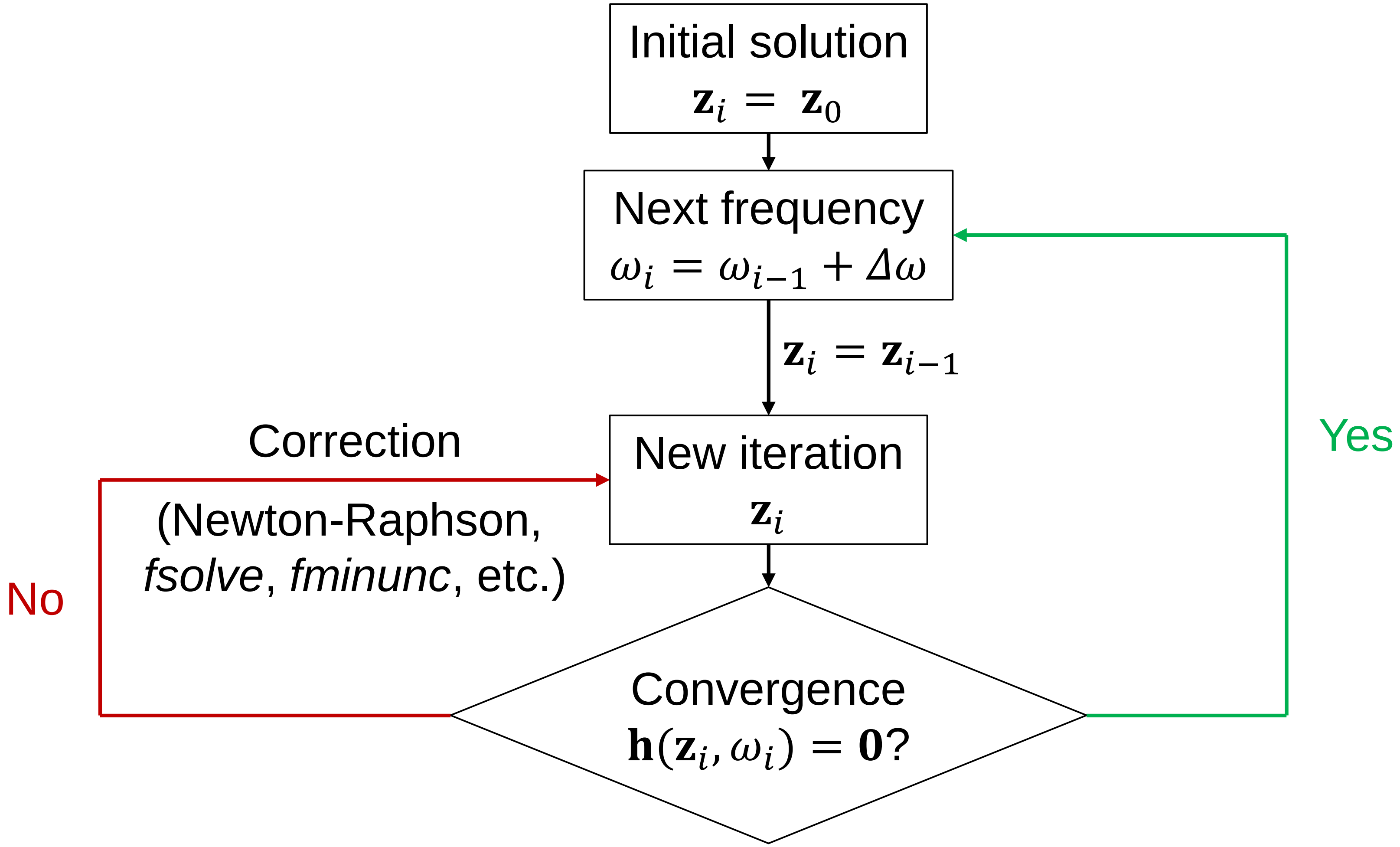
  Unforced and undamped system

Sequential Continuation – A Straightforward Approach

Increase the period and use the previously computed periodic solution as an initial guess for the next computation.

