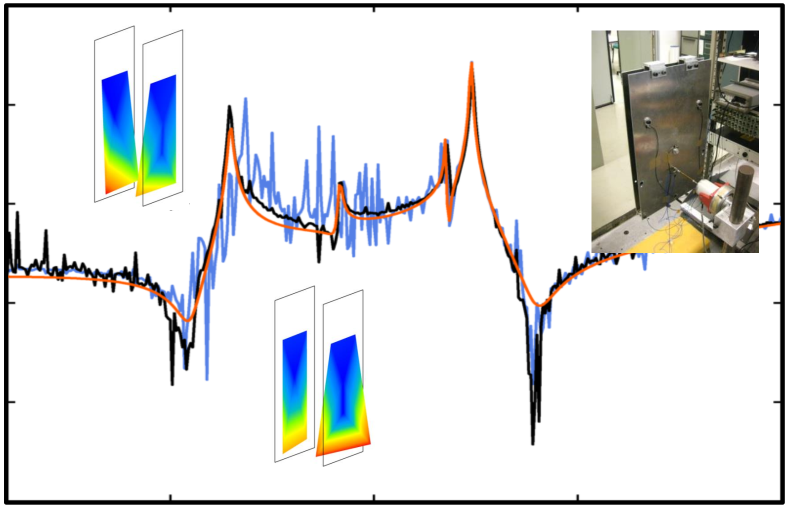
Nonlinear Vibrations of Aerospace Structures

University of Liège, Belgium

L09 Parameter Estimation

- Restoring force surface
- State-space models
- Subspace methods
- FNSI method



Ghislain Raze | g.raze@uliege.be
Jean-Philippe Noël

Nonlinear System Identification: a Three-Step Process

1 Detection

Do I observe nonlinear effects? *Yes.*

Should I build a nonlinear model? *Yes.*

2 Characterisation

Where is the nonlinearity located? *At the joint.*

What is the underlying physics? *Dry friction.*

How to model its effects? $f_{nl}(q, \dot{q}) = c\ sign(\dot{q})$.

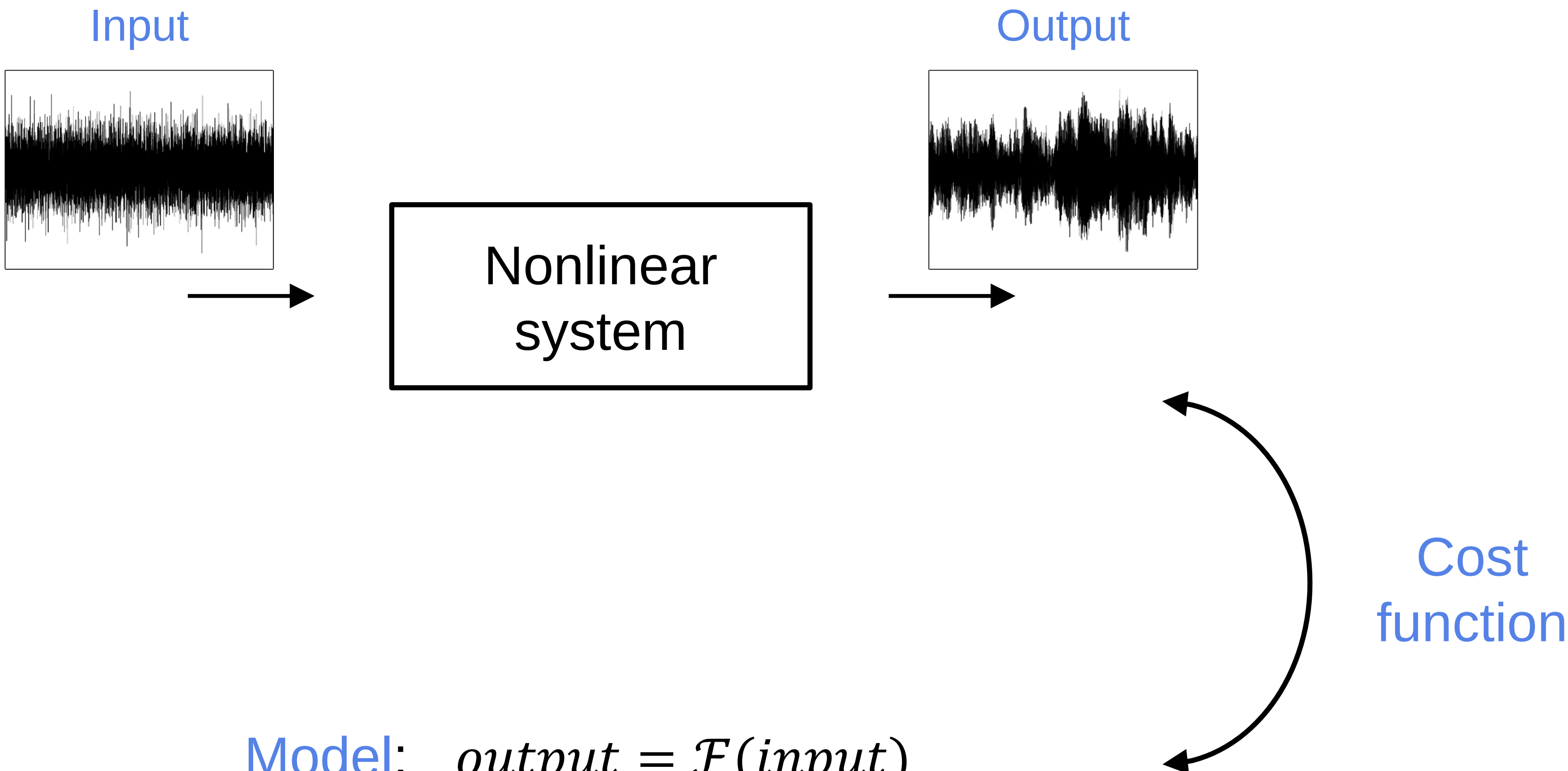
3 Estimation

Model parameters? $c = 5.47$. **This lecture**

How uncertain are they? $c = \mathcal{N}(5.47, 1)$.

Information vs. complexity

The Three Basic Ingredients in NL Parameter Estimation



Model: $output = \mathcal{F}(input)$

Outline of Lecture 9

The restoring force surface method.

Parameter estimation in linear structural models.

Nonlinear problem statement, state-space model structure and FNSI.

Numerical application to the SmallSat spacecraft.

Experimental application to a solar array structure.

The Restoring Force Surface Method: Sdof Case

Newton's second law for a single-degree-of-freedom oscillator reads

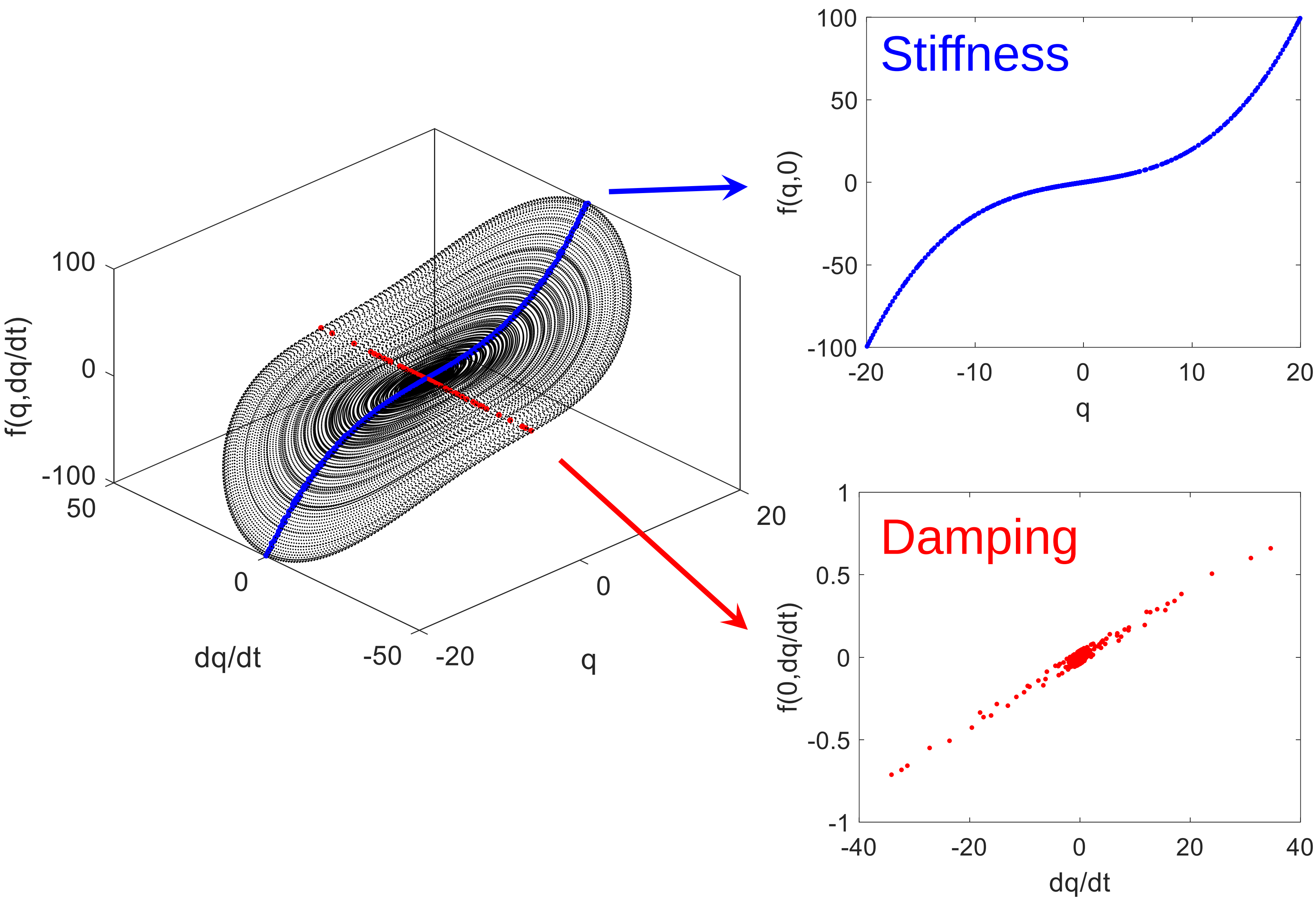
$$m\ddot{q} + f(q,\dot{q}) = f_{\text{ext}}$$

i.e.

$$f(q,\dot{q}) = f_{\text{ext}} - m\ddot{q}$$

If one knows m, and measures f_{ext} and either q, $\dot{q}$ or $\ddot{q}$, the restoring force $f(q,\dot{q})$ can be computed and visualized as a surface in the $(q,\dot{q},f)$ space. This is called the restoring force surface (RFS).

The RFS for a Duffing Oscillator



The RFS for Mdof Systems

For a multiple-degree-of-freedom system,

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{f}_{\text{ext}}$$

i.e.

$$\mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{f}_{\text{ext}} - \mathbf{M}\ddot{\mathbf{q}}$$

If one knows $\mathbf{M}$, and measures $\mathbf{f}_{\text{ext}}$ and either $\mathbf{q}$, $\dot{\mathbf{q}}$ or $\ddot{\mathbf{q}}$, the restoring force $\mathbf{f}(\mathbf{q}, \dot{\mathbf{q}})$ can be computed.

However, $\mathbf{f}$ cannot be plotted as a simple surface in general since it depends on $2N$ variables.

The RFS for Mdof Systems: Nonparametric Representation

Around the resonance of a mode,

$$\mathbf{q} \approx \boldsymbol{\phi}\eta$$

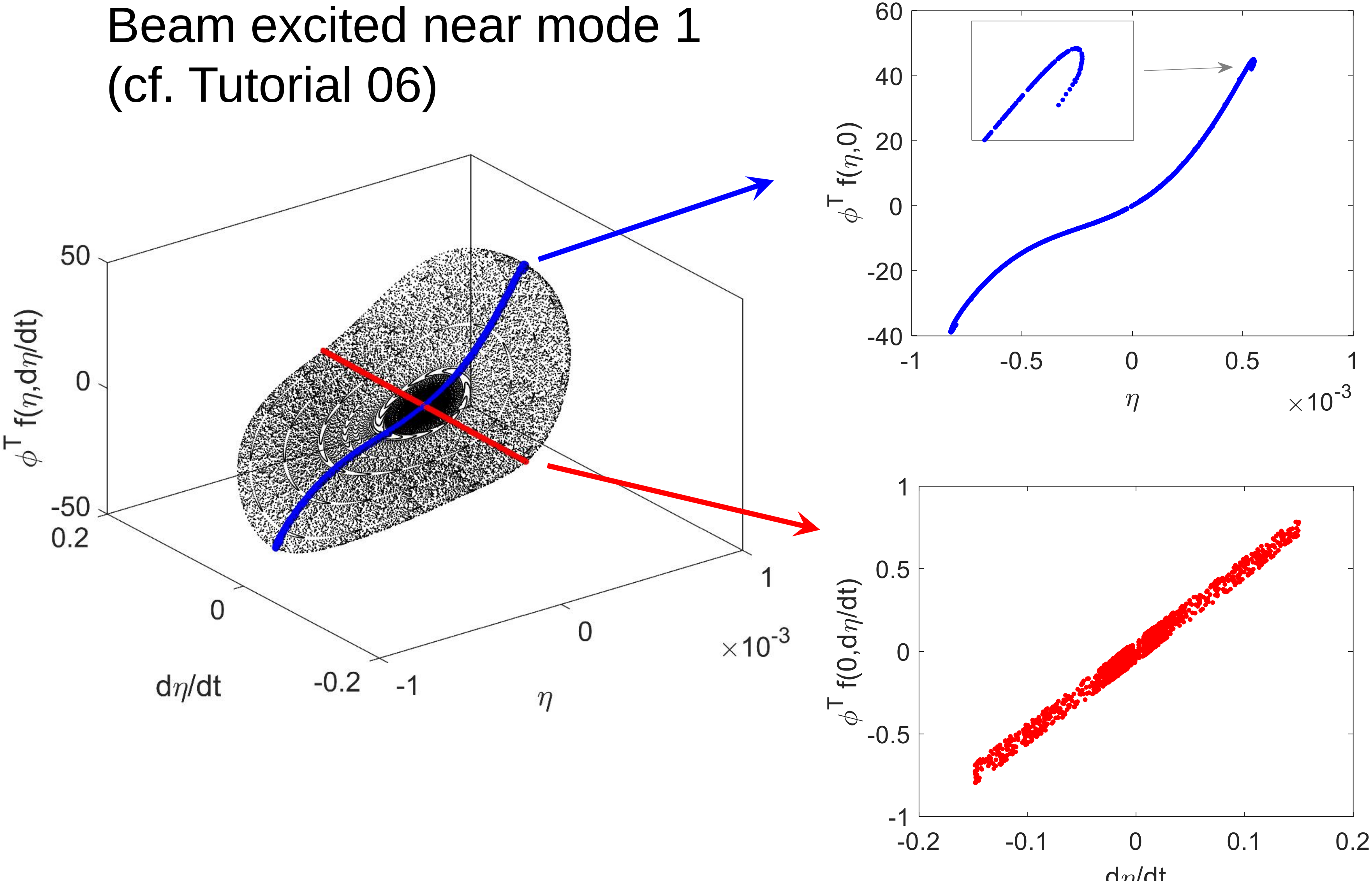
Projecting the equations of motion onto the mode shape,

$$\boldsymbol{\phi}^T\mathbf{f}(\boldsymbol{\phi}\eta, \boldsymbol{\phi}\dot{\eta}) \approx \boldsymbol{\phi}^T\mathbf{f}_{\text{ext}} - \boldsymbol{\phi}^T\mathbf{M}\boldsymbol{\phi}\ddot{\eta}$$

The (modal) restoring force $\boldsymbol{\phi}^T\mathbf{f}(\boldsymbol{\phi}\eta, \boldsymbol{\phi}\dot{\eta})$ can now be plotted as a surface in the $(\eta, \dot{\eta}, \boldsymbol{\phi}^T\mathbf{f})$ space.

The RFS with a Nonlinear Beam: Modal Force

Beam excited near mode 1
(cf. Tutorial 06)



The RFS for Mdof Systems: Justification for the ASM

The projection can be arbitrary. If we know the location of the nonlinearity, say, with a vector $\mathbf{l}$ such that

$$\mathbf{l} = [0, \dots, 0, \underset{i}{1}, 0, \dots, 0] \quad \text{or} \quad \mathbf{l} = [0, \dots, 0, \underset{i}{-1}, 0, \dots, 0, \underset{j}{1}, 0, \dots, 0]$$

we can project the equations using this vector

$$\mathbf{l}^T \mathbf{f}(\boldsymbol{\phi}\eta, \boldsymbol{\phi}\dot{\eta}) \approx \mathbf{l}^T \mathbf{f}_{\text{ext}} - \mathbf{l}^T \mathbf{M} \boldsymbol{\phi} \ddot{\eta}$$

Since $\mathbf{q} \approx \boldsymbol{\phi}\eta$, any dof is proportional to η and can be used as a coordinate for the RFS.

When $\mathbf{l}^{\mathrm{T}} \mathbf{f}_{\text{ext}} = 0$, this gives a more formal justification for the ASM.

The RFS for Mdof Systems: Parametric Representation

The RFS can be used for nonlinear parameter estimation if one assumes a functional form for $\mathbf{f}_{\mathrm{nl}}$ (obtained e.g. from nonlinear characterization)

$$\hat{\mathbf{f}}_{\mathrm{nl}}(\mathbf{q},\dot{\mathbf{q}}) = \sum_{i=1}^{M} k_i \mathbf{f}_i(\mathbf{q},\dot{\mathbf{q}})$$

assumed ($\mathbf{f}_i$), measured ($\mathbf{q},\dot{\mathbf{q}}$)

so

$$[\mathbf{f}_1(\mathbf{q},\dot{\mathbf{q}}) \quad \cdots \quad \mathbf{f}_M(\mathbf{q},\dot{\mathbf{q}})] \begin{bmatrix} k_1 \\ \vdots \\ k_M \end{bmatrix} = \mathbf{f}_{\mathrm{ext}} - \mathbf{M}\ddot{\mathbf{q}} - \mathbf{C}\dot{\mathbf{q}} - \mathbf{K}\mathbf{q}$$

The coefficients $k_1, \cdots, k_M$ can be estimated by fitting measurements.

The RFS for Mdof Systems: Least-Squares Fit

If one has Q measurement points at times $t_1, \ldots, t_Q$,

$$\begin{bmatrix} \mathbf{f}_1(\mathbf{q}(t_1), \dot{\mathbf{q}}(t_1)) & \cdots & \mathbf{f}_M(\mathbf{q}(t_1), \dot{\mathbf{q}}(t_1)) \\ & \vdots & \\ \mathbf{f}_1(\mathbf{q}(t_Q), \dot{\mathbf{q}}(t_Q)) & \cdots & \mathbf{f}_M(\mathbf{q}(t_Q), \dot{\mathbf{q}}(t_Q)) \end{bmatrix} \begin{bmatrix} k_1 \\ \vdots \\ k_M \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{f}_{\text{ext}}(t_1) - \mathbf{M}\ddot{\mathbf{q}}(t_1) - \mathbf{C}\dot{\mathbf{q}}(t_1) - \mathbf{K}\mathbf{q}(t_1) \\ \vdots \\ \mathbf{f}_{\text{ext}}(t_Q) - \mathbf{M}\ddot{\mathbf{q}}(t_Q) - \mathbf{C}\dot{\mathbf{q}}(t_Q) - \mathbf{K}\mathbf{q}(t_Q) \end{bmatrix}$$

and a least-squares solution is

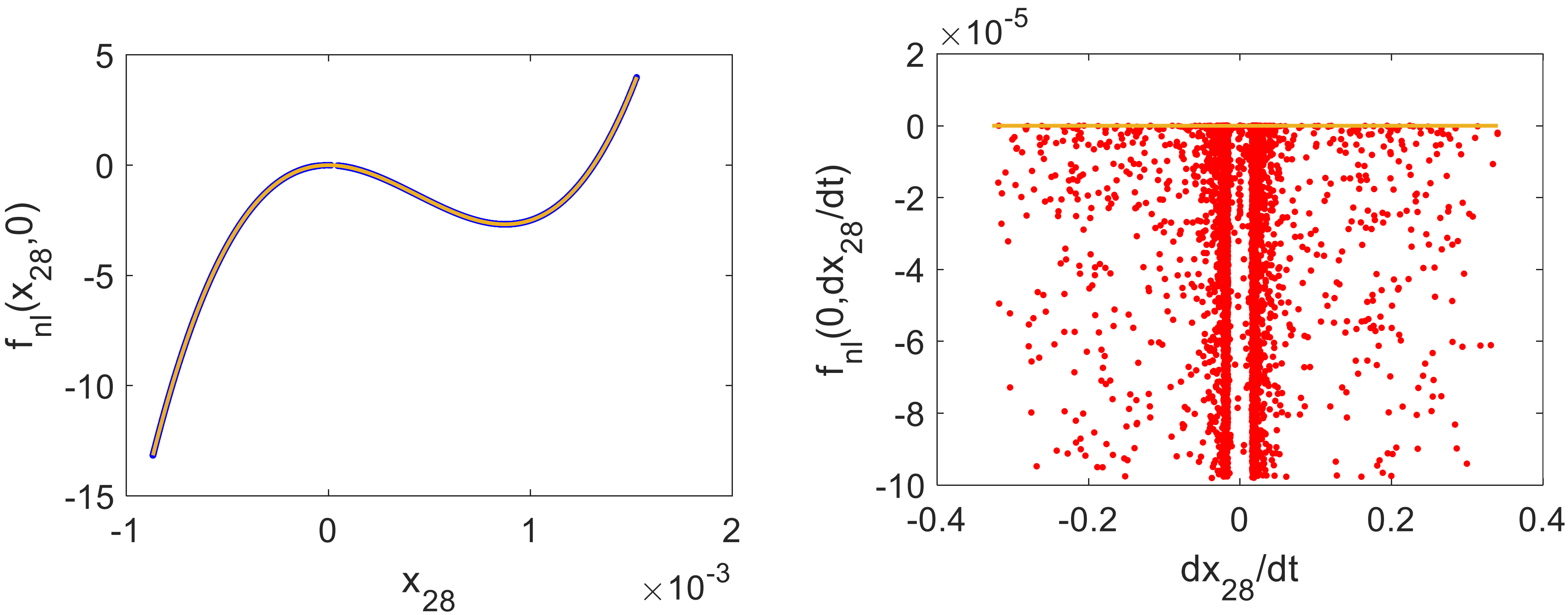
$$\begin{bmatrix} k_1 \\ \vdots \\ k_M \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1(\mathbf{q}(t_1), \dot{\mathbf{q}}(t_1)) & \cdots & \mathbf{f}_M(\mathbf{q}(t_1), \dot{\mathbf{q}}(t_1)) \\ & \vdots & \\ \mathbf{f}_1(\mathbf{q}(t_Q), \dot{\mathbf{q}}(t_Q)) & \cdots & \mathbf{f}_M(\mathbf{q}(t_Q), \dot{\mathbf{q}}(t_Q)) \end{bmatrix}^{\dagger}$$

$$\begin{bmatrix} \mathbf{f}_{\text{ext}}(t_1) - \mathbf{M}\ddot{\mathbf{q}}(t_1) - \mathbf{C}\dot{\mathbf{q}}(t_1) - \mathbf{K}\mathbf{q}(t_1) \\ \vdots \\ \mathbf{f}_{\text{ext}}(t_Q) - \mathbf{M}\ddot{\mathbf{q}}(t_Q) - \mathbf{C}\dot{\mathbf{q}}(t_Q) - \mathbf{K}\mathbf{q}(t_Q) \end{bmatrix}$$

The RFS with a Nonlinear Beam

For the nonlinear beam, if one assumes the correct nonlinearities, one retrives the correct coefficients from measurements.

$$\hat{f}_{nl}(x) = 8 \times 10^9 x^3 - 1.05 \times 10^7 x^2$$



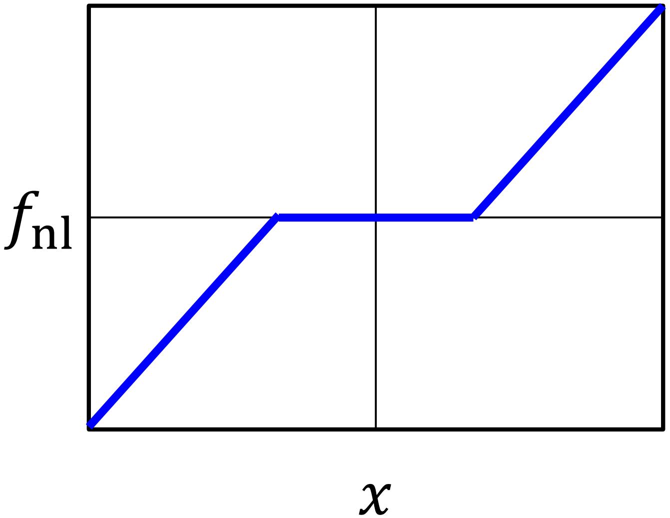
System with a Trilinear Stiffness

Consider the trilinear oscillator

$$\ddot{x} + 0.01\dot{x} + x + f_{\text{nl}}(x) = f_{\text{ext}}(t)$$

with

$$f_{\text{nl}}(x) = \begin{cases} 1 + x, & x \leq -1 \\ 0, & -1 < x < 1 \\ x - 1, & x \geq 1 \end{cases}$$



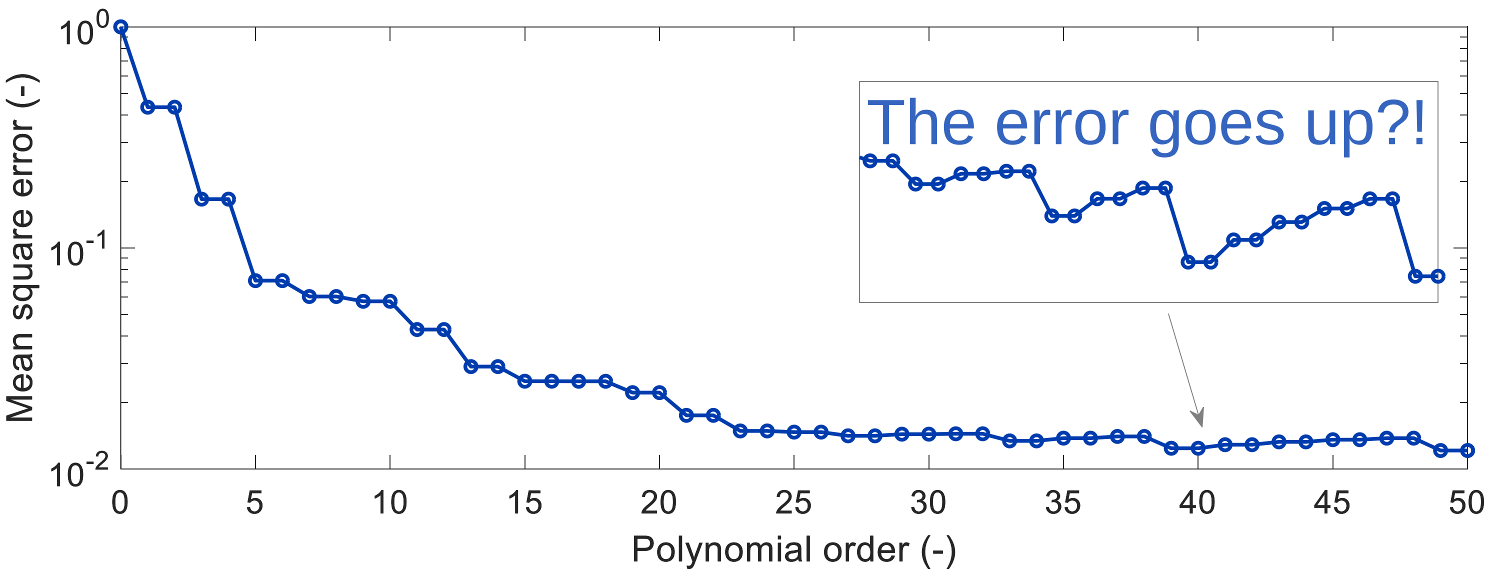
What happens if we do not know this functional form a priori?