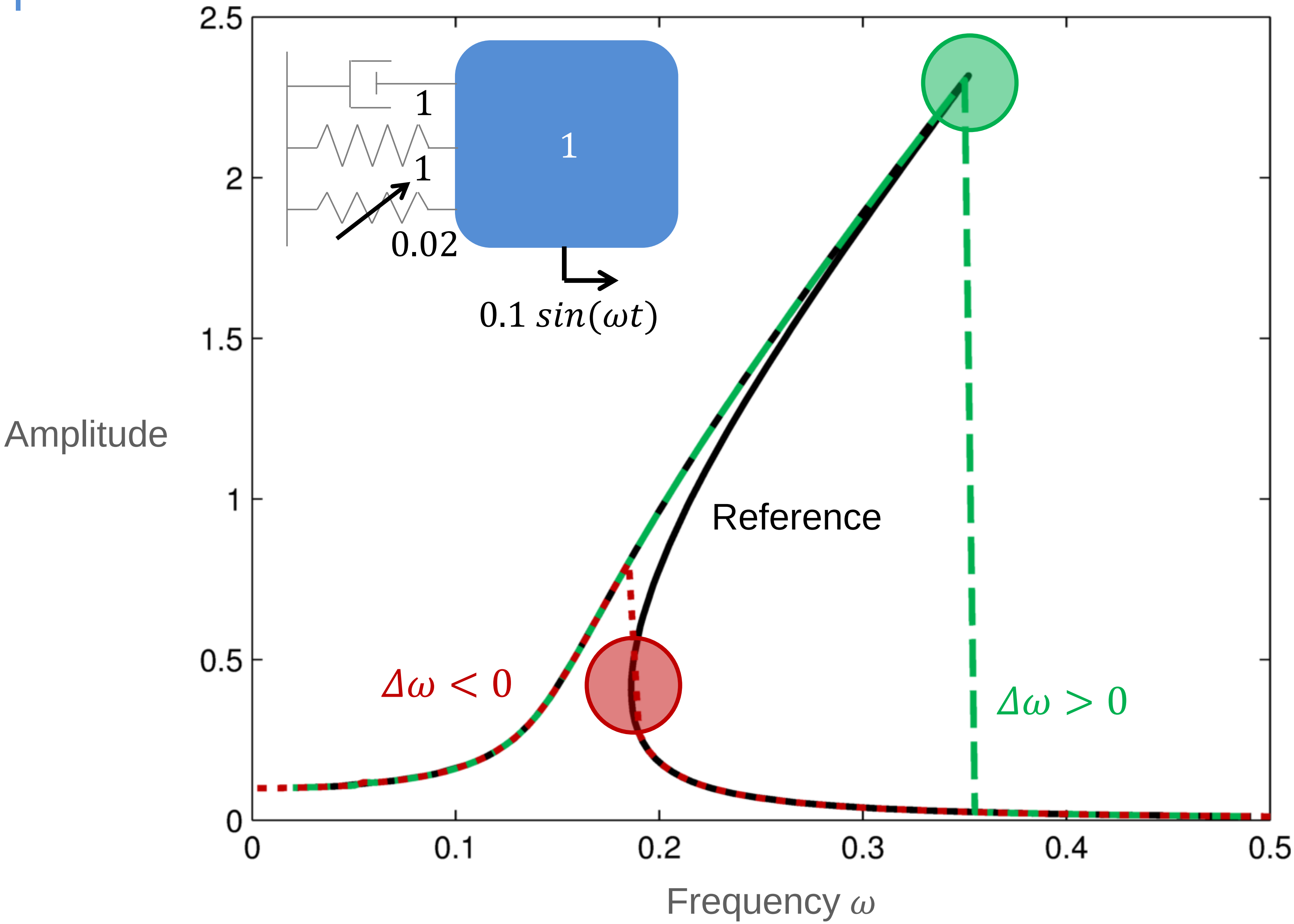


Sequential Continuation – Scheme



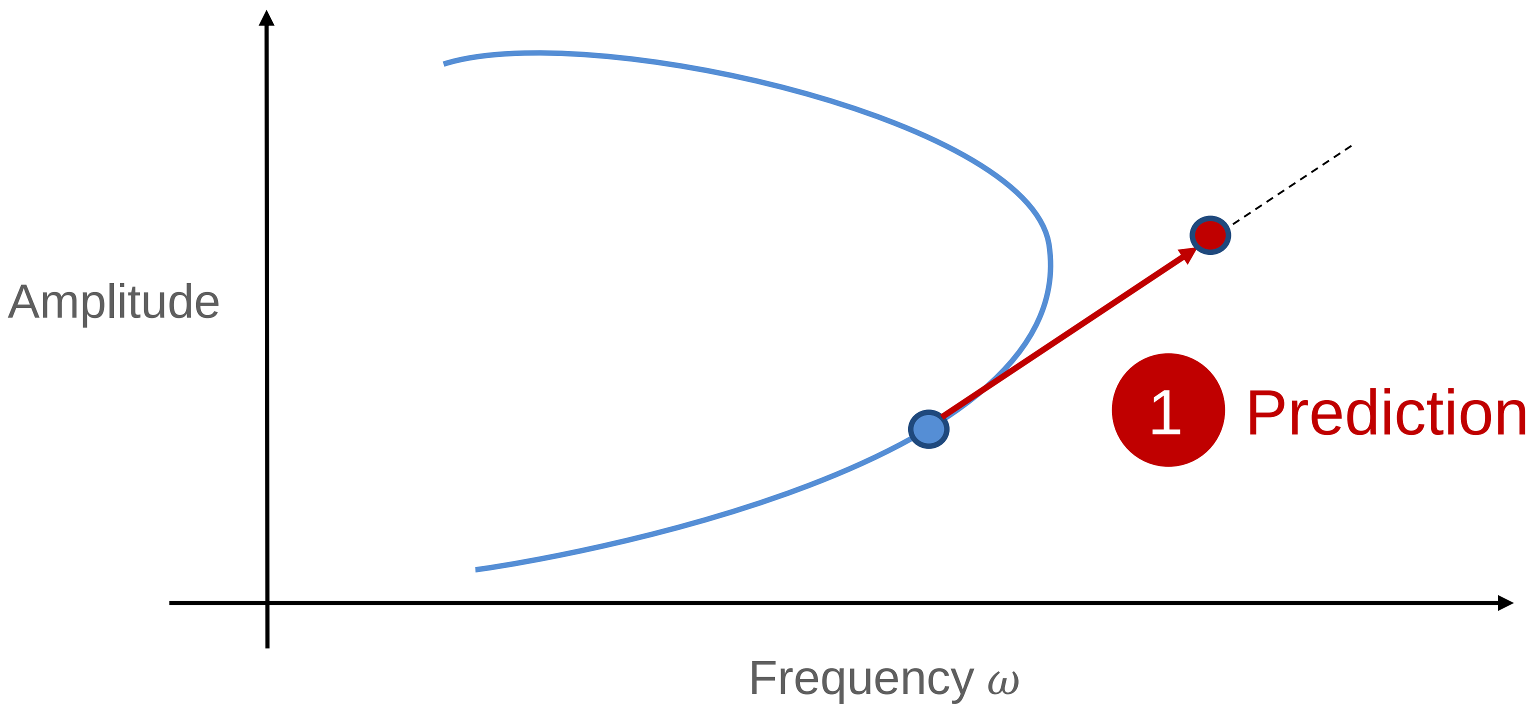
If HB method is already implemented, sequential continuation is programmed in a few lines.

Sequential Continuation Fails at Turning Points



A New Continuation Scheme

In order to pass through turning points, both the state $\mathbf{z}$ and the parameter ω should vary. This is done through a 2-step procedure:



A New Continuation Scheme

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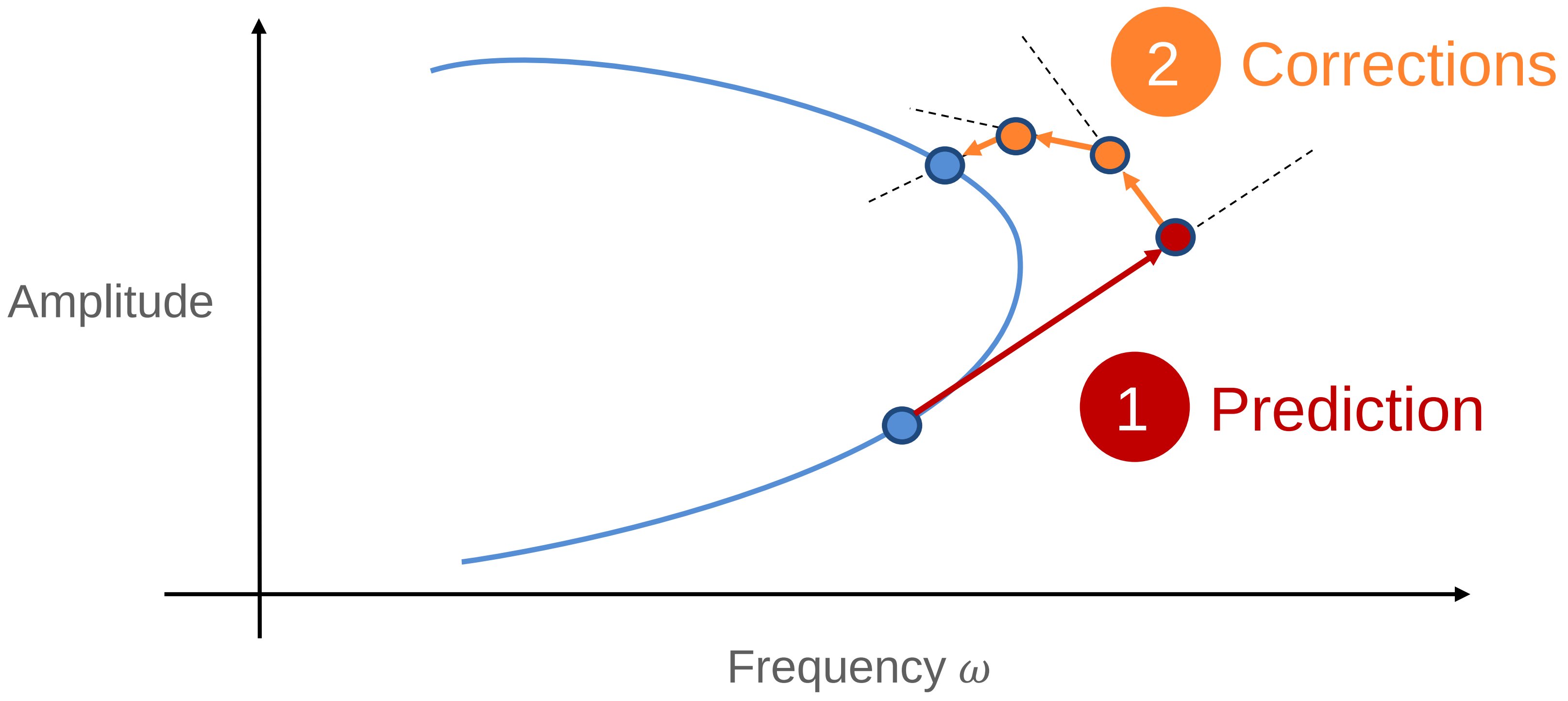


Predictor Step

Different predictors can be considered:

$$\mathbf{X}_{pred}^{i} = \mathbf{X}^{i-1} + s^{i}\mathbf{t}^{i}$$

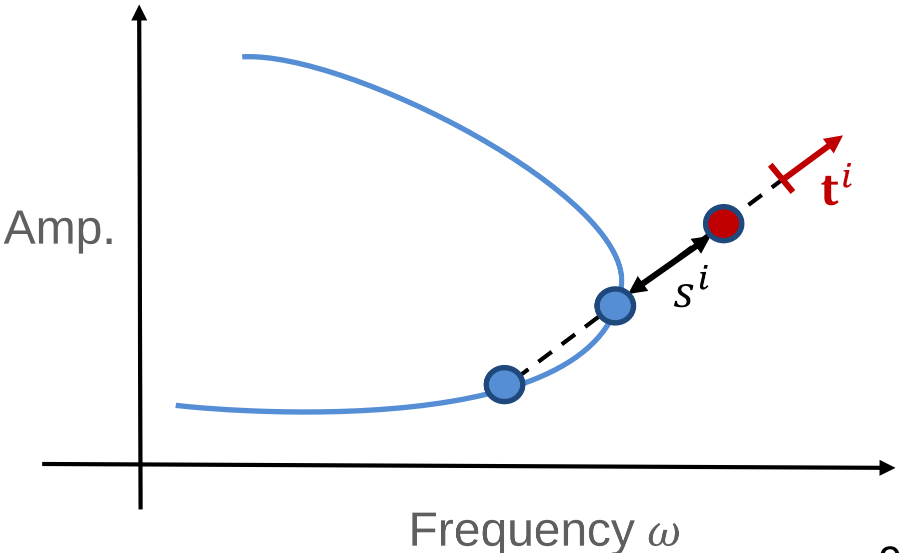
Unit vector ($\mathbf{t}^{i}$)

Stepsize (s^{i})

where $\mathbf{X} = [\mathbf{z} \quad \omega]^T$ denotes the unknown vector.

- Secant predictor

$$\mathbf{t}^{i} = \frac{\mathbf{X}^{i-1} - \mathbf{X}^{i-2}}{\|\mathbf{X}^{i-1} - \mathbf{X}^{i-2}\|}$$



Predictor Step

Different predictors can be considered:

$$\mathbf{X}^{i}_{pred} = \mathbf{X}^{i-1} + s^{i}\mathbf{t}^{i}$$

$\mathbf{t}^{i}$ → Unit vector

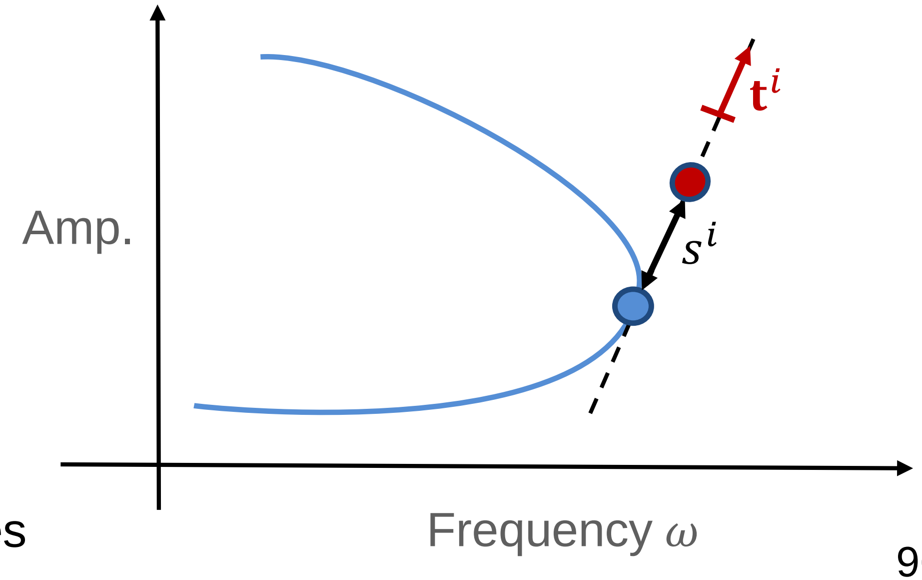
s^{i} → Stepsize

Tangent predictor

$$\begin{bmatrix} \mathbf{h_z} & \mathbf{h}_\omega \\ {\boldsymbol{t}^{i-1}}^{T} & \end{bmatrix} \mathbf{t}^{i} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}$$

${\boldsymbol{t}^{i-1}}^{T}$ → Imposed orientation

More accurate but requires the computation of the Jacobian matrices



Corrector Step

We are looking for a solution of $\mathbf{h}(\mathbf{z}, \omega) = \mathbf{0}$, with

$$\mathbf{h}(\mathbf{z}, \omega): \mathbf{R}^{n_z+1} \rightarrow \boldsymbol{R}^{n_z}$$

Two possibilities:

- Fix the parameter ω and only optimize $\mathbf{z}$.

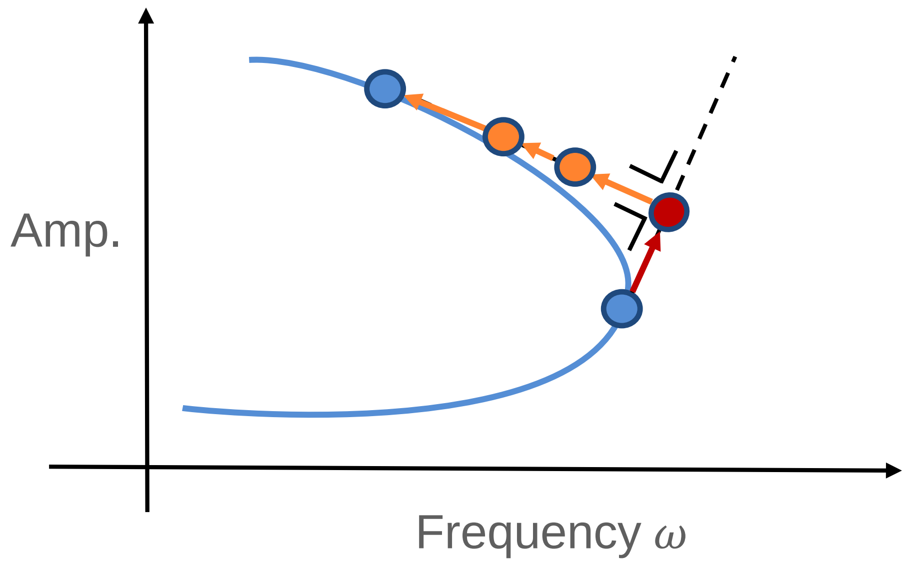
  Cf. sequential continuation

- Add another equation to the system.

  Pseudo-arclength and Moore-Penrose schemes

Pseudo-arclength Corrector Step

With the pseudo-arclength scheme, a solution is sought in the perpendicular direction w.r.t. the prediction.



Pseudo-arclength Corrector Step

With the pseudo-arclength scheme, a solution is sought in the perpendicular direction w.r.t. the prediction.

$$\mathbf{z}^i_{(j+1)} = \mathbf{z}^i_{(j)} + \boldsymbol{\Delta}\mathbf{z}_{(j)}$$
$$\omega^i_{(j+1)} = \omega^i_{(j)} + \Delta\omega_{(j)}$$