Let Us Try a Polynomial Fit

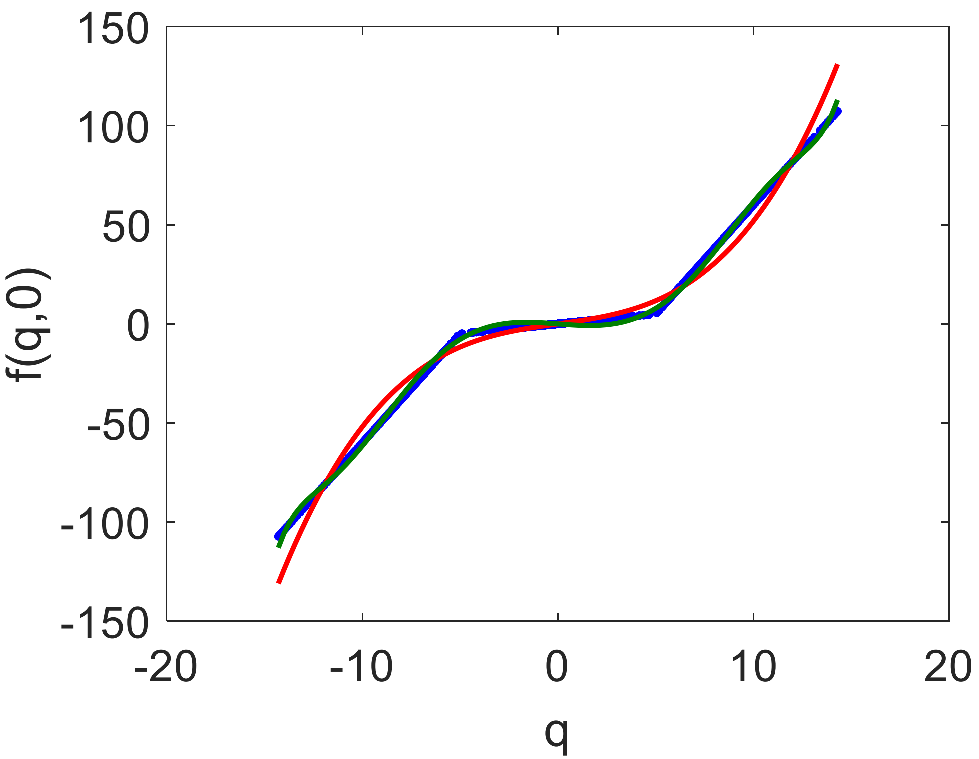
If we apply the RFS with polynomials of increasing order and consider the mean squared error (MSE)

$$\mathrm{MSE} = \frac{\left|\mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}) - \hat{\mathbf{f}}(\mathbf{q}, \dot{\mathbf{q}})\right|}{\left|\mathbf{f}(\mathbf{q}, \dot{\mathbf{q}})\right|}$$

we obtain



A Polynomial Fit of Order 9 Looks Alright

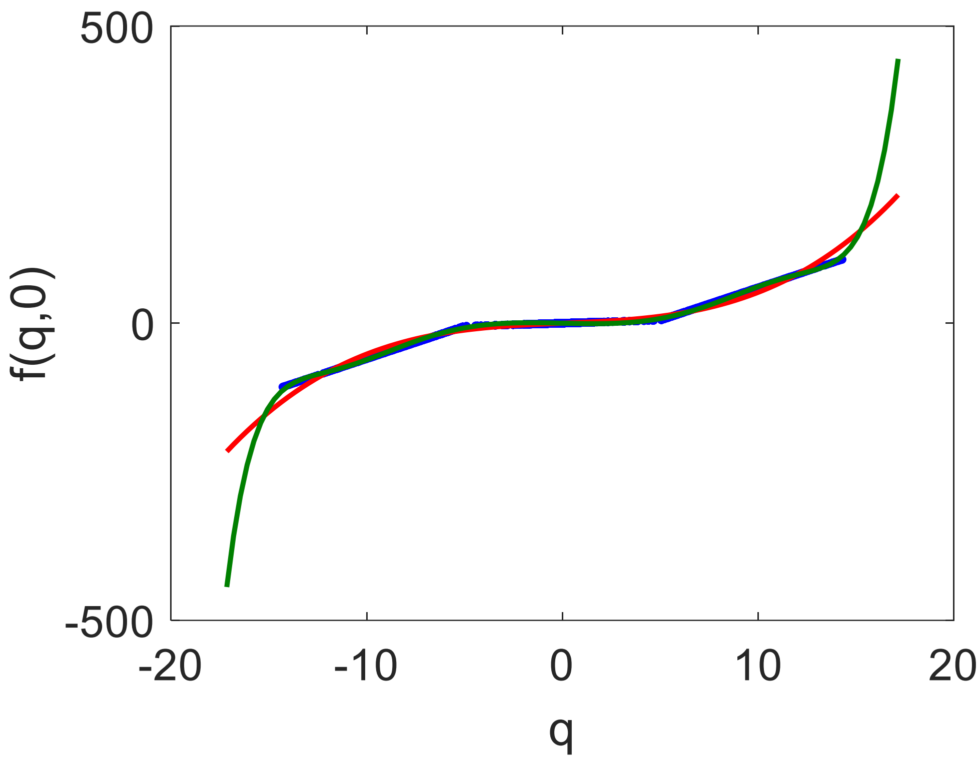


3rd order polynomial
MSE = 16.6%

9th order polynomial
MSE = 5.73%

Is this a good fit?

Yes but… Be Careful About Overfitting!



3rd order polynomial

9th order polynomial

The MSE does not give the full picture.

Be careful about extrapolation!

Nonlinearity characterization is crucial for parameter estimation.

The Parametric RFS for Mdof Systems in Summary

The parametric RFS:

is a very simple method.

works well (provided nonlinear characterization is correct).

requires to know the full $\mathbf{M}$ and potentially $\mathbf{C}$ and $\mathbf{K}$ as well (not easy in experiments).

can work with $\boldsymbol{\phi}^T\mathbf{M}\boldsymbol{\phi}$, which can be determined more easily, but the method becomes approximate.

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Parameter estimation in linear structural models.

Nonlinear problem statement, state-space model structure and FNSI.

Numerical application to the SmallSat spacecraft.

Experimental application to a solar array structure.

State-Space Formulation of the Estimation Problem

$$\boxed{\mathbf{M}\,\ddot{\mathbf{q}} + \mathbf{C}_v\,\dot{\mathbf{q}} + \mathbf{K}\,\mathbf{q}} = \boxed{\mathbf{p}}$$

linear system forcing function

state eq.
measurement eq.

$$\begin{cases} \dot{\mathbf{x}}(t) = \boxed{\mathbf{A}}\,\mathbf{x}(t) + \boxed{\mathbf{B}\,\mathbf{u}(t)} \\ \mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{u}(t) \end{cases}$$

The state-space matrices

$$\mathbf{A} \in \mathbb{R}^{n\times n} \qquad \mathbf{B} \in \mathbb{R}^{n\times i}$$

$$\mathbf{C} \in \mathbb{R}^{o\times n} \qquad \mathbf{D} \in \mathbb{R}^{o\times i}$$

are now the parameters to be estimated.

Why Use State-Space Models?

Most general representation of linear systems.

They are naturally applicable to the multi-input, multi-output case.

There exist efficient algorithms to solve linear state-space identification problems, *e.g.*, subspace methods.

Discrete-time Frequency-domain Representation

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{u}(t) \end{cases}$$

Discrete time
+
Frequency domain

$$\begin{cases} z_k\,\mathbf{X}(k) = \mathbf{A_d}\,\mathbf{X}(k) + \mathbf{B_d}\,\mathbf{U}(k) \\ \mathbf{Y}(k) = \mathbf{C_d}\,\mathbf{X}(k) + \mathbf{D_d}\,\mathbf{U}(k) \end{cases}$$

z-transform variable: $z_k = e^{j2\pi k/N}$

How to find the state-space matrices?

The state-space matrices can be found as a least-squares solution:

$$(\hat{\mathbf{A}}_\mathbf{d}, \hat{\mathbf{B}}_\mathbf{d}, \hat{\mathbf{C}}_\mathbf{d}, \hat{\mathbf{D}}_\mathbf{d}) = \arg\min \sum_k \left| \boxed{\mathbf{Y}(k)} - \left(\mathbf{C}_\mathbf{d}(z_k\mathbf{I} - \mathbf{A}_\mathbf{d})^{-1}\mathbf{B}_\mathbf{d} + \mathbf{D}_\mathbf{d}\right) \boxed{\mathbf{U}(k)} \right|^2$$

Measured (for $\mathbf{Y}(k)$) — Measured (for $\mathbf{U}(k)$)

They can be calculated using a (nonlinear) optimization procedure. However, this problem

- is difficult to initialize
- can converge to local minima, or not at all

Other procedures, such as **subspace methods** or PolyMAX can be used to overcome these issues.

Linear Subspace Methods: the Case Without Input

$$\begin{cases} z_k\,\mathbf{X}(k) = \mathbf{A_d}\,\mathbf{X}(k) + \mathbf{B_d}\,\mathbf{U}(k) \\ \mathbf{Y}(k) = \mathbf{C_d}\,\mathbf{X}(k) + \mathbf{D_d}\,\mathbf{U}(k) \end{cases}$$

$$\begin{aligned} z_k^l\mathbf{Y}(k) &= z_k^l\mathbf{C_d}\mathbf{X}(k) \\ &= z_k^{l-1}\mathbf{C_d}z_k\mathbf{X}(k) \\ &= z_k^{l-1}\mathbf{C_d}\mathbf{A_d}\mathbf{X}(k) \\ &= z_k^{l-2}\mathbf{C_d}\mathbf{A_d^2}\mathbf{X}(k) \\ &= \cdots \\ &= \mathbf{C_d}\mathbf{A_d^l}\mathbf{X}(k) \end{aligned}$$

Form a Hankel Matrix

$$z_k^l \mathbf{Y}(k) = \mathbf{C_d} \mathbf{A_d}^l \mathbf{X}(k)$$

$$\mathbf{Y} = \begin{bmatrix} \mathbf{Y}(1) & \cdots & \mathbf{Y}(F) \\ \vdots & \ddots & \vdots \\ z_1^{l-1}\mathbf{Y}(1) & \cdots & z_F^{l-1}\mathbf{Y}(F) \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{C_d X}(1) & \cdots & \mathbf{C_d X}(F) \\ \vdots & \ddots & \vdots \\ \mathbf{C_d A_d}^{l-1}\mathbf{X}(1) & \cdots & \mathbf{C_d A_d}^{l-1}\mathbf{X}(F) \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{C_d} \\ \vdots \\ \mathbf{C_d A_d}^{l-1} \end{bmatrix} \begin{bmatrix} \mathbf{X}(1) & \cdots & \mathbf{X}(F) \end{bmatrix}$$

Properties of the Hankel Matrix

$$\mathbf{Y} = \underbrace{\begin{bmatrix} \mathbf{C_d} \\ \vdots \\ \mathbf{C_d A_d^{l-1}} \end{bmatrix}}_{ol \times n} \underbrace{\begin{bmatrix} \mathbf{X}(1) & \cdots & \mathbf{X}(F) \end{bmatrix}}_{n \times F}$$

Now, recall that $\mathrm{rank}(\mathbf{UV}) \leq \min(\mathrm{rank}(\mathbf{U}), \mathrm{rank}(\mathbf{V}))$

So if $ol \geq n$ and $F \geq n$,

$$\mathrm{rank}(\mathbf{Y}) \leq n$$

It is thus possible to find the order of the system n in theory.

The SVD is Used to Get the Rank of the Hankel Matrix

$$\mathbf{Y} = \begin{bmatrix} \mathbf{C_d} \\ \vdots \\ \mathbf{C_d A_d^{\mathit{l}-1}} \end{bmatrix} [\mathbf{X}(1) \quad \cdots \quad \mathbf{X}(F)] \qquad \text{rank}(\mathbf{Y}) \leq n$$

$$= \mathbf{U\Sigma V}^H = [\mathbf{U}_1 \quad \mathbf{U}_2] \begin{bmatrix} \mathbf{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1^H \\ \mathbf{V}_2^H \end{bmatrix} = \mathbf{U}_1 \mathbf{\Sigma}_1 \mathbf{V}_1^H$$

Recalling that the states can be (re)defined arbitrarily, one can choose

$$[\mathbf{X}(1) \quad \cdots \quad \mathbf{X}(F)] = \mathbf{\Sigma}_1^{1/2} \mathbf{V}_1^H$$

$$\begin{bmatrix} \mathbf{C_d} \\ \vdots \\ \mathbf{C_d A_d^{\mathit{l}-1}} \end{bmatrix} = \mathbf{U}_1 \mathbf{\Sigma}_1^{1/2}$$

The State-space Matrices Can Be Retrieved from the SVD

$$\begin{bmatrix} \mathbf{C_d} \\ \vdots \\ \mathbf{C_d A_d}^{l-1} \end{bmatrix} = \mathbf{U}_1 \mathbf{\Sigma}_1^{1/2} = \mathbf{\Gamma}$$

The matrix $\hat{\mathbf{C}}_\mathbf{d}$ is obtained from the first o lines of $\mathbf{\Gamma}$.