i = continuation iteration
(j) = corrector iteration

with

Taylor series expansion

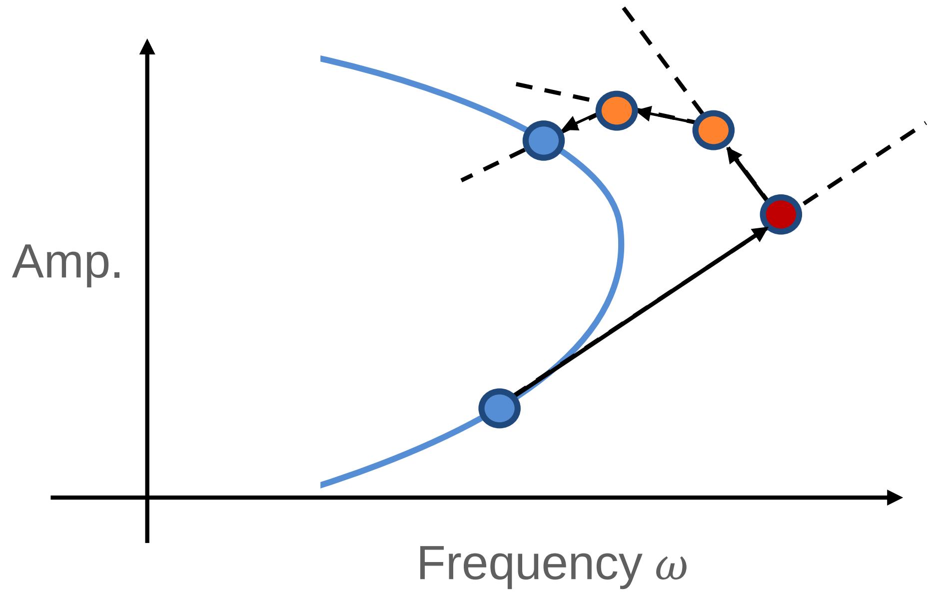
$$\begin{bmatrix} \mathbf{h}_{\mathbf{z}}\left(\mathbf{z}^i_{(j)}, \omega_{(j)}\right) & \mathbf{h}_{\omega}\left(\mathbf{z}^i_{(j)}, \omega_{(j)}\right) \\ {\mathbf{t}^i_{\mathbf{z}}}^T & \mathrm{t}^i_{\omega} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Delta}\mathbf{z}_{(\mathbf{j})} \\ \Delta\omega_{(j)} \end{bmatrix} = \begin{bmatrix} -\mathbf{h}\left(\mathbf{z}^i_{(j)}, \omega_{(j)}\right) \\ 0 \end{bmatrix}$$

Orthogonality condition

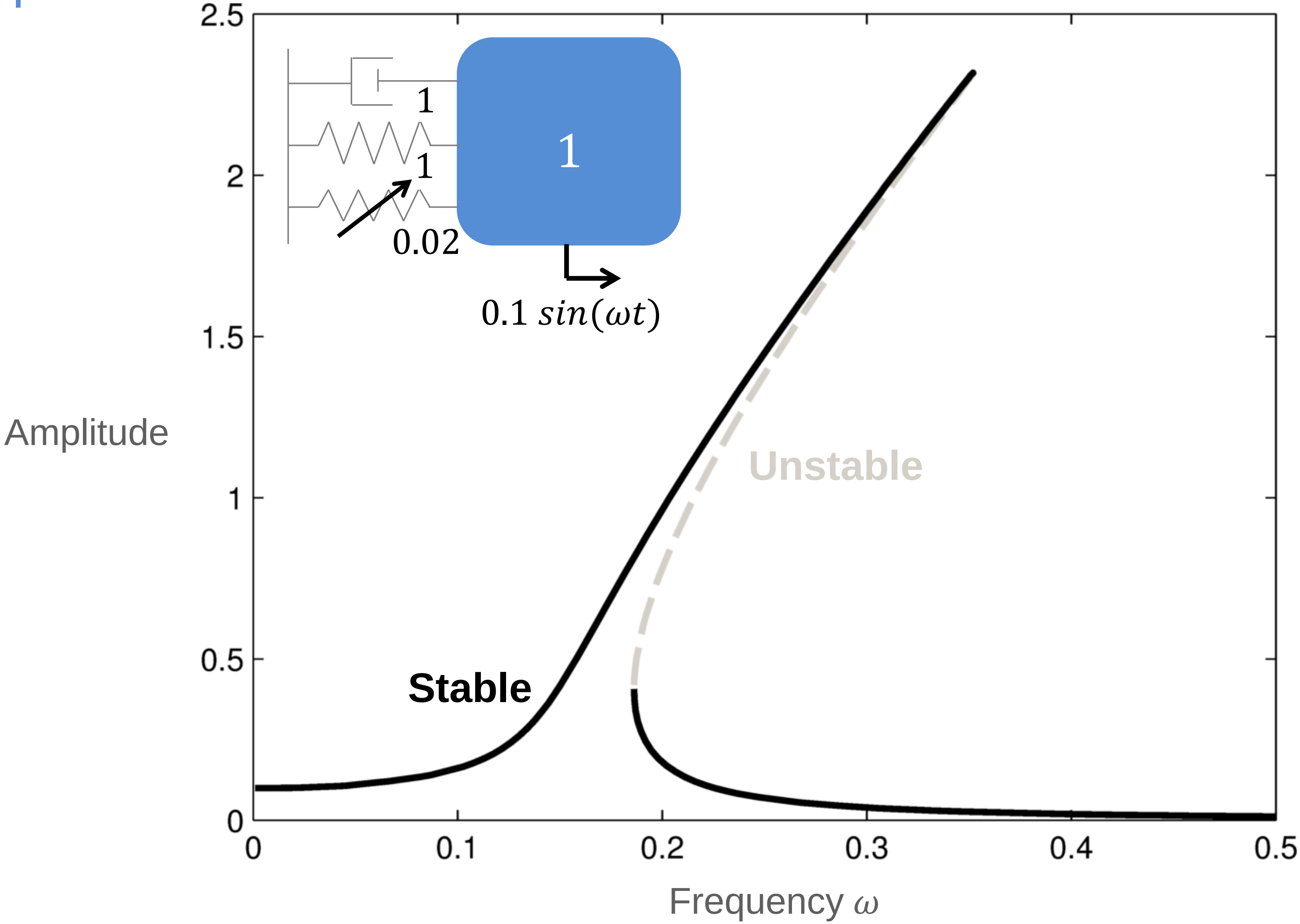
Other Correctors

Other corrector definitions can also be used.

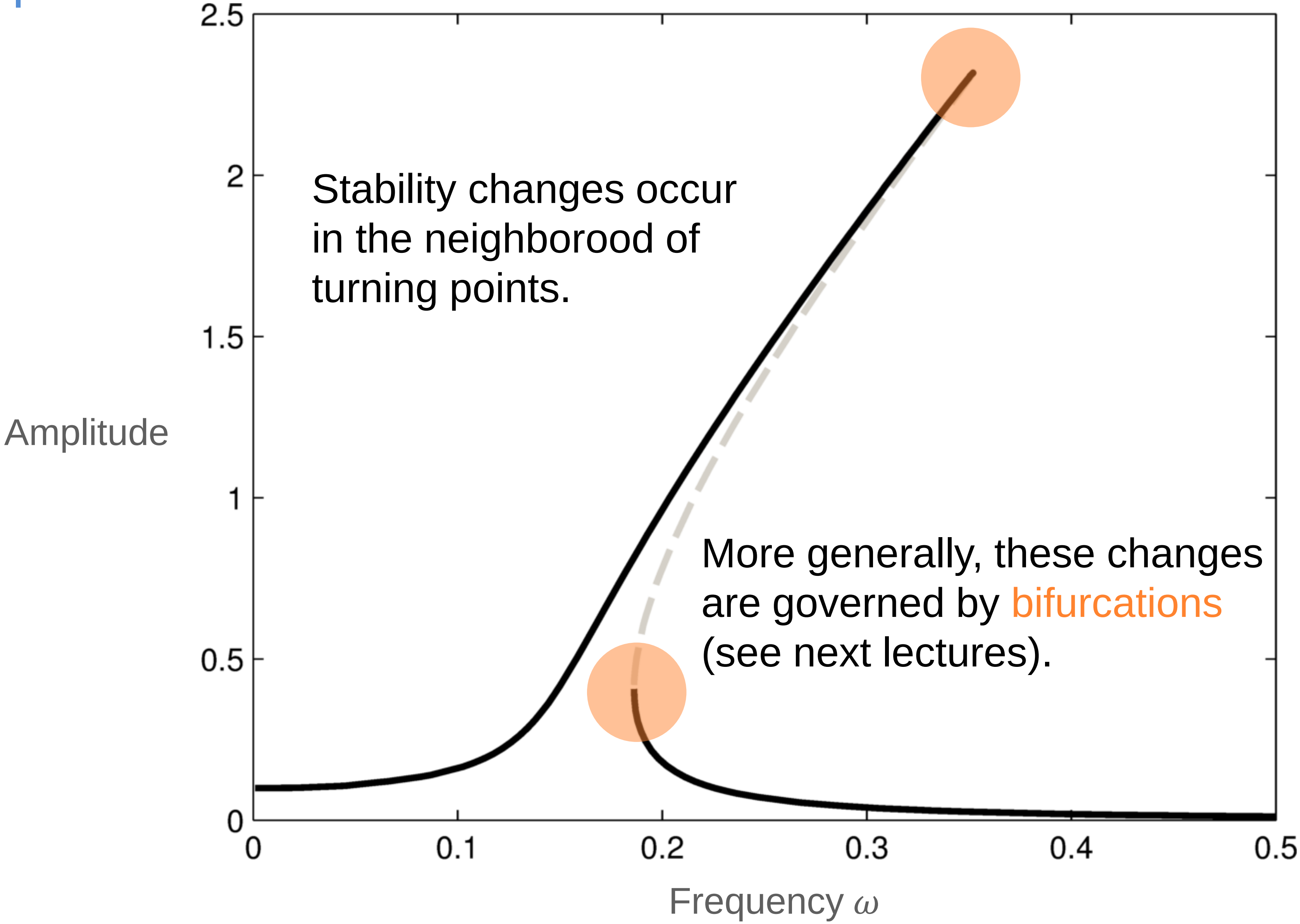
With the Moore-Penrose scheme for instance, the correction direction is updated at each corrector step.



Stability of Periodic Solutions Varies Along the Branch



Periodic Solutions Can be Stable or Unstable



Stability changes occur in the neighborood of turning points.

More generally, these changes are governed by bifurcations (see next lectures).

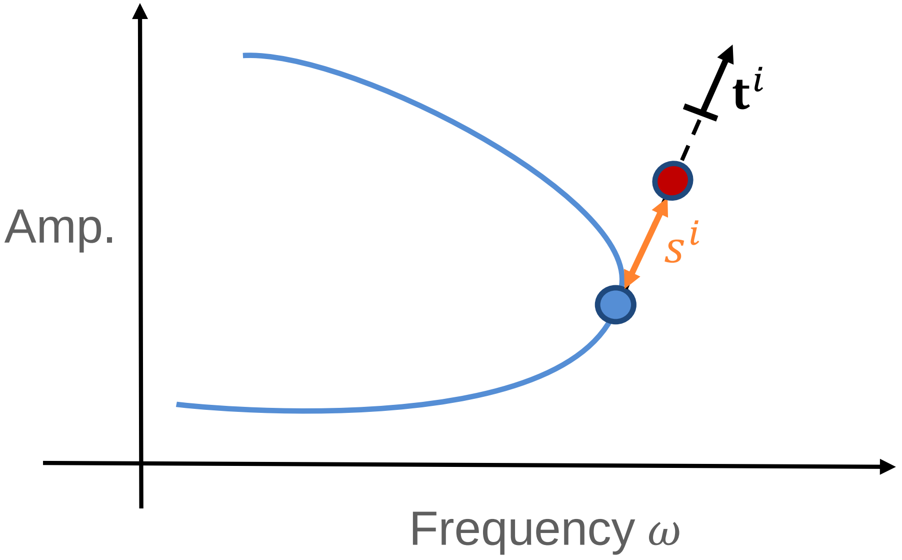
Influence of the Stepsize

Stepsize is a key parameter for the continuation procedure.