The matrix $\mathbf{\Gamma}$ has a special shift structure that can be exploited

$$\underline{\mathbf{\Gamma}} = \begin{bmatrix} \mathbf{C_d A_d} \\ \vdots \\ \mathbf{C_d A_d}^{l-1} \end{bmatrix} = \begin{bmatrix} \mathbf{C_d} \\ \vdots \\ \mathbf{C_d A_d}^{l-2} \end{bmatrix} \mathbf{A_d} = \overline{\mathbf{\Gamma}} \mathbf{A_d}$$

So, using a pseudo-inverse,

$$\widehat{\mathbf{A}}_\mathbf{d} = \left(\overline{\mathbf{\Gamma}}\right)^\dagger \underline{\mathbf{\Gamma}}$$

What If There Are Nonzero Inputs?

$$\begin{cases} z_k\,\mathbf{X}(k) = \mathbf{A_d}\,\mathbf{X}(k) + \mathbf{B_d}\,\mathbf{U}(k) \\ \mathbf{Y}(k) = \mathbf{C_d}\,\mathbf{X}(k) + \mathbf{D_d}\,\mathbf{U}(k) \end{cases}$$

$$z_k^l \mathbf{Y}(k)$$

$$= z_k^{l-1}\mathbf{C_d} z_k \mathbf{X}(k) + z_k^l \mathbf{D_d}\mathbf{U}(k)$$

$$= z_k^{l-1}\mathbf{C_d}\mathbf{A_d}\mathbf{X}(k) + z_k^{l-1}\mathbf{C_d}\mathbf{B_d}\mathbf{U}(k) + z_k^l \mathbf{D_d}\mathbf{U}(k)$$

$$= z_k^{l-2}\mathbf{C_d}\mathbf{A_d^2}\mathbf{X}(k) + \cdots$$

$$= \mathbf{C_d}\mathbf{A_d^l}\mathbf{X}(k) + \left(\sum_{m=1}^{l} z_k^{l-m}\mathbf{C_d}\mathbf{A_d^{m-1}}\mathbf{B_d} + z_k^l \mathbf{D_d}\right)\mathbf{U}(k)$$

Apply the Same Procedure and Remove the Effect of the Input

$$z_k^l \mathbf{Y}(k) = \mathbf{C_d} \mathbf{A_d}^l \mathbf{X}(k) + \left(\sum_{m=1}^{l} z_k^{l-m} \mathbf{C_d} \mathbf{A_d}^{m-1} \mathbf{B_d} + z_k^l \mathbf{D_d} \right) \mathbf{U}(k)$$

$$\mathbf{Y} = \begin{bmatrix} \mathbf{Y}(1) & \cdots & \mathbf{Y}(F) \\ \vdots & \ddots & \vdots \\ z_1^{l-1} \mathbf{Y}(1) & \cdots & z_F^{l-1} \mathbf{Y}(F) \end{bmatrix}$$

$$= \mathbf{\Gamma X} + \mathbf{\Lambda U}$$

Now, compute $\mathbf{U}^\perp$ (e.g., with a QR decomposition) such that

$$\mathbf{U}\mathbf{U}^\perp = \mathbf{0}$$

and thus

$$\mathbf{Y}\mathbf{U}^\perp = \mathbf{\Gamma X}\mathbf{U}^\perp$$

Estimation of the Full State-Space Model

$$\mathbf{Y}\mathbf{U}^{\perp} = \boldsymbol{\Gamma}\mathbf{X}\mathbf{U}^{\perp}$$

=> $\boldsymbol{\Gamma}$ can be estimated with a SVD, and then one gets $\widehat{\mathbf{A}}_{\mathbf{d}}$ and $\widehat{\mathbf{C}}_{\mathbf{d}}$.

Eventually, $\widehat{\mathbf{B}}_d$ and $\widehat{\mathbf{D}}_d$ are obtained as the solution of the **linear** least-squares problem

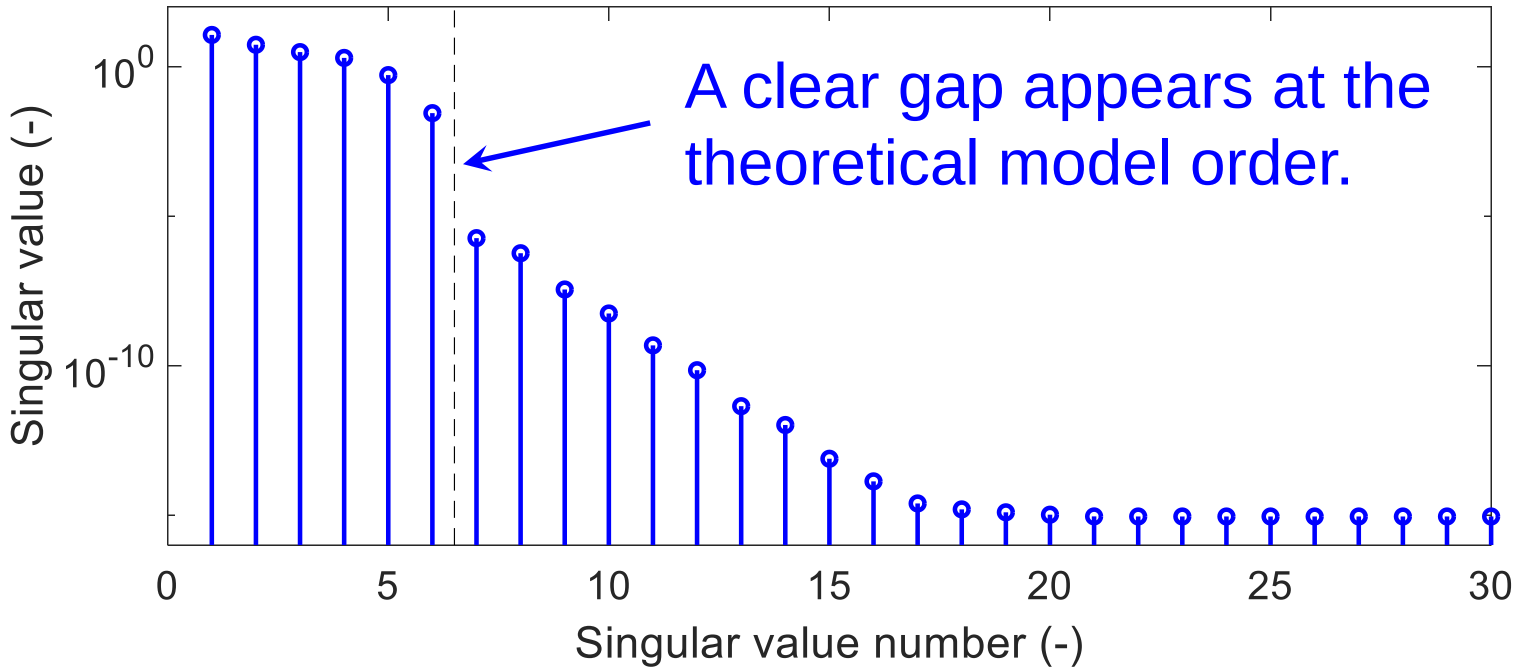
$$(\widehat{\mathbf{B}}_{\mathbf{d}}, \widehat{\mathbf{D}}_{\mathbf{d}}) = \arg\min \sum_k \left| \mathbf{Y}(k) - \left(\widehat{\mathbf{C}}_{\mathbf{d}} \left(z_k \mathbf{I} - \widehat{\mathbf{A}}_{\mathbf{d}} \right)^{-1} \mathbf{B}_{\mathbf{d}} + \mathbf{D}_{\mathbf{d}} \right) \mathbf{U}(k) \right|^2$$

Measured ($\mathbf{Y}(k)$), Known ($\widehat{\mathbf{C}}_{\mathbf{d}}$), Known ($\widehat{\mathbf{A}}_{\mathbf{d}}$), Measured ($\mathbf{U}(k)$)

Selecting the Model Order in a Practical Case

We consider the linear beam model, excited between 1 and 500 Hz (the first three modes are excited).

The singular values of the Hankel matrix can be used to select the model order.

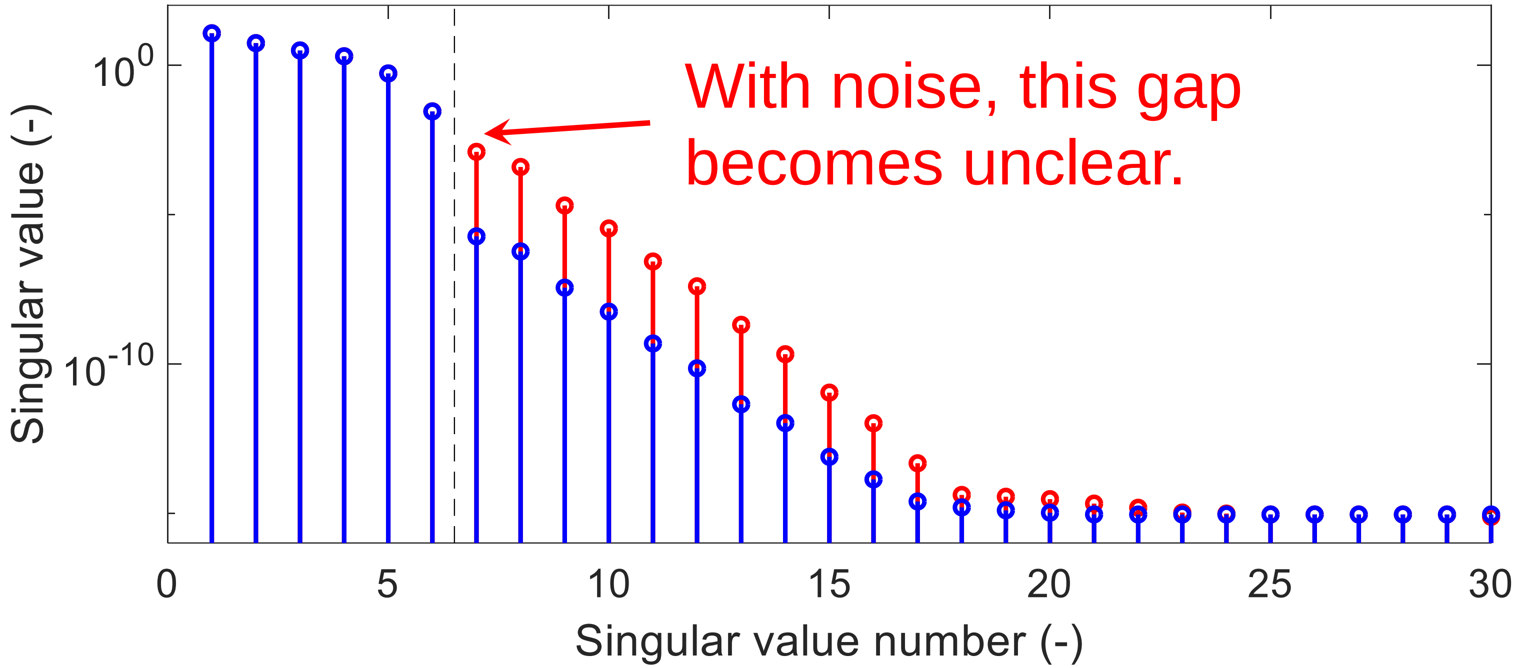


Although nonzero, most of the singular values can be discarded.

The Effect of 1% Input and Output Noise

We consider the linear beam model, excited between 1 and 500 Hz (the first three modes are excited).

The singular values of the Hankel matrix can be used to select the model order.

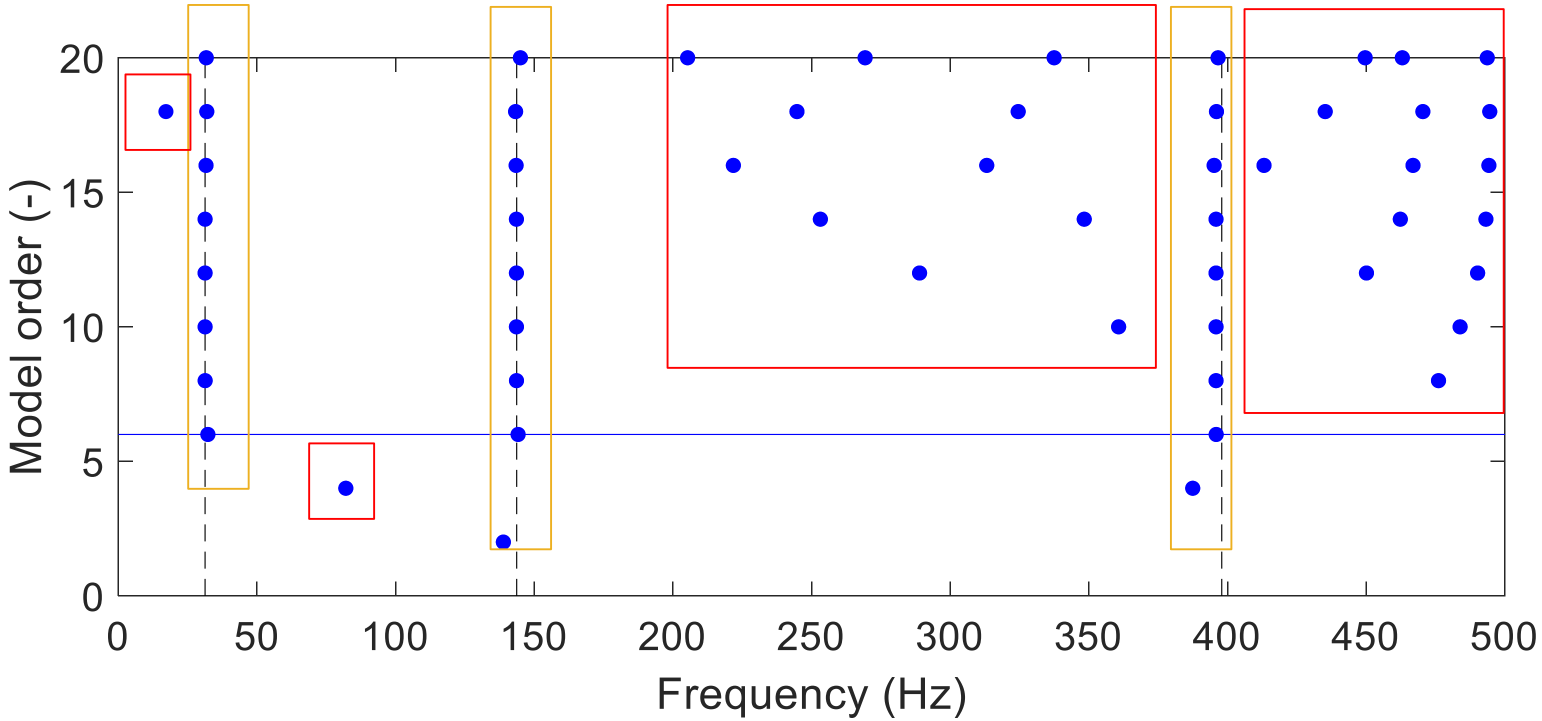


Model order selection with singular values can be complicated in practice.