$$\boldsymbol{X}^{i}_{pred} = \boldsymbol{X}^{i-1} + s^{i}\mathbf{t}^{i}$$

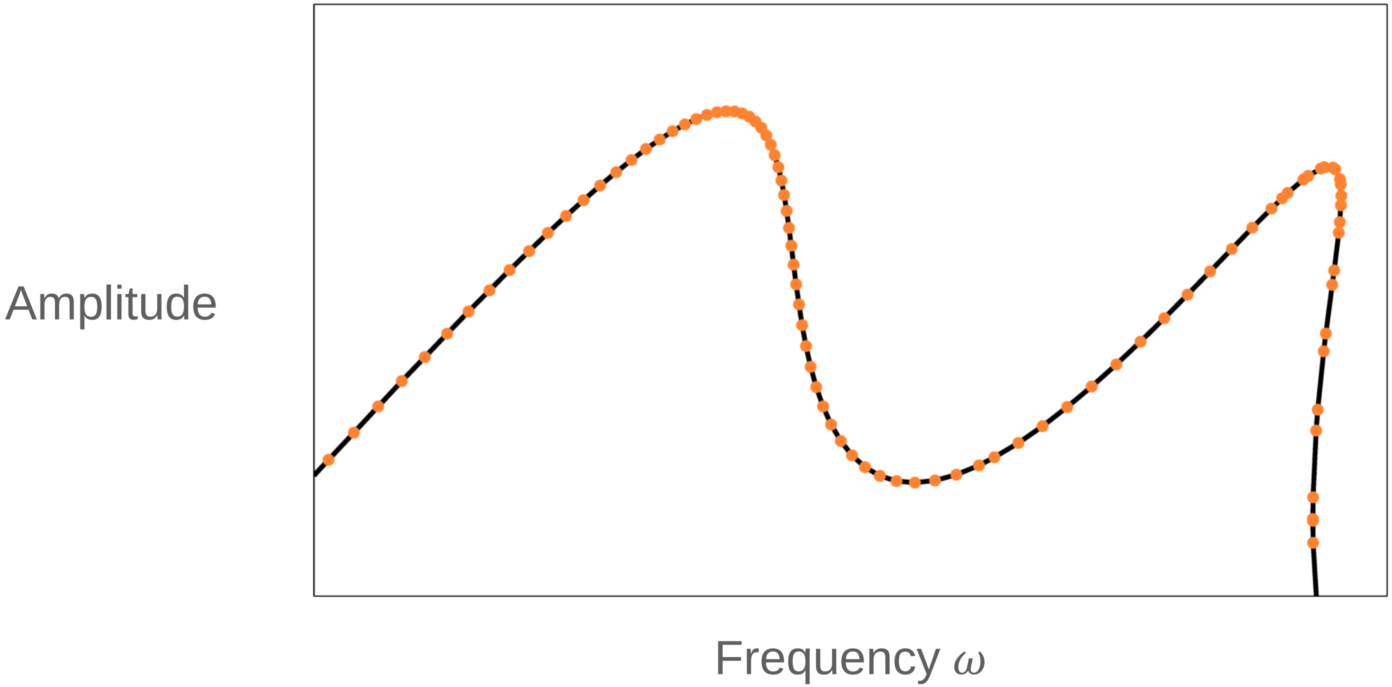
$\mathbf{t}^{i}$ → Unit vector

s^{i} → Stepsize



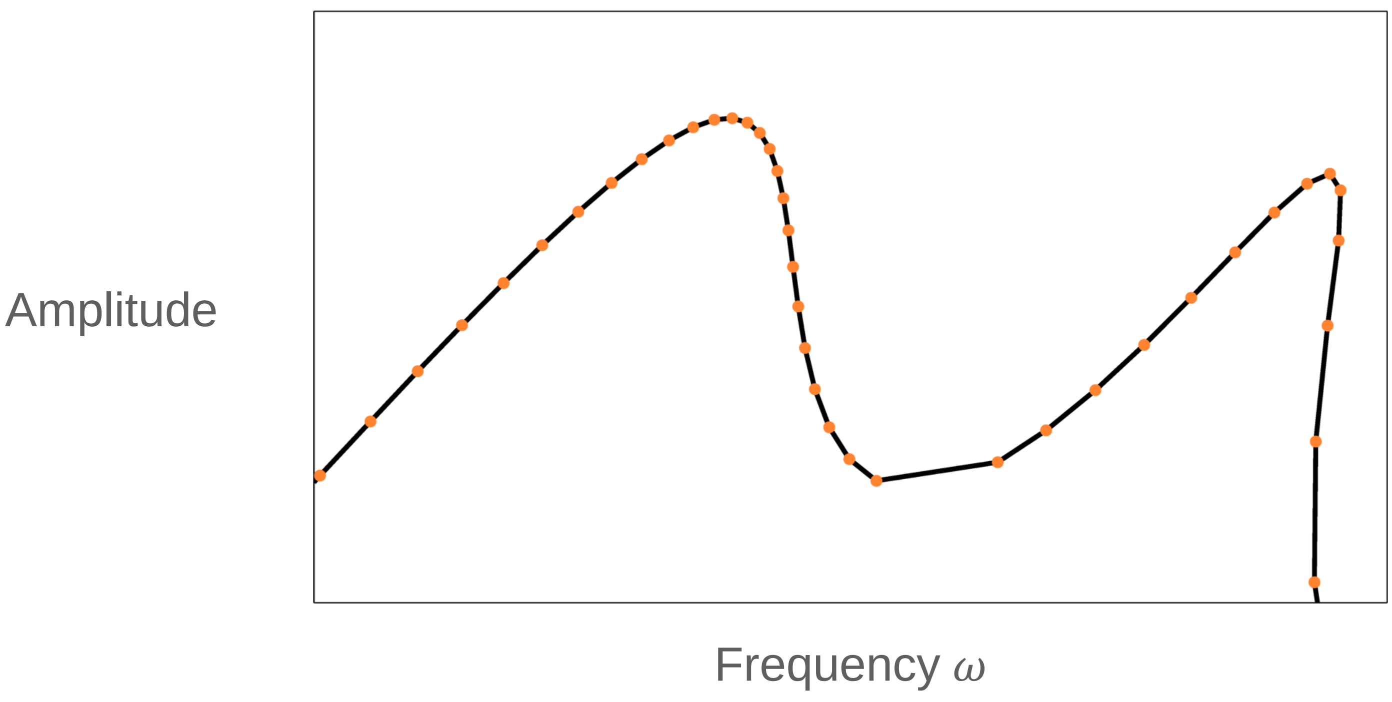
Small Stepsize

- ✓ Small number of corrections
- ✓ Good resolution for the branch
- ✗ Slow continuation procedure



Large Stepsize

- ✓ Fast continuation procedure
- ✗ Large number of corrections
- ✗ Poor resolution for the branch



Stepsize Strategy

- Fixed stepsize

$$s^i = \text{constant}$$

- Adaptative stepsize

$$s^i = \frac{M^*}{M} s^{i-1}$$

where M is the iteration number for the current correction, and M^* is the optimal iteration number.

Influence of Harmonic Balance Parameters

With the harmonic balance method, the displacements are approximated with Fourier series.

Number of harmonics

$$\mathbf{x}(t) = \mathbf{c}_0^{\mathbf{x}} + \sum_{k=1}^{N_H} \left(\mathbf{s}_k^{\mathbf{x}} \sin(k\omega t) + \mathbf{c}_k^{\mathbf{x}} \cos(k\omega t) \right)$$

$\mathbf{z}$

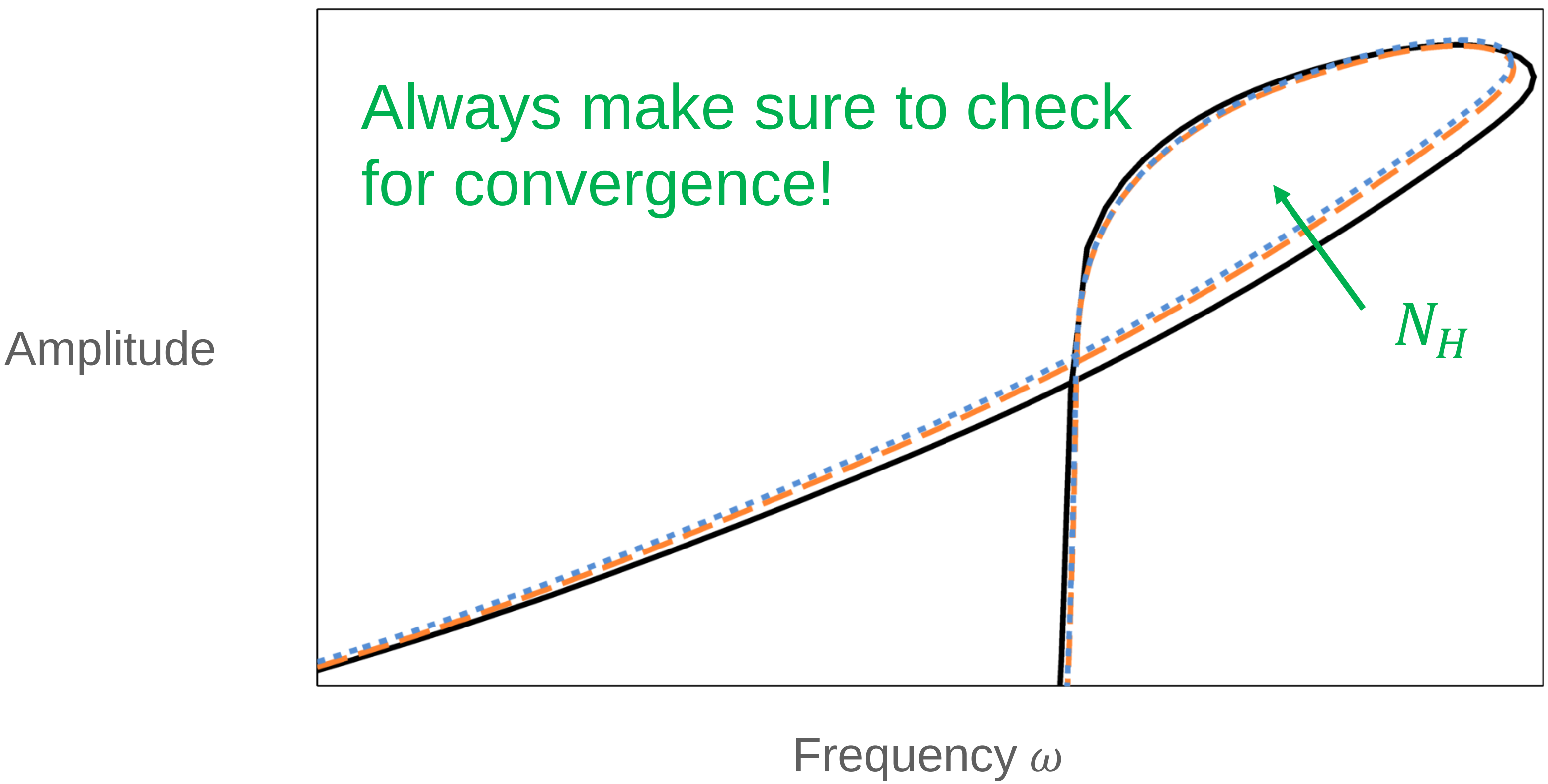
Fourier coefficients $\mathbf{z}$ are computed with the discrete Fourier transform:

$$\mathbf{z} = \mathbf{\Gamma}^{+}(N)\tilde{\mathbf{x}}$$

Number of time samples (power of 2)