What are the Poles of the Estimated System?

We can compute the poles of the system for different orders.

We observe **physical poles** as well as **spurious ones** which do not persist as we increase the order model.

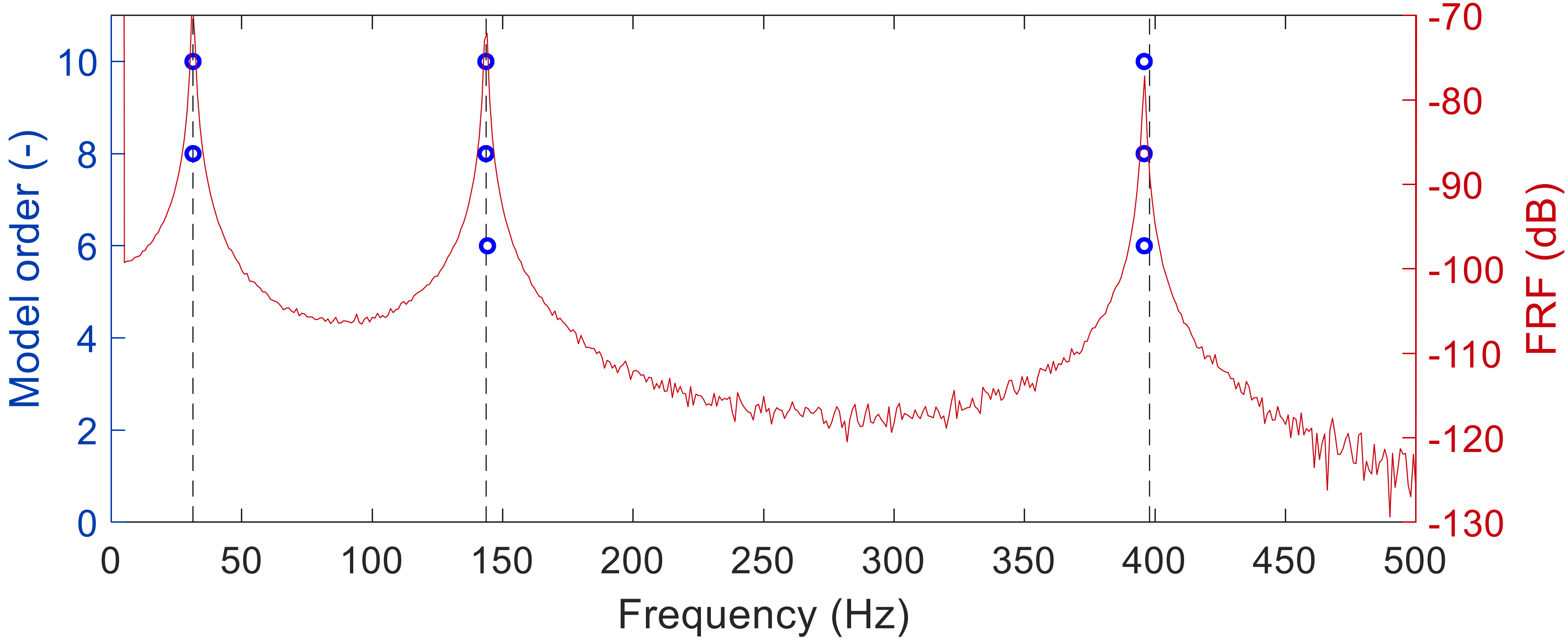


The Stabilization Diagram: a Decision-making Tool

We can compare the poles of different orders (e.g., their frequency and damping).

Those that do not change more than some tolerance are considered as **stabilized poles**, which are likely physical poles.

Plotting them yields the stabilization diagram.



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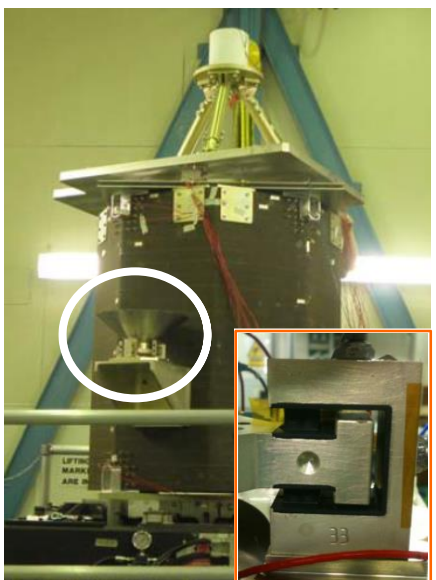
Nonlinear Problem Statement Assuming Characterisation

assumed

$$\mathbf{M}\,\ddot{\mathbf{q}} + \mathbf{C}_v\,\dot{\mathbf{q}} + \mathbf{K}\,\mathbf{q} + k_{nl}\,\mathbf{f}_{nl} = \mathbf{p}$$

measured (for $\ddot{\mathbf{q}}$) · measured (for $\mathbf{p}$)



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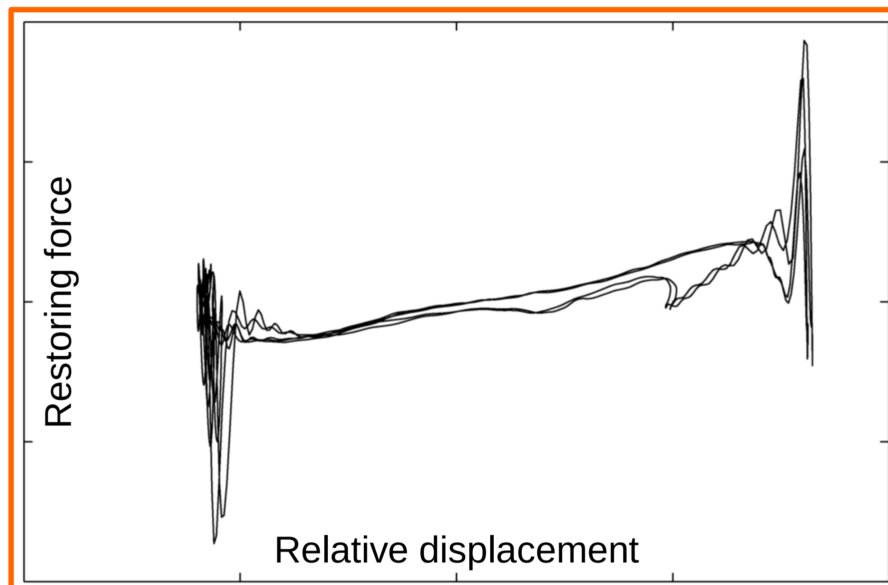


Nonlinear Coefficients Are New Parameters to Estimate

assumed

$$\mathbf{M}\boxed{\ddot{\mathbf{q}}} + \mathbf{C}_v\,\dot{\mathbf{q}} + \mathbf{K}\,\mathbf{q} + k_{nl}\boxed{\mathbf{f}_{nl}} = \boxed{\mathbf{p}}$$

measured　　　　measured

Parameters to be estimated:

nonlinearity strength k_{nl}

underlying linear FRFs $\mathbf{G}_p^{-1}(j\omega) = -\omega^2\mathbf{M} + j\omega\mathbf{C}_v + \mathbf{K}$

A Simple yet Powerful Reformulation

Move the nonlinear forces to the right-hand side of the EOM.

$$\mathbf{M}\,\ddot{\mathbf{q}} + \mathbf{C}_v\,\dot{\mathbf{q}} + \mathbf{K}\,\mathbf{q} = \mathbf{p} - k_{nl}\,\mathbf{f}_{nl}$$

Equivalent Linear State-Space Identification Problem

$$\boxed{\mathbf{M}\,\ddot{\mathbf{q}} + \mathbf{C}_v\,\dot{\mathbf{q}} + \mathbf{K}\,\mathbf{q}} = \boxed{\mathbf{p} - k_{nl}\,\mathbf{f}_{nl}}$$

underlying linear system

extended
forcing function

state eq.
measurement eq.

$$\begin{cases} \dot{\mathbf{x}}(t) = \boxed{\mathbf{A}}\,\mathbf{x}(t) + \boxed{\mathbf{B}\,\mathbf{e}(t)} \\ \mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{e}(t) \end{cases}$$

Reminder: the Extended Input Term Is Known

$$\mathbf{M}\,\ddot{\mathbf{q}} + \mathbf{C}_v\,\dot{\mathbf{q}} + \mathbf{K}\,\mathbf{q} = \mathbf{p} - k_{nl}\,\mathbf{f}_{nl}$$

underlying linear system

extended
forcing function

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{e}(t) \\ \mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{e}(t) \end{cases}$$

$$\mathbf{B}\,\mathbf{e}(t) = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{M}^{-1} & k_{nl}\mathbf{M}^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{p}(t) \\ -\mathbf{f}_{nl}(t) \end{pmatrix}$$

known

Linear Subspace "Machinery" Can Be Applied to our Problem

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{e}(t) \\ \mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{e}(t) \end{cases}$$

extended input

Discrete time
+
Frequency domain

$$\begin{cases} z_k\,\mathbf{X}(k) = \mathbf{A_d}\,\mathbf{X}(k) + \mathbf{B_d}\,\mathbf{E}(k) \\ \mathbf{Y}(k) = \mathbf{C_d}\,\mathbf{X}(k) + \mathbf{D_d}\,\mathbf{E}(k) \end{cases}$$

Frequency-domain nonlinear subspace identification (FNSI) method

z-transform variable: $z_k = e^{j2\pi k/N}$

Outline of Lecture 9

The restoring force surface method.

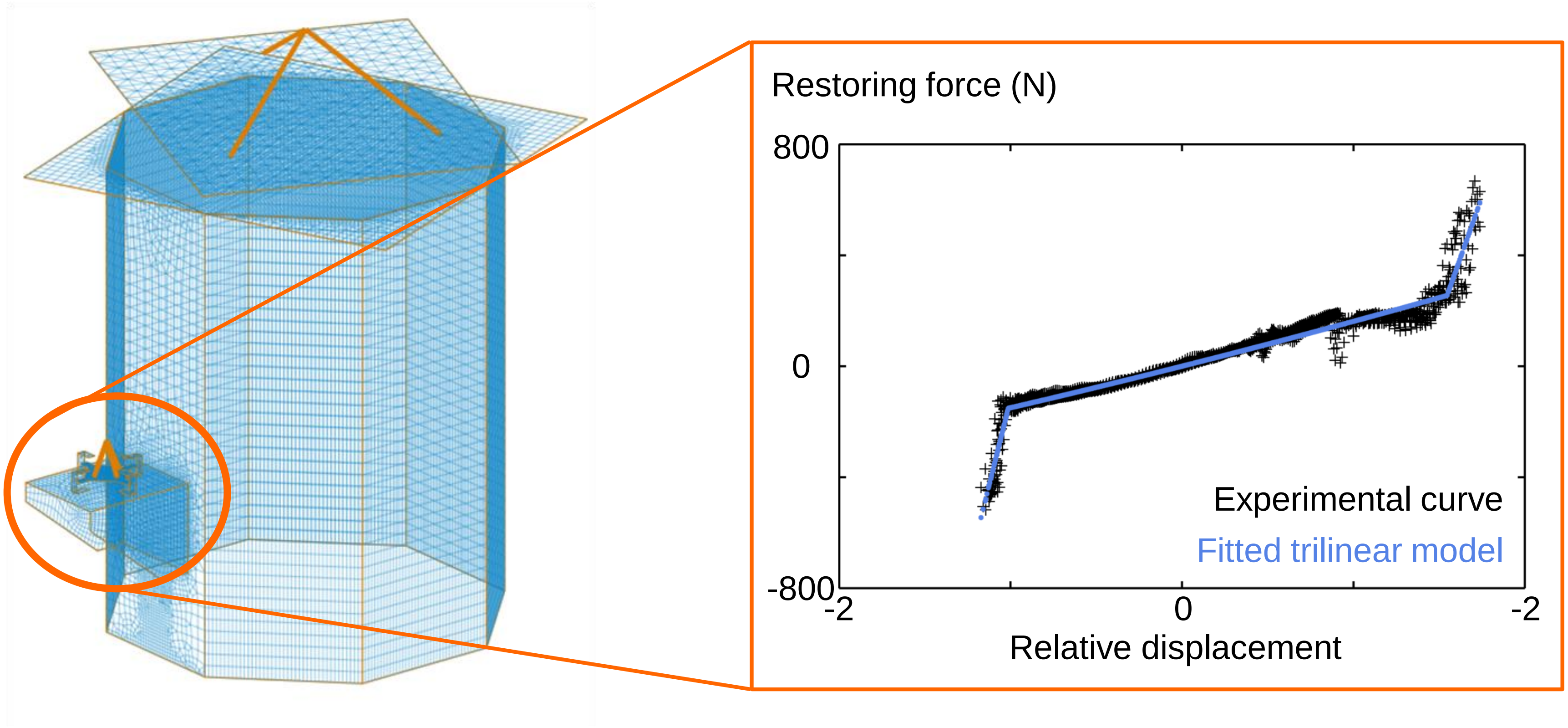
Parameter estimation in linear structural models.

Nonlinear problem statement, state-space model structure and FNSI.

Numerical application to the SmallSat spacecraft.

Experimental application to a solar array structure.

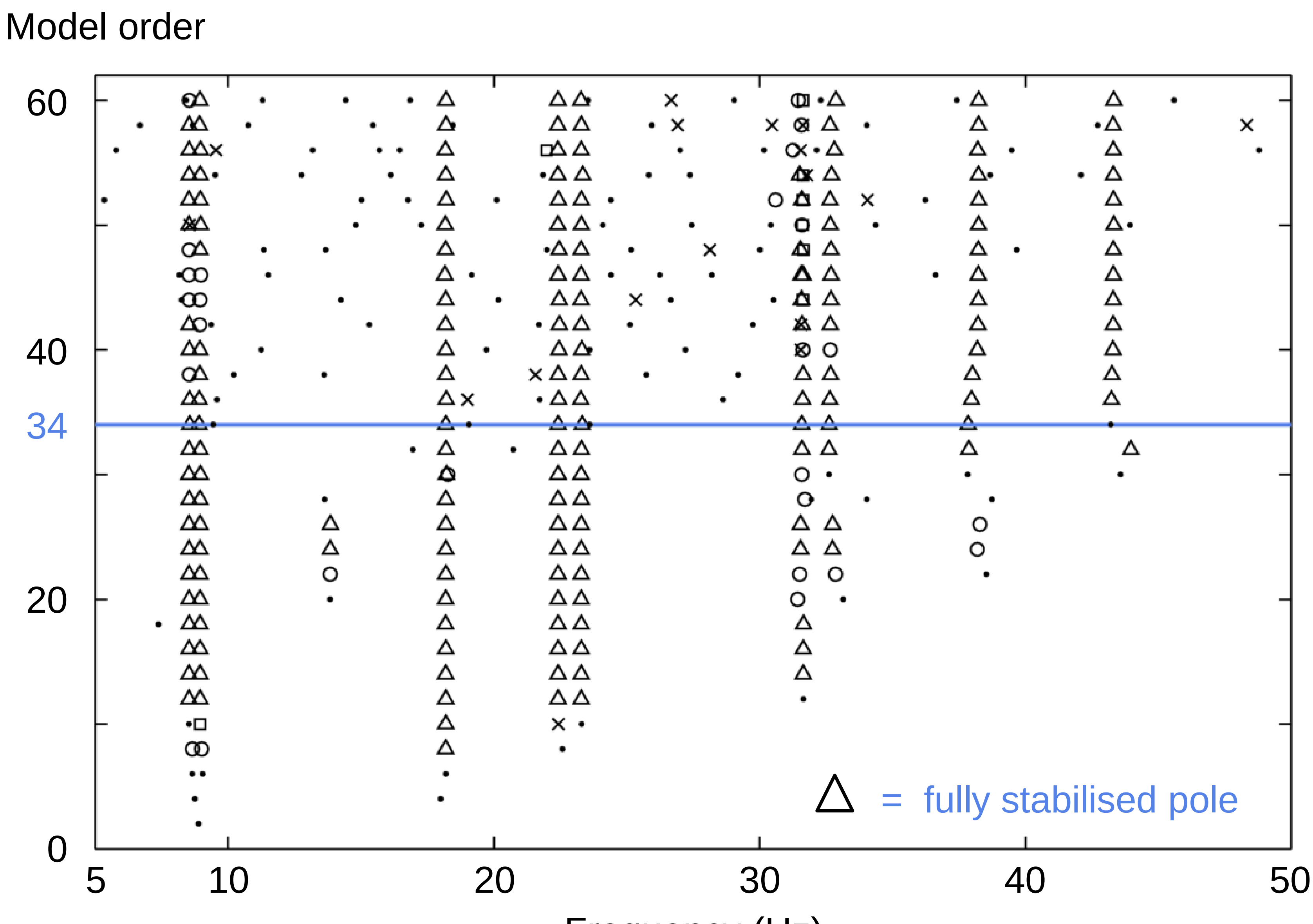
SmallSat FEM with Experimental Nonlinearities



Reduced FE model
accurate between 0 – 100 Hz

12 piecewise nonlinearities
activated under a 200 N excitation

Model Order Selection via a Stabilisation Diagram



Accurate ID of the Lateral Nonlinear Coefficients

NC	Exact value	Real part	Error (%)	Log_{10} (R/I)
1 – X (neg.)	26.76	26.82	0.22	1.70
2 – X (pos.)	46.23	47.27	2.20	2.33
3 – Y (neg.)	26.76	26.78	0.05	2.06
4 – Y (pos.)	46.23	46.58	0.76	2.11

Outline of Lecture 9

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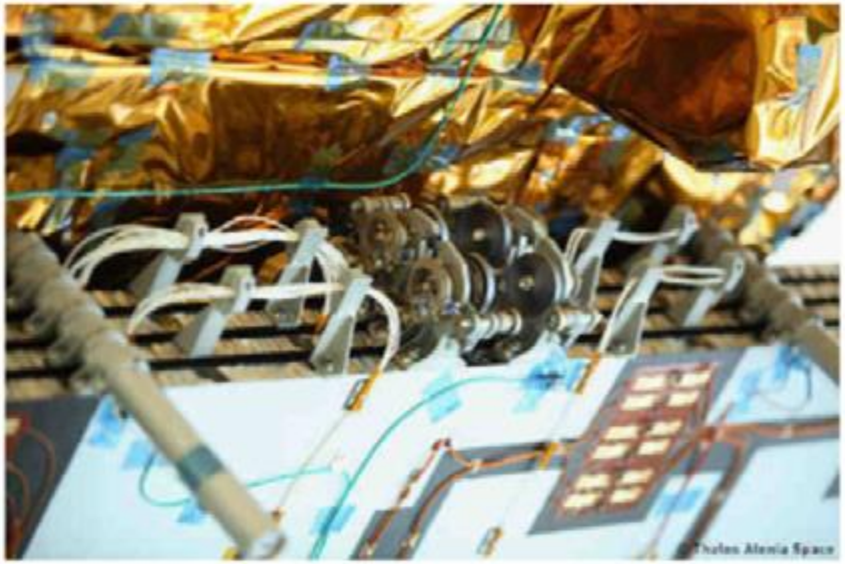
Solar Array Dynamics in Folded Configuration

Opening test

Complex connections between panels

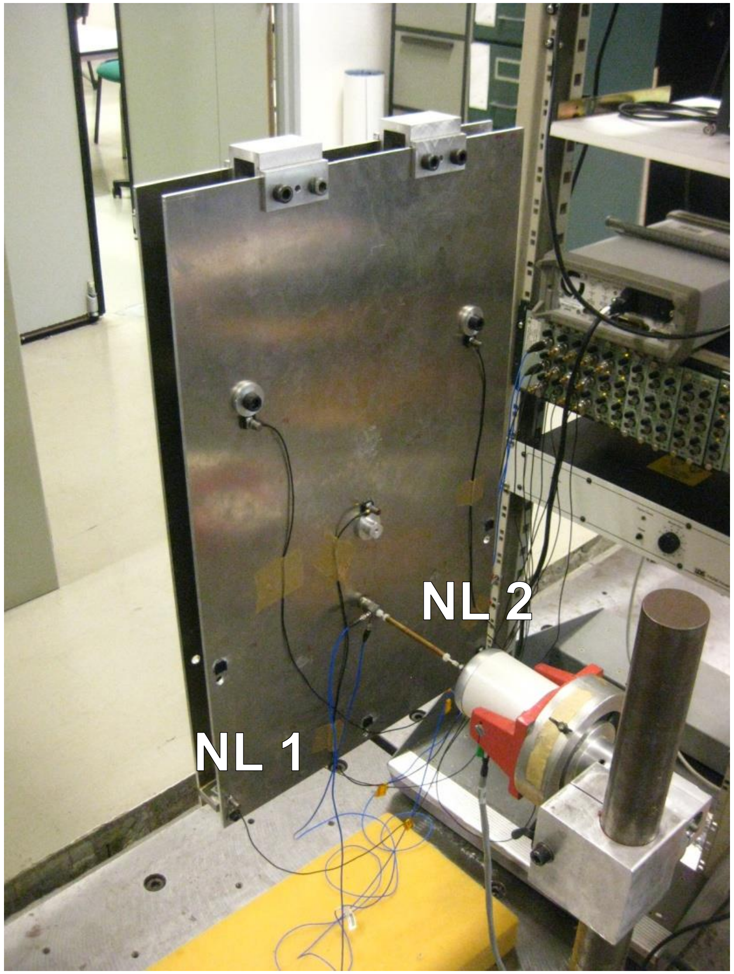


Gaps, friction, impacts, large displacements may be triggered.

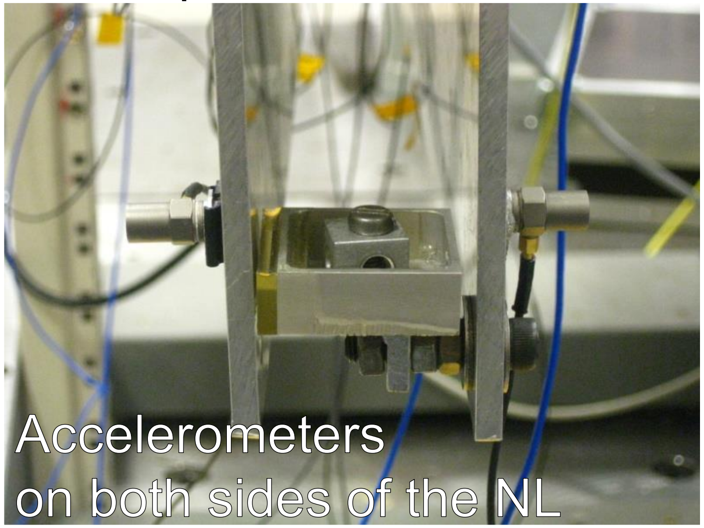
Start of a study led by Thales Cannes and CNES (France).

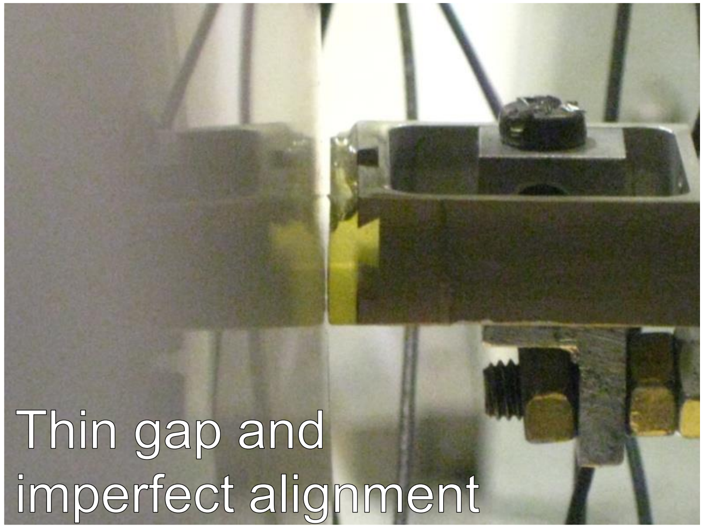
Development of a Simplified Test Rig in Besançon, France

Panels in free-free conditions



Close-up of a solithane snubber





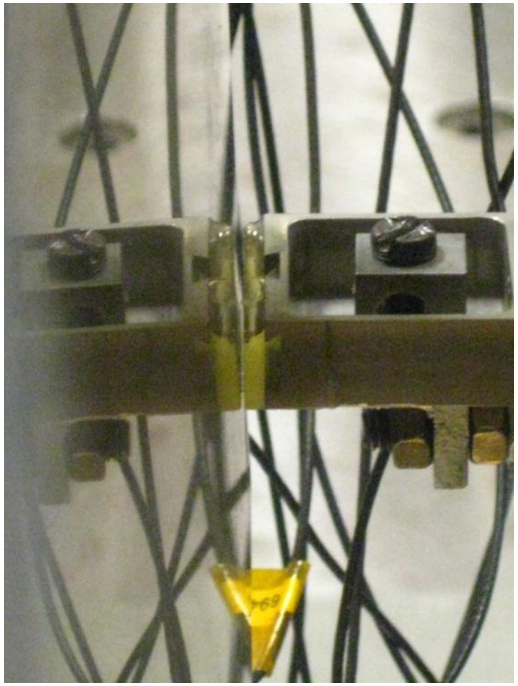
Cubic Splines May Outperform Ordinary Polynomials

Complex NL mechanisms are commonly captured using high-order polynomials that may not be stable throughout.

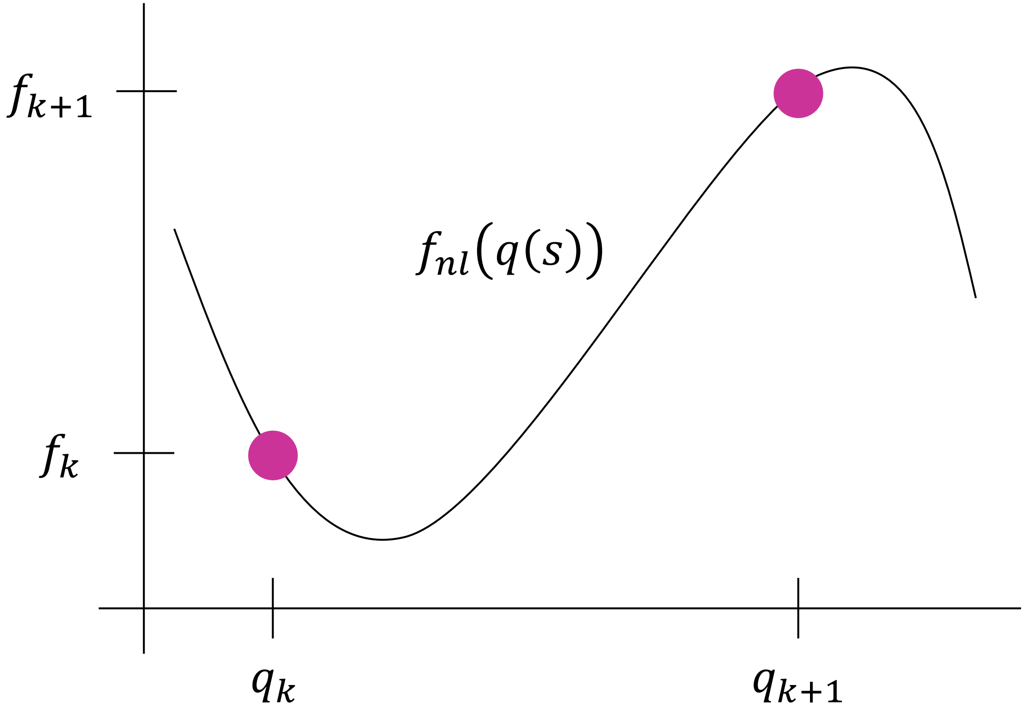
Cubic splines are simple, stable, flexible and intuitive.

The FNSI method can calculate a large number of parameters.