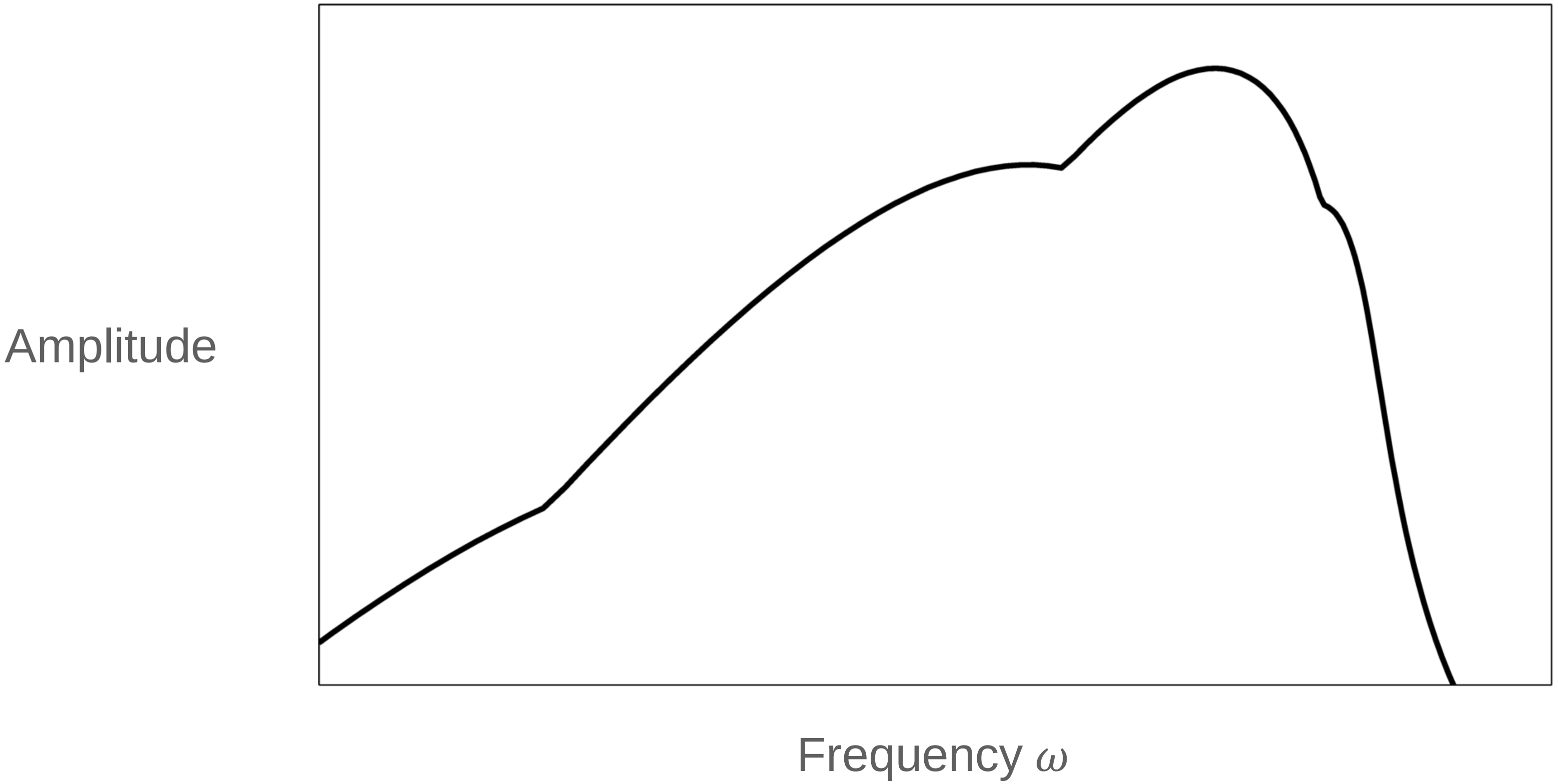
Influence of the Number of Harmonics N_H

N_H has a direct influence on the accuracy of the harmonic balance solution, and hence on the accuracy of the branch.



Influence of the Number of Time Samples N

N has a direct influence on the discrete Fourier transform, and the accuracy of the alternating frequency/time-domain method.



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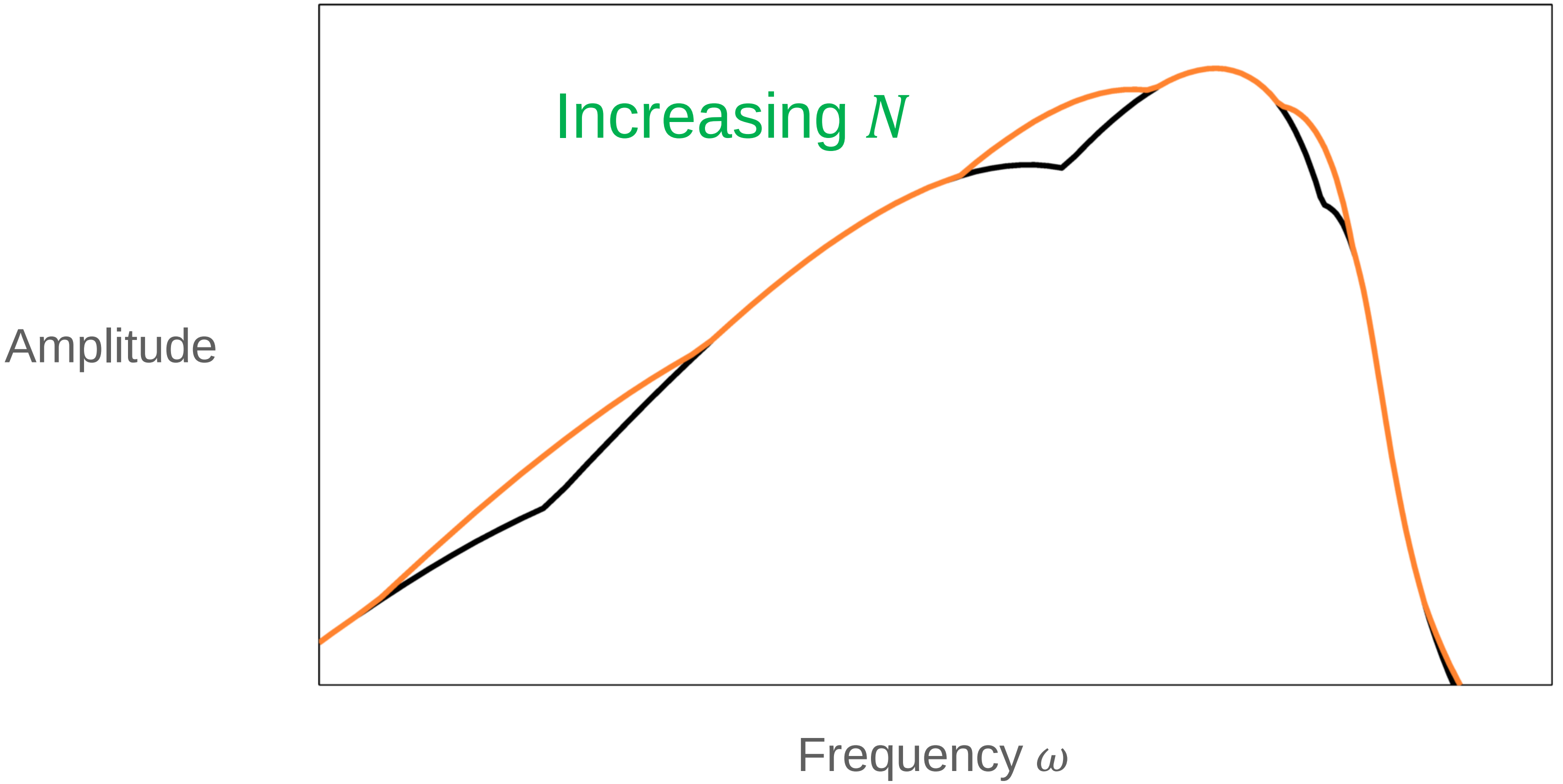