Nonlinear Characterisation using Cubic Splines

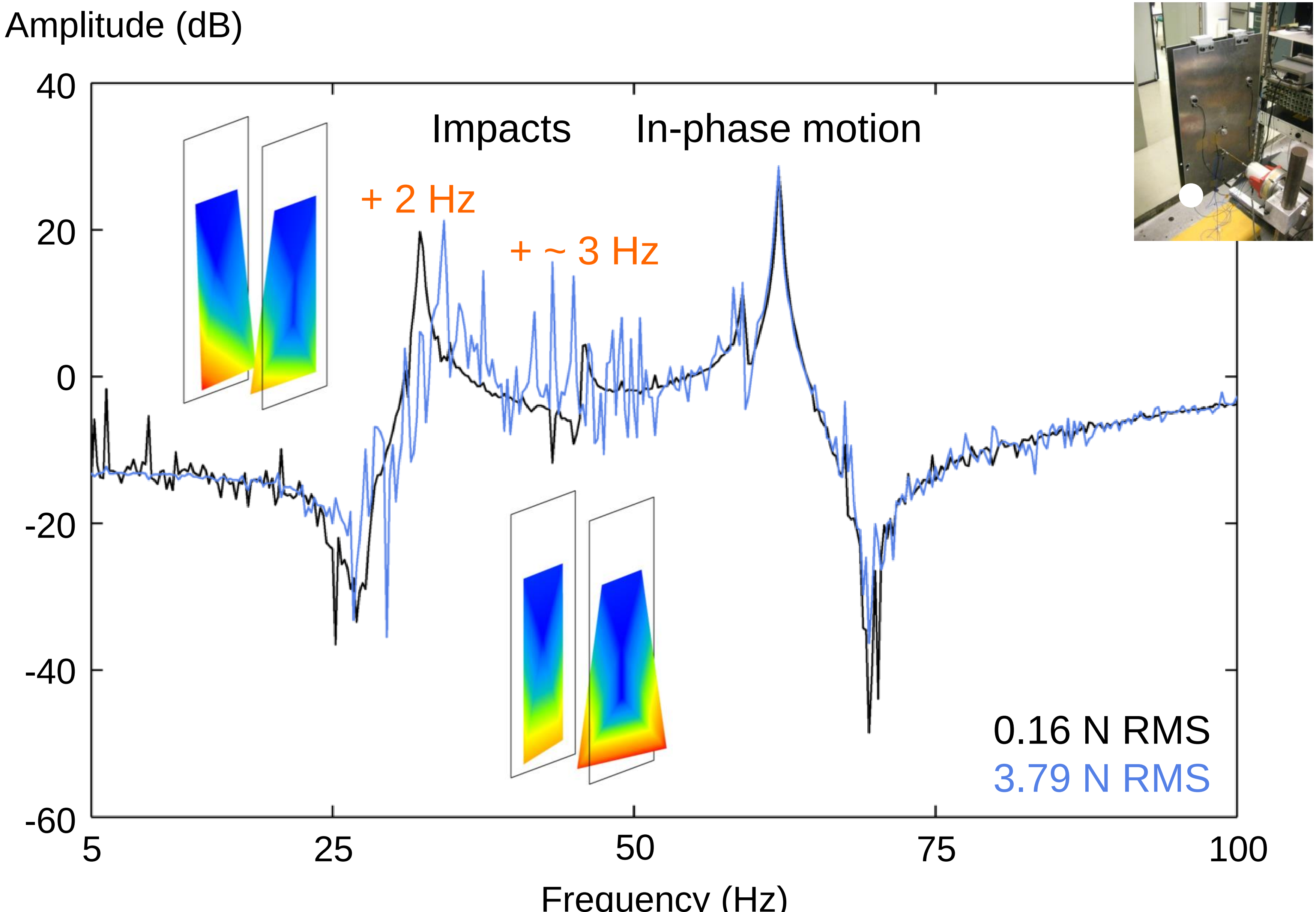


Difficult to assume a functional form a priori

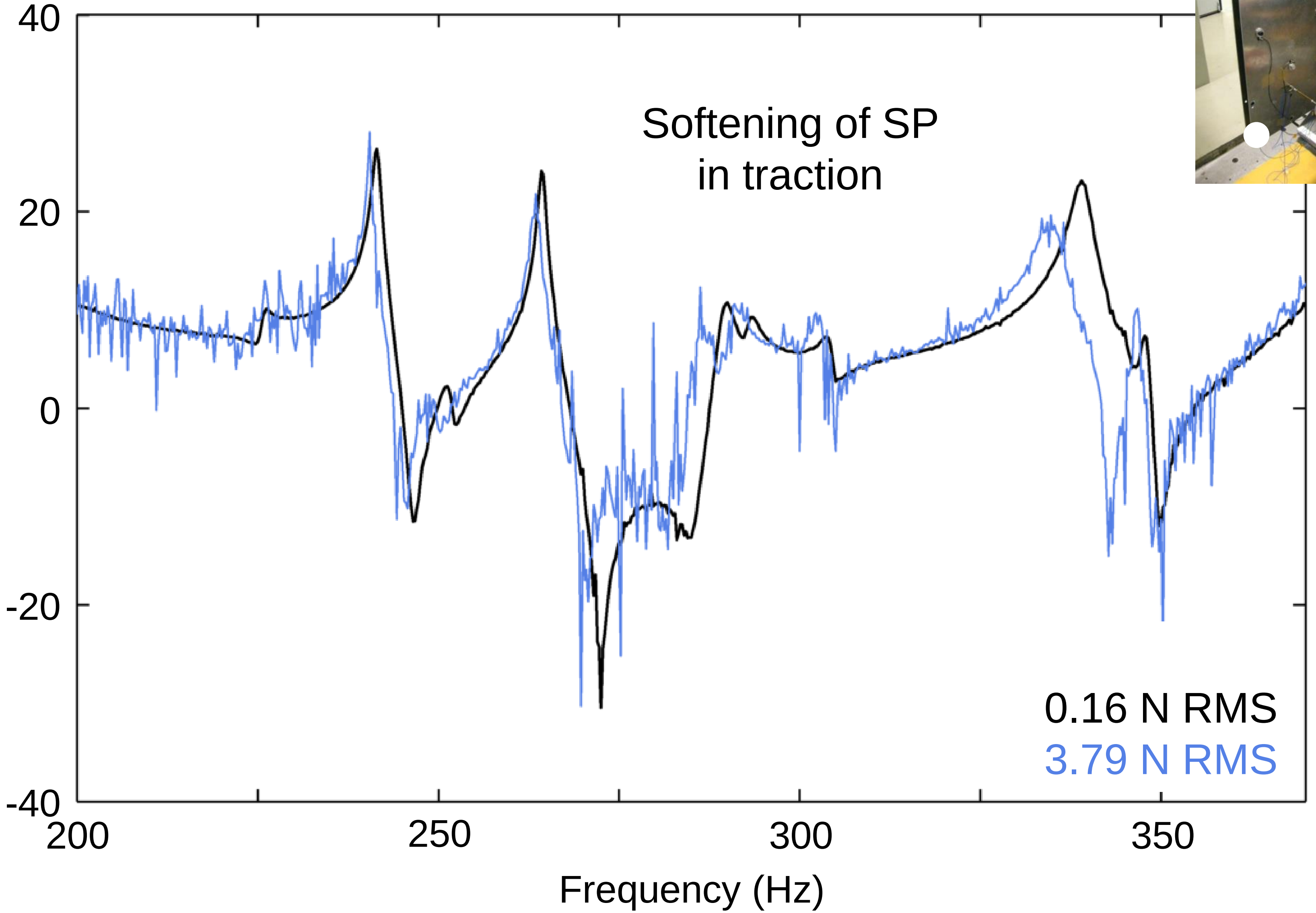


$$f_{nl}(q(s)) = (2s^3 - 2s^2 + 1)f_k + (-2s^3 + 3s^2)f_{k+1}$$
$$+ (s^3 - 2s^2 + s)(q_{k+1} - q_k)f'_k$$
$$+ (s^3 - s^2)(q_{k+1} - q_k)f'_{k+1}$$

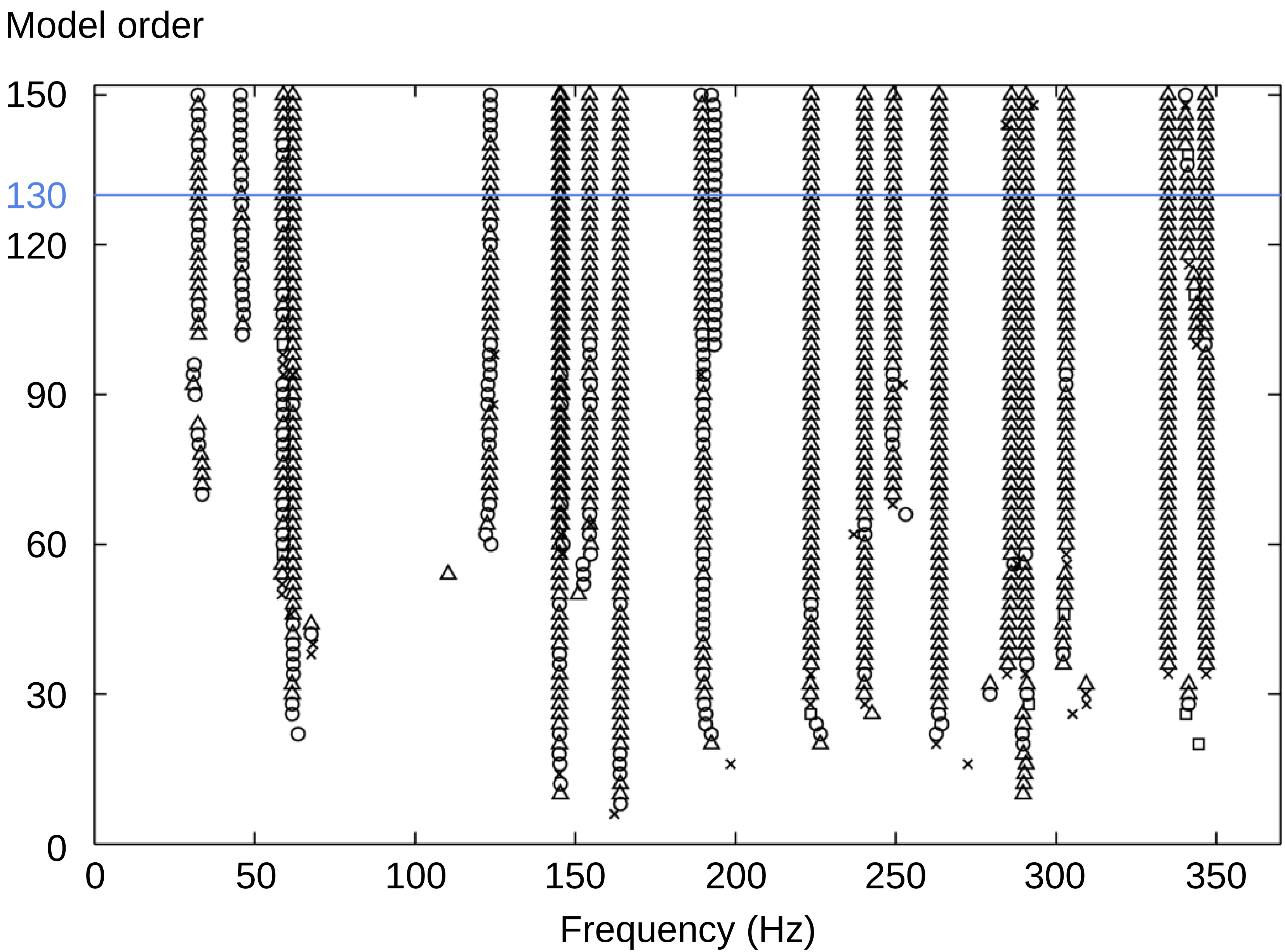
Nonlinearity Detection using FRFs at Low Frequency



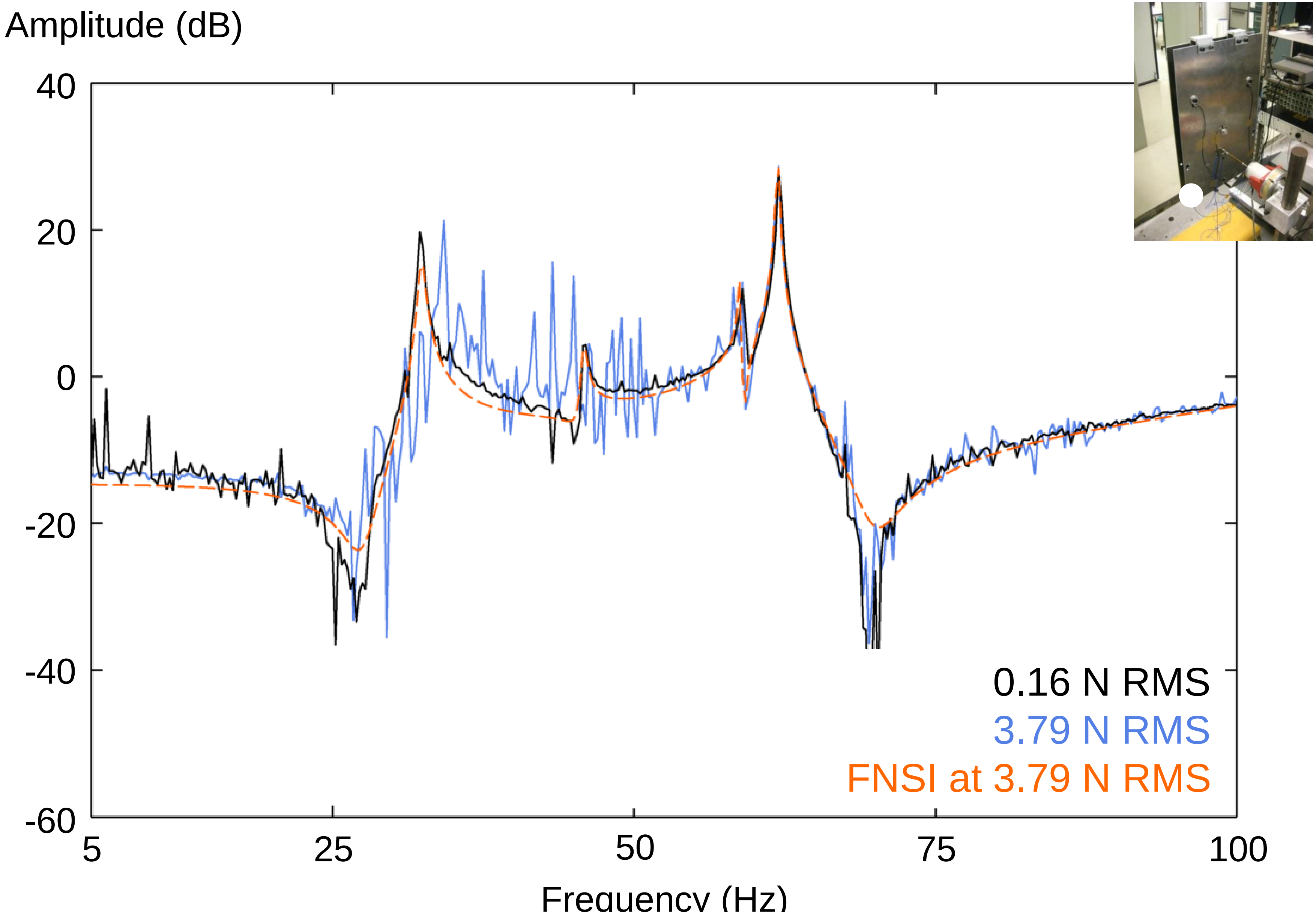
Nonlinearity Detection using FRFs at High Frequency



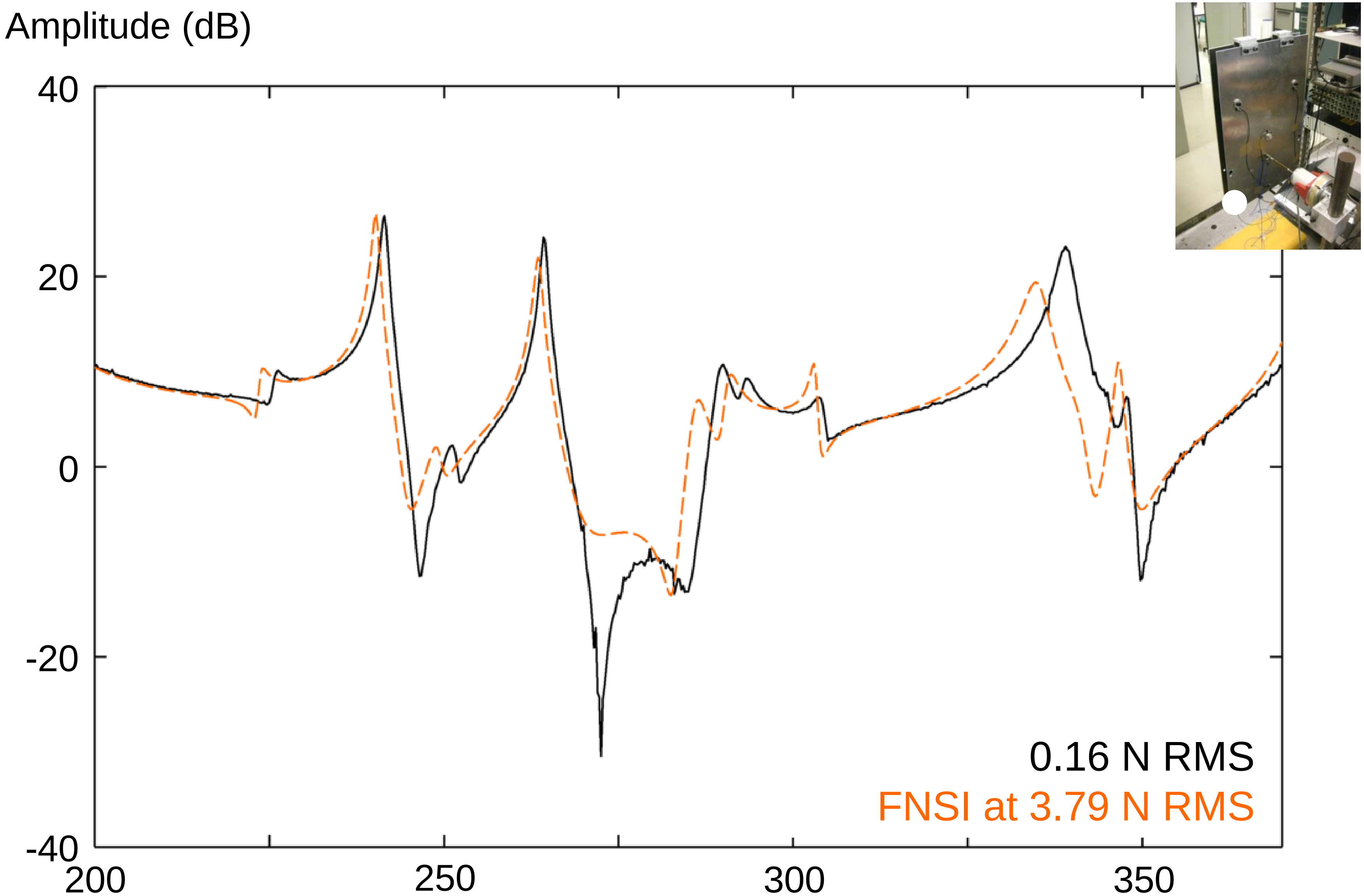
Stabilisation Diagram with Spline Nonlinearities



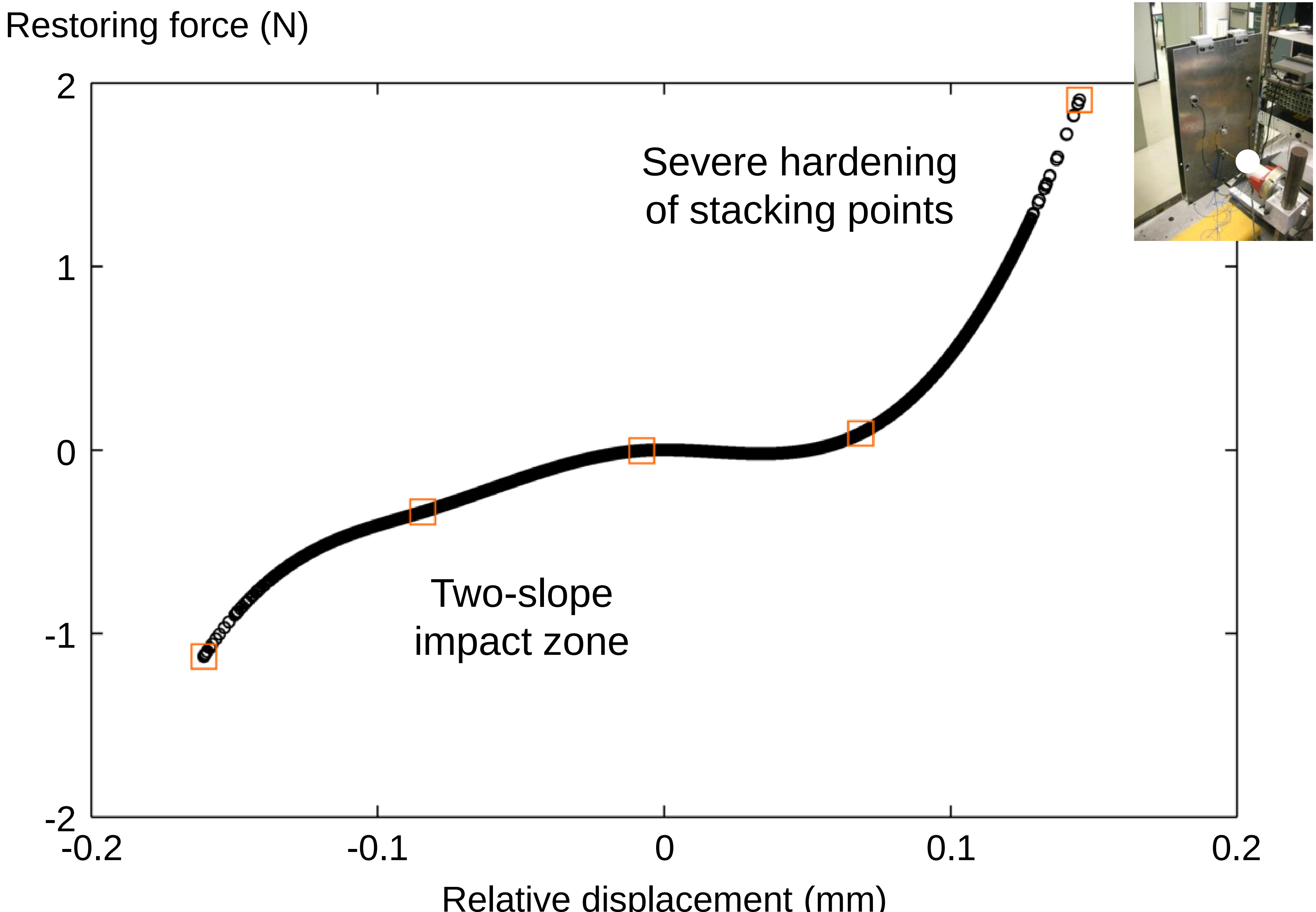
Successful Reconstruction of FRFs Below 100 Hz



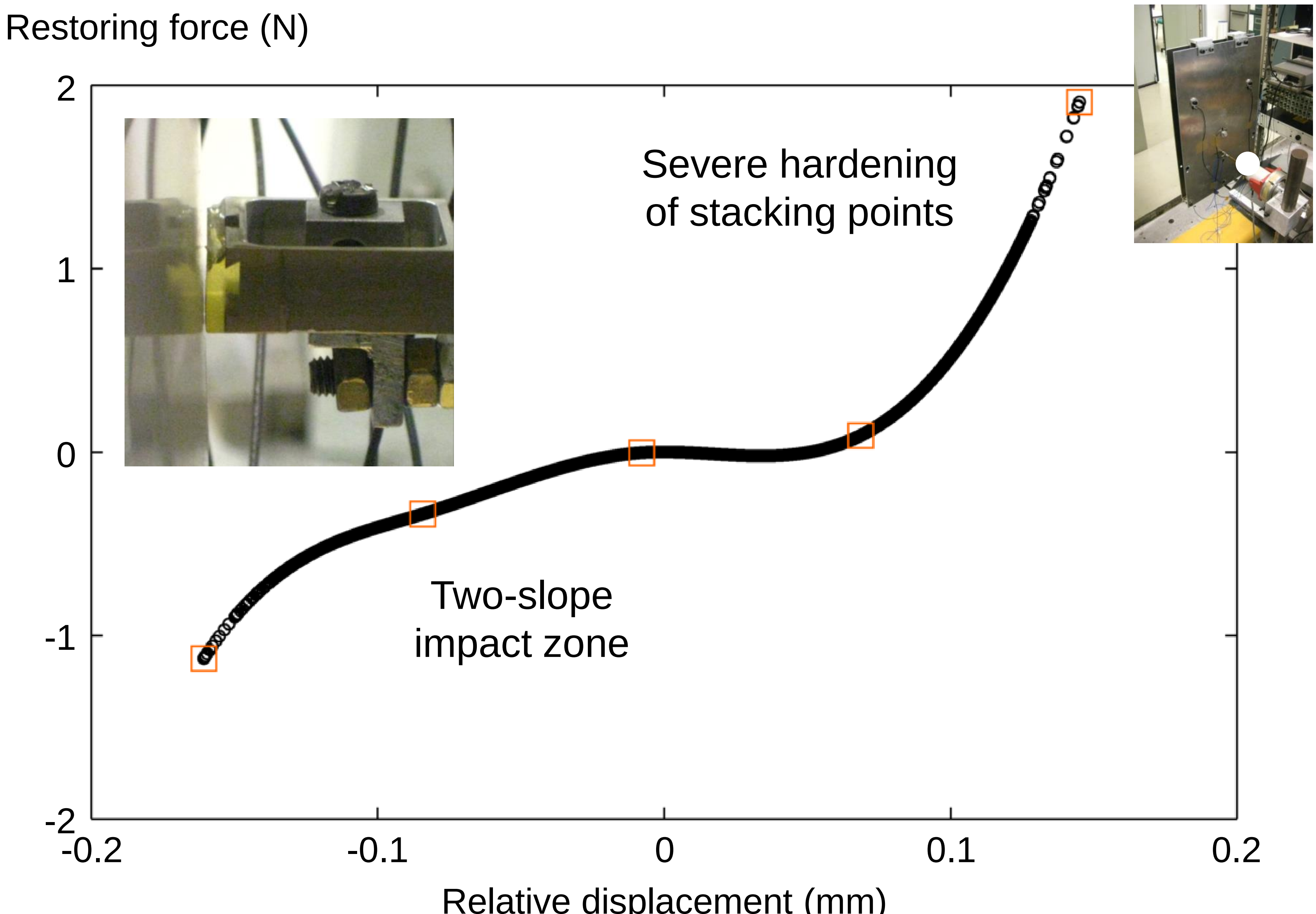
Bolt Loosening at High Frequency Is not Captured



Spline-based Nonlinear Stiffness Force with 5 Knots



Two-Slope Impacts Are Due to Snubber Misalignment



Concluding Remarks and Learning Outcomes

The RFS is a simple method but requires a lot of information.

Linear systems can be identified with (e.g.) a subspace method, and this can be extended to nonlinear systems.

The FNSI method can identify state-space models of complex nonlinear structures, assuming an accurate characterisation.

It can calculate accurately a great number of parameters.

It is compatible with stabilisation diagrams and cubic splines.

Further Readings

J.P. Noël, G. Kerschen, **Frequency-domain subspace identification for nonlinear mechanical systems**, Mechanical Systems and Signal Processing, 40, 701-717, 2013.

J.P. Noël, S. Marchesiello, G. Kerschen, **Subspace-based identification of a nonlinear spacecraft in the time and frequency domains**, Mechanical Systems and Signal Processing, 43, 217-236, 2014.

J.P. Noël, G. Kerschen, E. Foltête, S. Cogan, **Grey-box identification of a nonlinear solar array structure using cubic splines**, International Journal of Non-linear Mechanics, 67, 106-119, 2014.