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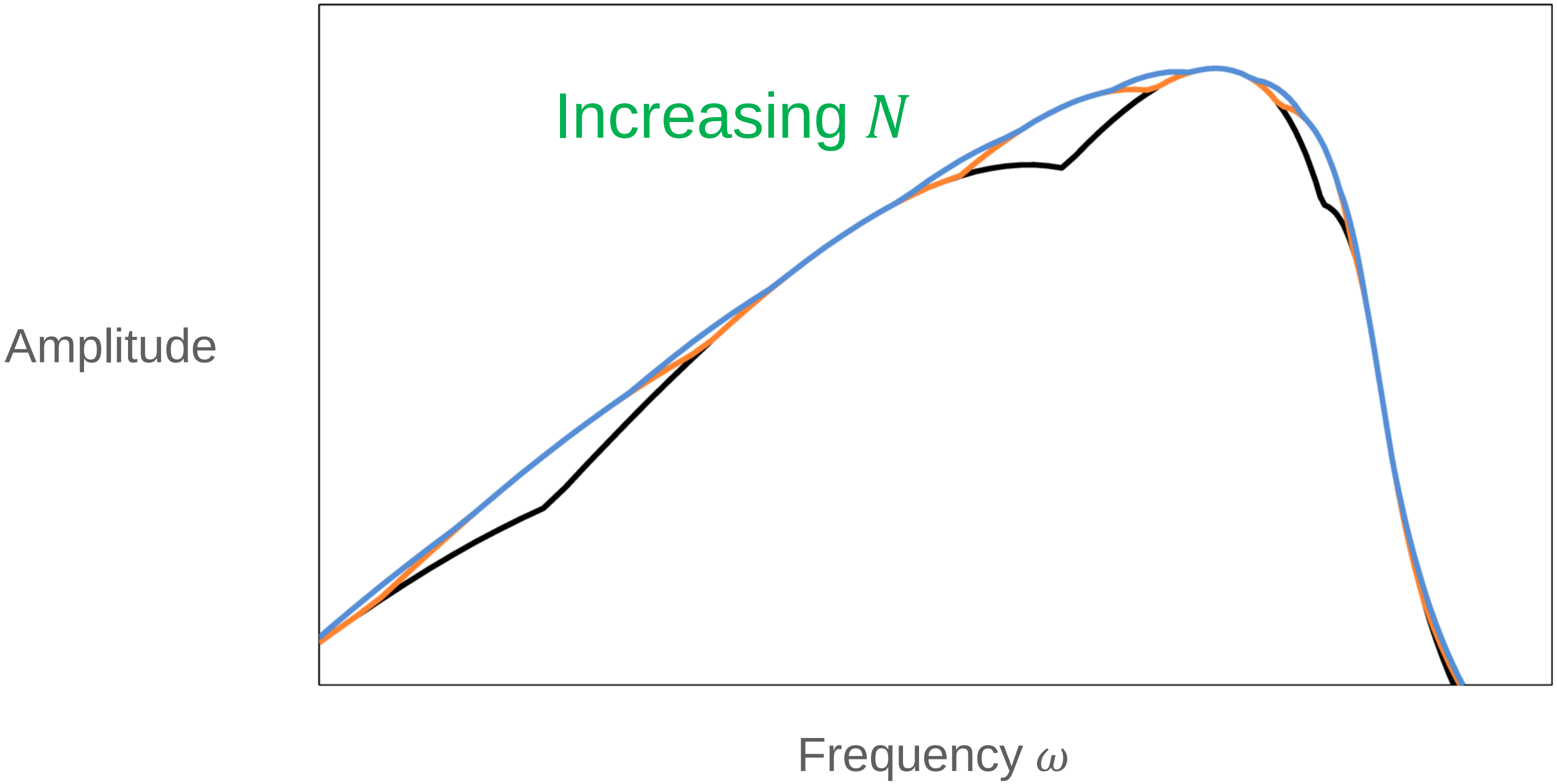

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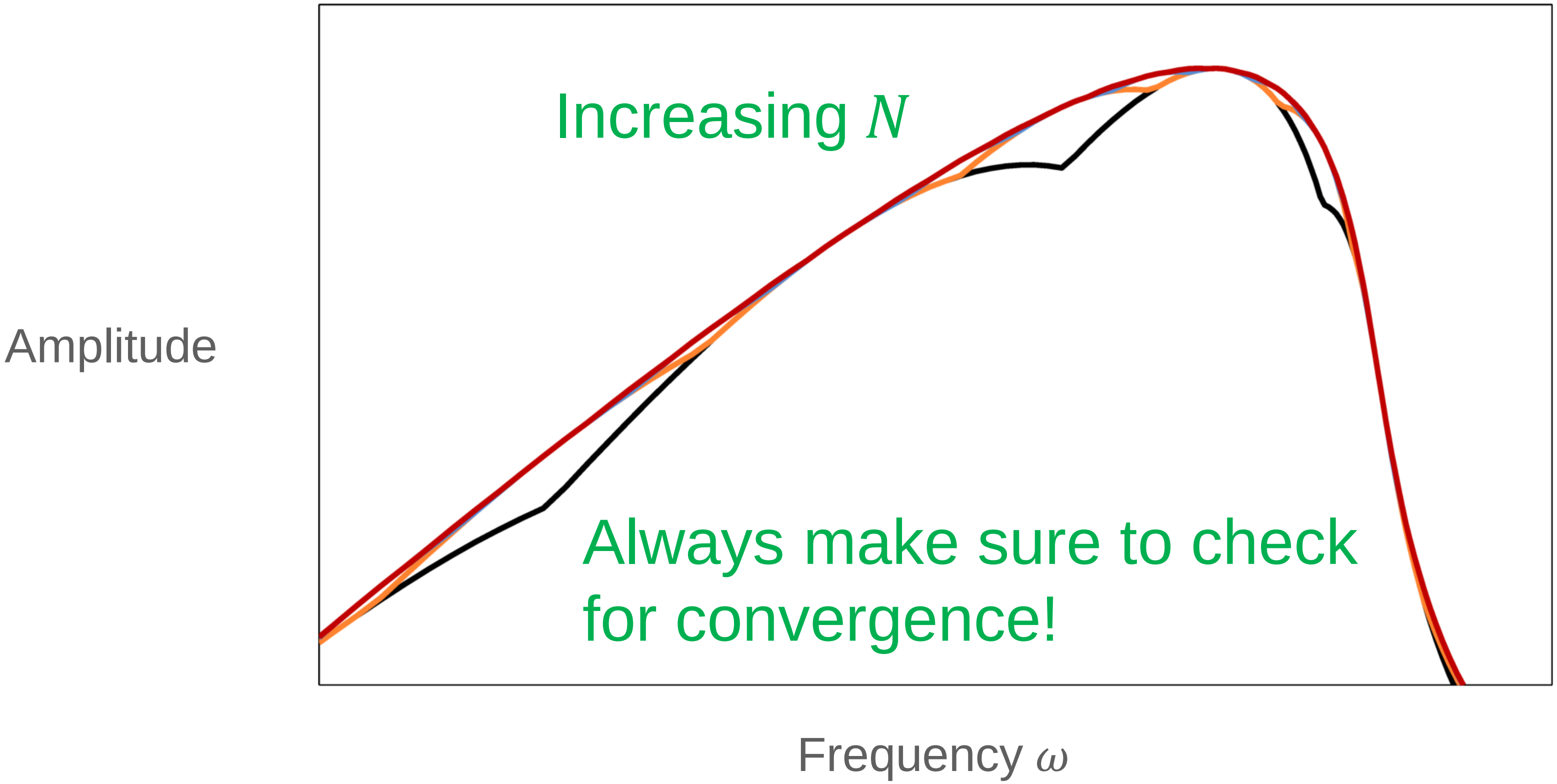

Continuation: In Summary

Sequential continuation can be easily implemented to represent the evolution of the periodic solutions w.r.t. to the frequency ω but it fails at turning points.

Continuation schemes based on predictor/corrector steps give the evolution of the periodic solutions in both stable and unstable regions.

HB and continuation parameters have to be carefully selected to ensure accuracy and good resolution of the branches.

Further Readings

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